THE CONSENSUS VALUE: A NEW SOLUTION CONCEPT FOR COOPERATIVE GAMES

By Y. Ju, P.E.M. Borm, P.H.M. Ruys

May 2004

ISSN 0924-7815
The Consensus Value:
a New Solution Concept for Cooperative Games\textsuperscript{1}

Yuan Ju\textsuperscript{2} \quad Peter Borm\textsuperscript{3} \quad Pieter H.M. Ruys\textsuperscript{4}

April 2004

\textsuperscript{1}The authors thank René van den Brink and Eric van Damme for helpful discussions and comments.
\textsuperscript{2}CentER for Economic Research and Department of Econometrics and Operations Research, Tilburg University, P.O.Box 90153, 5000 LE, Tilburg, the Netherlands. E-mail: Y.Ju@uvt.nl
\textsuperscript{3}CentER and Department of Econometrics and Operations Research, Tilburg University, E-mail: P.E.M.Borm@uvt.nl
\textsuperscript{4}TILEC, Department of Econometrics and Operations Research, and Tias Business School, Tilburg University. E-mail: ruys@uvt.nl
Abstract

By generalizing the standard solution for 2-person games into $n$-person cases, this paper develops a new solution concept for cooperative games: the consensus value. We characterize the consensus value as the unique function that satisfies efficiency, symmetry, the quasi dummy property and additivity. By means of the transfer property, a second characterization is provided. By defining the stand-alone reduced game, a recursive formula for the value is established. We also show that this value is the average of the Shapley value and the equal surplus solution. Furthermore, we discuss a possible generalization.

JEL classification codes: C71; D71; D63.
Keywords: Shapley value; equal surplus solution; consensus value.
1 Introduction

In this paper, we study the problem of sharing the joint gains of cooperation and try to find a solution concept which can not only be axiomatically characterized but which is also constructive (based on an explicit process of sharing gains of cooperation). Following a simple and natural way of generalizing the standard solution for 2-person games into \( n \)-person cases, we obtain a new solution concept for TU (transferable utility) games: the consensus value.

The consensus value is related to two well established solution concepts: the equal surplus solution (cf. Moulin (2003)) and the Shapley value (Shapley (1953)).

The equal surplus solution, also known as the CIS-value (Driessen and Funaki (1991), van den Brink and Funaki (2003)), assigns to every player her individual value, and distributes the remainder of the value of the grand coalition equally among all players. Thus, the equal surplus solution is a central solution concept in terms of egalitarianism. Moreover, it is particularly useful for a class of games where the only possible final outcomes are either the complete cooperation of all players or the complete breakdown of cooperation\(^1\). However, since the equal surplus solution rules out the consideration on partial cooperation, it fails to explain the interaction between coalitions and leaves the evolution process from complete breakdown to complete cooperation as a blackbox. Consequently, this solution concept seems insufficient for general \( n \)-person cooperative games but could well serve as a specific benchmark.

The Shapley value, on the other hand, takes all coalitional values into account and somehow corresponds to the players’ expected marginal contributions. Moreover, the Shapley value is characterized as the unique function that satisfies efficiency, symmetry, the dummy property and additivity. Although the Shapley value serves as the central solution concept for TU games, there is still critique. For instance, Luce and Raiffa (1957) criticize the efficiency postulate and the additivity postulate. A recent critique on the efficiency postulate can be found in Maskin (2003). In our paper, the justification of the dummy property is considered. Generally speaking, there may exist two extreme opinions about the gain of a dummy player. From the individualist point of view, we do get the classical dummy property requiring that no more value is allocated to a dummy player than her own value \( v(\{i\}) \). However, from the egalitarian or collectivistic perspective, one can argue that all members of a society including dummies should share the joint surplus equally among them. This distinction opens up the possibility to relax this postulate. In this spirit we

\(^1\)Another possible interpretation could be that we only have information about the two extreme ends of a game: the individual values and the value of the grand coalition; or simply when we do not care about partial cooperation.
introduce and discuss a so-called \textit{quasi dummy property}.

In addition, in our opinion, the constructive interpretation of the Shapley value, i.e., the marginal contribution approach, is not so convincing. Here, the terminology of “marginal contribution” is somewhat misleading. In fact, the marginal contribution is jointly created by the existing coalition of players and the entrant, but not by the entrant solely. Following this reasoning, it seems too much to give a later entrant the whole marginal value in superadditive games. Similarly, this rule is hard to implement if the marginal contribution is less than the entrant’s individual value if the loss is caused by the interaction between the entrant and the incumbents. Of course, those aspects are smoothed out in some sense by taking the average of the marginal contributions over all different orders.

Although basically we follow the same line as the Shapley value to study the problem of sharing gains of cooperation, i.e., using an average serial method, we propose to replace the allocation of marginal contributions by a method which is based on the standard solution for 2-person games. Given an ordering of players, we take a bilateral perspective and consider that any surplus is the joint contribution between an existing coalition of players (i.e., the incumbents) and an entrant. By taking the incumbents as one party and the entrant as a second party, the standard solution for 2-person games can be applied all the way with consensus. That is, all the joint surpluses are always equally split between the corresponding two parties. Since no specific ordering is pre-determined, we average over all possible permutations. Such a constructive process is helpful to solve practical problems. Ju, Ruys and Borm (2004) apply it to study loss compensation and surplus sharing problems in project-allocation situations.

We characterize the consensus value as the unique one-point solution concept for TU games that satisfies efficiency, symmetry, the quasi dummy property and additivity. By means of the transfer property, an alternative characterization for the consensus value is provided. We also establish a recursive formula for the consensus value by defining a stand-alone reduced game. Moreover, surprisingly, we find that the consensus value is the average of the Shapley value and the equal surplus solution. Furthermore, by introducing a share parameter on the splitting of joint surpluses, we obtain a generalization of the consensus value. In particular, the Shapley value and the equal surplus solution are the two polar cases of these generalized consensus values.

In addition to this section introducing the paper and reviewing the seminal works briefly, the remaining part has the following structure. In the next section, we formally define the consensus value and establish a recursive formula. As an illustration we consider glove games. In section 3, we characterize the consensus value in an axiomatic way and discuss
the properties under consideration. Moreover, it is shown that the consensus value is the average of the Shapley value and the equal surplus solution. An alternative characterization using the transfer property is provided. Generalizations are discussed in the final section.

2 The consensus value

Let us consider an arbitrary 2-person cooperative TU game with player set \( N = \{1, 2\} \) and characteristic function \( v \) determined by the values: \( v(\{1\}), v(\{2\}) \) and \( v(\{1, 2\}) \). A reasonable solution is that player 1 gets

\[
v(\{1\}) + \frac{v(\{1, 2\}) - v(\{1\}) - v(\{2\})}{2}
\]

and player 2 gets

\[
v(\{2\}) + \frac{v(\{1, 2\}) - v(\{2\}) - v(\{1\})}{2}.
\]

That is, the (net) surplus generated by the cooperation between player 1 and 2, \( v(\{1, 2\}) - v(\{1\}) - v(\{2\}) \), is equally shared between the two players. This solution is called the standard solution for 2-person cooperative games.

Now, we provide a generalization of the standard solution for 2-person games into \( n \)-person cases. It follows the following line of reasoning.

Consider a 3-person game \((N, v)\) with player set \( N = \{1, 2, 3\} \). Assume we have the order \((1, 2, 3)\): player 1 shows up first, then player 2, and finally player 3. When player 2 joins player 1, we in fact have a 2-person situation, and following the principles of the standard solution, the surplus \( v(\{1, 2\}) - v(\{1\}) - v(\{2\}) \) will be equally split among them.

Next player 3 enters the scene, who would like to cooperate with player 1 and 2. Because coalition \( \{1, 2\} \) has already been formed before she enters the game, player 3 will actually cooperate with the existing coalition \( \{1, 2\} \) instead of simply cooperating with 1 and 2 individually. If \( \{1, 2\} \) agrees to cooperate with 3 as well, the coalitional value \( v(\{1, 2, 3\}) \) will be generated. Now, the question is how to share it?

Again, following the standard solution to 2-person games, one can argue that both parties should get half of the joint surplus \( v(\{1, 2, 3\}) - v(\{1, 2\}) - v(\{3\}) \) in addition to their individual values. The reason is simple: coalition \( \{1, 2\} \) can be regarded as one player instead of two players because they have already formed a cooperating coalition. Internally, 1 and 2 will receive equal shares of the surplus because this part is obtained extra by the coalition \( \{1, 2\} \) cooperating with coalition \( \{3\} \).

One can also tell this story in the reverse way, yielding the same outcome in terms of surplus sharing. Initially, three players cooperate with each other and \( v(\{1, 2, 3\}) \) is
obtained. We now consider players leaving the existing coalition one by one in the opposite order \((3, 2, 1)\). So, player 3 leaves first. By the standard solution for 2-person games, player 3 should get half of the joint surplus/loss plus her individual payoff, i.e.,

\[
v(\{3\}) + \frac{v(\{1, 2, 3\}) - v(\{3\}) - v(\{1, 2\})}{2},
\]

as 1 and 2 remain as one cooperating coalition \(\{1, 2\}\). Thus, the value left for coalition \(\{1, 2\}\), which we call the *standardized remainder* (the value left for the corresponding remaining coalition) for \(\{1, 2\}\), is

\[
v(\{1, 2\}) + \frac{v(\{1, 2, 3\}) - v(\{1, 2\}) - v(\{3\})}{2}.
\]

In the same fashion, the standardized remainder for \(\{1\}\) will be

\[
v(\{1\}) + \frac{v(\{1, 2\}) + \frac{v(\{1, 2, 3\}) - v(\{1, 2\}) - v(\{3\})}{2}}{2} - v(\{1\}) - v(\{2\})
\]

when player 2 leaves the coalition \(\{1, 2\}\) next.

Extending this argument to an \(n\)-person case, we obtain a general method, which can be understood as a *standardized remainder rule* since we take the later entrant (or earlier leaver) and all her pre-entrants (or post-leavers) as two parties and apply the standard solution for 2-person games all the way. Furthermore, since no ordering is pre-determined for a TU game, we will average over all possible orderings.

Formal definitions are provided below. We denote by \(TU^N\) the class of all TU games with player set \(N\). Let \(\Pi(N)\) be the set of all bijections \(\sigma : \{1, 2, ..., |N|\} \rightarrow N\). For a given \(\sigma \in \Pi(N)\) and \(k \in \{1, 2, ..., |N|\}\) we define \(S^\sigma_k = \{\sigma(1), \sigma(2), ..., \sigma(k)\} \subset N\) and \(S^\sigma_0 = \emptyset\). Let \(v \in TU^N\). Recursively, we define

\[
r(S^\sigma_k) = \begin{cases} v(N) & \text{if } k = |N| \\ v(S^\sigma_k) + \frac{1}{2} (r(S^\sigma_{k+1}) - v(S^\sigma_k) - v(\{\sigma(k+1)\})) & \text{if } k \in \{1, ..., |N| - 1\}
\end{cases}
\]

where \(r(S^\sigma_k)\) is the *standardized remainder* for coalition \(S^\sigma_k\): the value left for \(S^\sigma_k\) after allocating surpluses to earlier leavers \(N \setminus S^\sigma_k\).

We then construct the *individual standardized remainder vector* \(s^\sigma(v)\), which corresponds to the situation where the players leave the game one by one in the order \((\sigma(|N|), \sigma(|N| - 1), ..., \sigma(1))\) and assign to each player \(\sigma(k)\), besides her individual payoff \(v(\{\sigma(k)\})\), half of the net surplus from the standardized remainder \(r(S^\sigma_k)\). Formally,
\[
\sigma(k)(\nu) = \begin{cases} 
\nu(\{\sigma(k)\}) + \frac{1}{2} \left( r(S_k^\sigma) - \nu(S_{k-1}^\sigma) - \nu(\{\sigma(k)\}) \right) & \text{if } k \in \{2, \ldots, |N|\} \\
r(S_1^\sigma) & \text{if } k = 1
\end{cases}
\]

**Definition 2.1** For every \( v \in TU^N \), the consensus value \( \gamma(v) \) is defined as the average of the individual standardized remainder vectors, i.e.,

\[
\gamma(v) = \frac{1}{|N|!} \sum_{\sigma \in \Pi(N)} s^\sigma(v).
\]

Hence, the consensus value can be interpreted as the expected individual standardized remainder a player can get by participating in coalitions.

Following the process of obtaining the consensus value, a more descriptive name for this solution concept could be the average serial standardized remainder value\(^2\).

**Example 2.2** Consider the 3-person TU game described below.

<table>
<thead>
<tr>
<th>( S )</th>
<th>{1}</th>
<th>{2}</th>
<th>{3}</th>
<th>{12}</th>
<th>{13}</th>
<th>{23}</th>
<th>{123}</th>
</tr>
</thead>
<tbody>
<tr>
<td>( v(S) )</td>
<td>10</td>
<td>0</td>
<td>0</td>
<td>18</td>
<td>23</td>
<td>0</td>
<td>30</td>
</tr>
</tbody>
</table>

With \( \sigma : \{1, 2, 3\} \rightarrow N \) defined by \( \sigma(1) = 2 \), \( \sigma(2) = 1 \) and \( \sigma(3) = 3 \), which is shortly denoted by \( \sigma = (2 1 3) \), we get

\[
s_3^\sigma(v) = s_{\sigma(3)}^\sigma(v) = \nu(\{3\}) + \frac{1}{2}(\nu(N) - \nu(\{1, 2\}) - \nu(\{3\})) = 6,
\]

\[
s_1^\sigma(v) = s_{\sigma(2)}^\sigma(v) = \nu(\{1\}) + \frac{1}{2}(r(\{2, 1\}) - \nu(\{2\}) - \nu(\{1\})) = 17,
\]

\[
s_2^\sigma(v) = s_{\sigma(1)}^\sigma(v) = r(\{2\}) = \nu(\{2\}) + \frac{1}{2}(r(\{2, 1\}) - \nu(\{2\}) - \nu(\{1\})) = 7.
\]

All individual standardized remainder vectors\(^3\) are given by

<table>
<thead>
<tr>
<th>( \sigma )</th>
<th>( s_1^\sigma )</th>
<th>( s_2^\sigma )</th>
<th>( s_3^\sigma )</th>
</tr>
</thead>
<tbody>
<tr>
<td>(123)</td>
<td>17</td>
<td>7</td>
<td>6</td>
</tr>
<tr>
<td>(132)</td>
<td>18\frac{1}{4}</td>
<td>3\frac{1}{2}</td>
<td>8\frac{1}{4}</td>
</tr>
<tr>
<td>(213)</td>
<td>17</td>
<td>7</td>
<td>6</td>
</tr>
<tr>
<td>(231)</td>
<td>20</td>
<td>5</td>
<td>5</td>
</tr>
<tr>
<td>(312)</td>
<td>18\frac{1}{4}</td>
<td>3\frac{1}{2}</td>
<td>8\frac{1}{4}</td>
</tr>
<tr>
<td>(321)</td>
<td>20</td>
<td>5</td>
<td>5</td>
</tr>
</tbody>
</table>

Hence, \( \gamma(v) = (18\frac{5}{12}, 5\frac{1}{6}, 6\frac{5}{12}) \) whereas the Shapley value of this game is given by \( \Phi(v) = (20\frac{1}{6}, 3\frac{2}{3}, 6\frac{1}{6}) \).

\(^2\)In the same spirit, an alternative name for the Shapley value could be the average serial marginal contribution value.

\(^3\)The fact that two permutations like (123) and (213) yield the same payoff vector only holds for the class of all 3-person TU games.
To further illustrate the consensus value, we consider glove games.

**Example 2.3** *(A glove game)*

Let \( N = \{1, 2, 3\} \) be the set of players. Player 1 has one left hand glove. Player 2 and 3 have one right hand glove each. A single glove is worth nothing. A (left-right) pair is worth 1 Euro. The corresponding TU game \((N, v)\) is determined by the values: \( v(\{i\}) = 0 \) for all \( i \) in \( N \), \( v(\{2, 3\}) = 0 \), and \( v(\{1, 2\}) = v(\{1, 3\}) = v(\{1, 2, 3\}) = 1 \).

One can readily check that the consensus value of this game is
\[
\gamma(v) = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right).
\]

In the more general case where \(|N| > 3\) but there still only one left hand glove player while all the others have one right hand glove each, the consensus value yields that the left hand glove player gets \( \frac{1}{2} \) and each right hand glove player gets \( \frac{1}{2}(|N| - 1) \).

Next it is shown that the consensus value satisfies the basic property of relative invariance with respect to strategic equivalence.

**Lemma 2.4** The consensus value \( \gamma \) is relative invariant with respect to strategic equivalence, i.e., for \( \alpha > 0 \) and \( \beta \in \mathbb{R}^N \), we have
\[
\gamma_i(\alpha v + \beta) = \alpha \gamma_i(v) + \beta_i
\]
for all \( i \in N \) and \( v \in TU^N \).

**Proof.** Let \( v \in TU^N \), \( \sigma \in \Pi(N) \), \( \alpha > 0 \) and \( \beta \in \mathbb{R}^N \).

**Claim 1:** \( r^{\alpha v + \beta}(S^\sigma_k) = \alpha r^v(S^\sigma_k) + \sum_{i \in S^\sigma_k} \beta_i \) for all \( k \in \{1, 2, ..., |N|\} \).

For \( k = |N| \), this is obvious.

For \( k \in \{1, ..., |N| - 1\} \), using induction we have
\[
r^{\alpha v + \beta}(S^\sigma_k) = \left(\alpha v(S^\sigma_k) + \sum_{i \in S^\sigma_k} \beta_i \right) + \frac{1}{2} \left( r^{\alpha v + \beta}(S^\sigma_{k+1}) - \left(\alpha v(S^\sigma_k) + \sum_{i \in S^\sigma_k} \beta_i \right) - (\alpha v(\sigma(k + 1)) + \beta_{\sigma(k+1)}) \right)
\]
\[
= \alpha \left( v(S^\sigma_k) + \frac{1}{2} (r^v(S^\sigma_{k+1}) - v(S^\sigma_k) - v(\{\sigma(k + 1)\})) \right) + \sum_{i \in S^\sigma_k} \beta_i
\]
\[
= \alpha r^v(S^\sigma_k) + \sum_{i \in S^\sigma_k} \beta_i
\]

**Claim 2:** \( s^\sigma_{\sigma(k)}(\alpha v + \beta) = \alpha s^\sigma_{\sigma(k)}(v) + \beta_{\sigma(k)} \) for all \( k \in \{1, 2, ..., |N|\} \).

For \( k \in \{2, ..., |N|\} \), by the definition of individual standardized remainder and claim 1 we
\[ s^\sigma_{\sigma(k)}(\alpha v + \beta) = (\alpha v(\sigma(k)) + \beta_{\sigma(k)}) + \frac{1}{2} \left( r^{\alpha v + \beta}(S^\sigma_k) - \left( \alpha v(S^\sigma_{k-1}) + \sum_{i \in S^\sigma_{k-1}} \beta_i \right) - (\alpha v(\sigma(k)) + \beta_{\sigma(k)}) \right) \]
\[ = \alpha \left( v(\{\sigma(k)\}) + \frac{1}{2} (r(S^\sigma_k) - v(S^\sigma_{k-1}) - v(\{\sigma(k)\})) \right) + \beta_{\sigma(k)} \]
\[ = \alpha s^\sigma_{\sigma(k)}(v) + \beta_{\sigma(k)} \]

For \( k = 1 \), this is obvious.

From claim 1 and claim 2 it immediately follows that

\[ \gamma_i(\alpha v + \beta) = \alpha \gamma_i(v) + \beta_i \]

for all \( i \in N \).

The consensus value can be reformulated by means of a recursive formula, adopting the same technique as in the paper by O’Neill (1982) for the so-called Run to the Bank rule.

Formally, let \( f : TU^N \rightarrow \mathbb{R}^N \) be a solution concept. For \((N, v) \in TU^N\) and \(i \in N\), we introduce the game \((N \setminus \{i\}, v^{-i})\) defined by

\[ v^{-i}(S) = \begin{cases} v(S) & \text{if } S \subseteq N \setminus \{i\} \\ v(N \setminus \{i\}) + \frac{v(N) - v(N \setminus \{i\}) - v(\{i\})}{2} & \text{if } S = N \setminus \{i\} \end{cases} \]

and call \( v^{-i} \) the stand-alone reduced game of \((N, v)\) with respect to player \(i\).

We say that a solution concept \( f \) satisfies stand-alone recursion if for every game \((N, v) \in TU^N\) with \(|N| \geq 3\) we have

\[ f_i(N, v) = \frac{1}{|N|} \left( \sum_{j \in N \setminus \{i\}} f_i(N \setminus \{j\}, v^{-j}) + (v(\{i\}) + \frac{v(N) - v(N \setminus \{i\}) - v(\{i\})}{2}) \right) \]

for all \( i \in N \).

One can readily check that the consensus value is the unique one-point solution concept on the class of all \(n\)-person TU games with \(n \geq 2\) which is standard for 2-person games and satisfies stand-alone recursion.
3 Characterizations

Let $f : TUN \to \mathbb{R}^N$ be a one-point solution concept. We consider the following properties.

- **Efficiency:** $\sum_{i \in N} f_i(v) = v(N)$ for all $v \in TUN$;
- **Symmetry:** for two players $i, j \in N$, if $v(S \cup \{i\}) = v(S \cup \{j\})$ for any $S \subseteq N \setminus \{i, j\}$, then $f_i(v) = f_j(v)$ for all $v \in TUN$;
- **The quasi dummy property:** if for some player $i \in N$, $v(S \cup \{i\}) = v(S) + v(\{i\})$ for any $S \subseteq N \setminus \{i\}$, then $f_i(v) = \frac{1}{2}v(\{i\}) + \frac{1}{2} \left( v(\{i\}) + \frac{v(N) - \sum_{j \in N} v(\{j\})}{|N|} \right)$ for all $v \in TUN$;
- **Additivity:** $f(v + w) = f(v) + f(w)$ for all $v, w \in TUN$.

The properties of efficiency, symmetry, and additivity are clear by themselves. The quasi dummy property is a modification of the classical dummy property.

As is argued in the introduction, the classical dummy property is utilitarian oriented, or, put differently, individualism oriented. However, from the egalitarian point of view or from the collectivistic perspective, one can argue that all members of a society including dummies should share the joint surplus equally among them. Thus, assigning a dummy player either $v(\{i\})$ or $v(\{i\}) + \frac{v(N) - \sum_{j \in N} v(\{j\})}{|N|}$ can be viewed as consequences of two contrastive viewpoints. Concerning the tradeoff between these two extreme cases, we make a fair compromise and take the average as the gain of a dummy player, which results in the so-called quasi dummy property.

We show that the consensus value is the unique one-point solution concept that satisfies these four properties.

**Theorem 3.1** Let $f : TUN \to \mathbb{R}^N$. Then, $f$ equals the consensus value if and only if it satisfies efficiency, symmetry, the quasi dummy property and additivity.

**Proof.**

We first show that the consensus value satisfies those four properties.

(i) Efficiency is obvious since, by construction, $s^\sigma(v)$ is efficient for all $\sigma \in \Pi(N)$.

(ii) Now, let us check symmetry. Let $i, j$ be two symmetric players in $v \in TUN$. Consider $\sigma \in \Pi(N)$, and set, without loss of generality, $\sigma(k) = i, \sigma(l) = j$, where $k, l \in \{1, ..., |N|\}$. Let $\bar{\sigma} \in \Pi(N)$ be the permutation which is obtained from $\sigma$ by interchanging the positions of $i$ and $j$, i.e.,

$$
\bar{\sigma}(m) = \begin{cases} 
\sigma(m) & \text{if } m \neq k, l \\
i & \text{if } m = l \\
j & \text{if } m = k
\end{cases}
$$

\textsuperscript{4}Cultural and philosophical factors may affect the propensity or choice between the two polar opinions.
As $\sigma \mapsto \tilde{\sigma}$ is bijective, it suffices to prove that $s_i^\sigma(v) = s_j^\sigma(v)$.

**Case 1:** $1 < k < l$.

By definition, we know

$$s_i^\sigma(v) = s_{\sigma(k)}^\sigma(v) = v(\{\sigma(k)\}) + \frac{1}{2}(r(S_k^\sigma) - v(S_{k-1}^\sigma) - v(\{\sigma(k)\}))$$

$$s_j^\sigma(v) = s_{\sigma(k)}^\sigma(v) = v(\{\sigma(k)\}) + \frac{1}{2}(r(S_k^\sigma) - v(S_{k-1}^\sigma) - v(\{\tilde{\sigma}(k)\}))$$

Note that, $v(\{\sigma(k)\}) = v(\{i\}) = v(\{j\}) = v(\{\tilde{\sigma}(k)\})$, $S_{k-1}^\sigma = S_{k-1}^\sigma$, and thus $v(S_{k-1}^\sigma) = v(S_{k-1}^\sigma)$. It remains to show that $r(S_k^\sigma) = r(S_k^\sigma)$.

Clearly, $r(S_m^\sigma) = r(S_m^\sigma)$ for $m \geq l$. By induction, we can show that $r(S_{l-t}^\sigma) = r(S_{l-t}^\sigma)$ for $t \in \{1, \ldots, l - k - 1\}$ as

$$r(S_{l-t}^\sigma) = v(S_{l-t}^\sigma) + \frac{1}{2}(r(S_{l-t+1}^\sigma) - v(S_{l-t}^\sigma) - v(\{\sigma(l-t+1)\}))$$

and

$$r(S_{l-t}^\sigma) = v(S_{l-t}^\sigma) + \frac{1}{2}(r(S_{l-t+1}^\sigma) - v(S_{l-t}^\sigma) - v(\{\tilde{\sigma}(l-t+1)\}))$$

Here, we also use the fact that $\sigma(\{l-t\}) = \tilde{\sigma}(\{l-t\})$. Moreover, $S_{l-t}^\sigma \setminus \{i\} = S_{l-t}^\sigma \setminus \{j\}$, we know that $v(S_{l-t}^\sigma) = v(S_{l-t}^\sigma)$.

Then, using symmetry (twice),

$$r(S_k^\sigma) = v(S_k^\sigma) + \frac{1}{2}(r(S_{k+1}^\sigma) - v(S_k^\sigma) - v(\{\sigma(k+1)\}))$$

$$= v(S_k^\sigma) + \frac{1}{2}(r(S_{k+1}^\sigma) - v(S_k^\sigma) - v(\{\tilde{\sigma}(k+1)\}))$$

$$= r(S_k^\sigma).$$

**Case 2:** $1 < l < k$. The proof is analogous to Case 1.

**Case 3:** $1 = k < l$.

In this case,

$$s_i^\sigma(v) = s_{\sigma(1)}^\sigma(v) = r(S^\sigma_1)$$

$$s_j^\sigma(v) = s_{\tilde{\sigma}(1)}^\sigma(v) = r(S^\sigma_1)$$

What remains is identical to Case 1.

**Case 4:** $1 = l < k$. Analogously to Case 3.

(iii) As for additivity, it is immediate to see that $s_{\sigma(i)}^\sigma(v + w) = s_{\sigma(i)}^\sigma(v) + s_{\sigma(i)}^\sigma(w)$ for all $v, w \in TU^N$ and for all $i \in \{1, 2, \ldots, |N|\}$.

(iv) By relative invariance with respect to strategic equivalence (Lemma 2.4), it suffices to
prove that the consensus value \( \gamma \) satisfies the quasi dummy property for zero-normalized games. Let \( v \in TU^N \) be zero-normalized and \( i \in N \) a dummy in \( v \). It suffices to show that 

\[
\gamma_i(v) = \frac{v(N)}{2N}.
\]

For \( \sigma \in \Pi(N) \), we have 

\[
\begin{align*}
\gamma_i(v) & = \frac{r(S^\sigma_{|N|})}{2N} \\
r(S^\sigma_{|N|-1}) & = \frac{1}{2} v(N) + \frac{1}{2} r(S^\sigma_{|N|-1}) \\
r(S^\sigma_{|N|-2}) & = \frac{1}{4} v(N) + \frac{1}{4} r(S^\sigma_{|N|-1}) + \frac{1}{2} v(S^\sigma_{|N|-2}) \\
r(S^\sigma_{|N|-3}) & = \frac{1}{8} v(N) + \frac{1}{8} r(S^\sigma_{|N|-1}) + \frac{1}{4} r(S^\sigma_{|N|-2}) + \frac{1}{2} v(S^\sigma_{|N|-3}) \\
& \vdots \\
r(S^\sigma_2) & = \frac{1}{2} (r(S^\sigma_3) + v(S^\sigma_2)) \\
r(S^\sigma_1) & = \frac{1}{2} (r(S^\sigma_2))
\end{align*}
\]

A general expression is provided below.

\[
r(S^\sigma_k) = \begin{cases} 
    v(N) & \text{if } k = |N| \\
    \frac{1}{2} |N|-k v(N) + \sum_{l=k}^{N-1} \left( \frac{1}{2} l-k+1 \right) v(S^\sigma_l) & \text{if } 2 \leq k \leq |N| - 1 \\
    \frac{1}{2} |N|-1 v(N) + \sum_{l=2}^{N-1} \left( \frac{1}{2} l \right) v(S^\sigma_l) & \text{if } k = 1
\end{cases}
\]

Let \( i \in N \) be a dummy player in \( v \). Let \( \sigma(k) = i \). Then,

\[
s^\sigma_i(v) = s^\sigma_{\sigma(k)}(v) = \begin{cases} 
    \frac{1}{2} (r(S^\sigma_k) - v(S^\sigma_{k-1})) & \text{if } k \geq 2 \\
    r(S^\sigma_1) & \text{if } k = 1
\end{cases}
\]

Hence, for \( k \in \{1, \ldots, |N|\} \), 

\[
\begin{align*}
\sigma_{|N|}(v) & = 0 \\
\sigma_{|N|-1}(v) & = \frac{1}{4} v(N) + \frac{1}{4} v(S^\sigma_{|N|-1}) - \frac{1}{2} v(S^\sigma_{|N|-2}) \\
\sigma_{|N|-2}(v) & = \frac{1}{8} v(N) + \frac{1}{8} v(S^\sigma_{|N|-1}) + \frac{1}{4} v(S^\sigma_{|N|-2}) - \frac{1}{2} v(S^\sigma_{|N|-3}) \\
\sigma_{|N|-3}(v) & = \frac{1}{16} v(N) + \frac{1}{16} v(S^\sigma_{|N|-1}) + \frac{1}{8} v(S^\sigma_{|N|-2}) + \frac{1}{4} v(S^\sigma_{|N|-3}) - \frac{1}{2} v(S^\sigma_{|N|-4}) \\
& \vdots \\
\sigma_3(v) & = \frac{1}{2|N|-2} v(N) + \frac{1}{2|N|-2} v(S^\sigma_{|N|-1}) + \frac{1}{2|N|-3} v(S^\sigma_{|N|-2}) + \frac{1}{2|N|-3} v(S^\sigma_{|N|-3}) + \frac{1}{2} v(S^\sigma_3) - \frac{1}{2} v(S^\sigma_2) \\
\sigma_2(v) & = \frac{1}{2|N|-1} v(N) + \frac{1}{2|N|-1} v(S^\sigma_{|N|-1}) + \frac{1}{2|N|-2} v(S^\sigma_{|N|-2}) + \frac{1}{2} v(S^\sigma_3) + \frac{1}{4} v(S^\sigma_2) \\
\sigma_1(v) & = \frac{1}{2|N|-1} v(N) + \frac{1}{2|N|-1} v(S^\sigma_{|N|-1}) + \frac{1}{2|N|-2} v(S^\sigma_{|N|-2}) + \frac{1}{2} v(S^\sigma_3) + \frac{1}{4} v(S^\sigma_2)
\end{align*}
\]
And a general expression is
\[
s_{\sigma}^\sigma(v) = s_{\sigma(i)}^\sigma(v) = \begin{cases} 
0 & \text{if } k = |N| \\
(\frac{1}{2})^{|N|-k+1}v(N) + \sum_{l=k}^{|N|-1} \left(\frac{1}{2}\right)^{l-k+2}v(S_{k}^\sigma) - \frac{1}{2}v(S_{k-1}^\sigma) & \text{if } 2 \leq k \leq |N| - 1 \\
(\frac{1}{2})^{|N|-1}v(N) + \sum_{l=2}^{|N|-1} \left(\frac{1}{2}\right)^{l}v(S_{l}^\sigma) & \text{if } k = 1
\end{cases}
\]

Consider a class \( P \) of \( |N| \) permutations \( \sigma \in \Pi(|N|) \) such that for \( \sigma, \tau \in P \) it holds that for all \( j_1, j_2 \in N \{i\} \)
\[
\sigma^{-1}(j_1) < \sigma^{-1}(j_2) \iff \tau^{-1}(j_1) < \tau^{-1}(j_2).
\]
That is, given an ordering of the players \( N \{i\} \), let the dummy player \( i \) move from the end to the beginning without changing the other players’ relative positions. Summing over the above equations, one readily checks that all the terms except \( v(N) \) will cancel out. That is,
\[
\sum_{\sigma \in P} s_{\sigma}^\sigma(v) = \left(\frac{1}{2} \right)^2 + \left(\frac{1}{2}\right)^3 + \ldots + \left(\frac{1}{2}\right)^{|N|-1} + \left(\frac{1}{2}\right)^{|N|-1} \right) v(N) = \frac{1}{2} v(N).
\]
Since \( \Pi(|N|) \) can be partitioned in \((|N| - 1)!\) of these classes, it follows that
\[
\gamma_i(v) = \frac{1}{|N|!} \sum_{\sigma \in \Pi(|N|)} s_{\sigma}^\sigma(v) = \frac{(|N| - 1)!}{|N|!} \cdot \frac{v(N)}{2} = v(N) \frac{1}{2|N|}.
\]

Conversely, let \( f : TU^N \rightarrow \mathbb{R}^N \) satisfy efficiency, symmetry, and the quasi dummy property. It easily follows that \( f \) is uniquely determined for (multiples of) unanimity games. Hence requiring a solution \( f \) to be additive too, it follows that \( f \) is uniquely determined for any game in \( TU^N \), since the class of unanimity games \( \{(N, u_T)|T \in 2^N \{\emptyset\}\} \) constitutes a basis of \( TU^N \).

Note that the quasi dummy property can be reformulated as \( f_i(v) = \frac{1}{2}\Phi_i(v) + \frac{1}{2}E_i(v) \) for all \( v \in TU^N \) and every dummy player \( i \) in \( v \). Here, \( \Phi(v) \) is the Shapley value of game \( v \) and \( E(v) \) denotes the equal surplus solution of game \( v \), i.e., \( E_i(v) = v(\{i\}) + \frac{v(N)-\sum_{j \in N} v(\{j\})}{|N|} \) for all \( i \in N \).

In fact, this property carries over to all players as is seen in Theorem 3.2.

**Theorem 3.2** For every \( v \) in \( TU^N \) it holds that
\[
\gamma(v) = \frac{1}{2} \Phi(v) + \frac{1}{2} E(v).
\]
Proof. It is readily shown that \( f(v) := \frac{1}{2}\Phi(v) + \frac{1}{2}E(v) \) satisfies the four characterizing properties: efficiency, symmetry, quasi dummy property and additivity.

We now provide an alternative characterization for the consensus value by means of the transfer property.

The transfer property (Dubey (1975)) in some sense substitutes for additivity. It is defined as follows. For any two games \( v, w \in TU_N \), we first define the games \( (v \vee w) \) and \( (v \wedge w) \) by \( (v \vee w)(S) = \max\{v(S), w(S)\} \) and \( (v \wedge w)(S) = \min\{v(S), w(S)\} \) for all \( S \subseteq N \). Let \( f : TU_N \rightarrow \mathbb{R}^N \) be a solution concept on the class of TU games. Then, \( f \) satisfies the transfer property if \( f(v \vee w) + f(v \wedge w) = f(v) + f(w) \) for all \( v, w \in TU_N \).

Dubey (1975) characterized the Shapley value as the unique value on the class of monotonic simple games satisfying efficiency, symmetry, the dummy property, and transfer property. Feltkamp (1995) generalized this result to the class of all TU games. More specifically, the Shapley value is the unique value on the class of TU games satisfying efficiency, symmetry, the dummy property and the transfer property (cf. Feltkamp (1995, p.134, Theorem 9.1.5)).

We now have an alternative characterization of the consensus value for TU games.

**Theorem 3.3** The consensus value is the only one-point solution on the class of TU games that satisfies efficiency, symmetry, the quasi dummy property and the transfer property.

**Proof.** As is well known, a solution concept \( f : TU_N \rightarrow \mathbb{R}^N \) satisfying additivity on \( TU_N \) also satisfies the transfer property on \( TU_N \). Therefore, the consensus value satisfies the transfer property. In addition, requiring a solution concept \( f : TU_N \rightarrow \mathbb{R}^N \) to satisfy efficiency, symmetry, and the quasi dummy property, it easily follows that \( f \) is uniquely determined for (multiples of) unanimity games. Moreover, by Feltkamp (1995, Lemma 9.1.4), it also follows that if the solution concept \( f \) satisfies the transfer property too, it is uniquely determined for any game in \( TU_N \).

4 A generalization of the consensus value

By relaxing the way of sharing remainders, we get a generalization of the consensus value: the **generalized consensus value**.

Let \( v \in TU_N \). We define the generalized remainder, with respect to an order \( \sigma \in \Pi(N) \) and \( \theta \in [0, 1] \), recursively by

\[
r_\theta(S^\sigma_k) = \begin{cases} v(N) & \text{if } k = |N| \\ v(S^\sigma_k) + (1 - \theta) \left( r_\theta(S^\sigma_{k+1}) - v(S^\sigma_k) - v(\{\sigma(k+1)\}) \right) & \text{if } k \in \{1, \ldots, |N| - 1\} \end{cases}
\]
Correspondingly, the individual generalized remainder vector $s_{\theta}^\sigma(v)$ is the vector in $\mathbb{R}^N$ defined by

$$(s_{\theta}^\sigma)_{\sigma(k)}(v) = \begin{cases} v(\{\sigma(k)\}) + \theta \left( r_{\theta}(S_{\sigma}^\sigma) - v(S_{\sigma k-1}^\sigma) - v(\{\sigma(k)\}) \right) & \text{if } k \in \{2, \ldots, |N|\} \\ r_{\theta}(S_{\sigma}^\sigma) & \text{if } k = 1 \end{cases}$$

**Definition 4.1** For every $v \in TU^N$ and $\theta \in [0, 1]$, the generalized consensus value $\gamma_{\theta}(v)$ is defined as the average of the individual generalized remainder vectors, i.e.,

$$\gamma_{\theta}(v) = \frac{1}{|N|!} \sum_{\sigma \in \Pi(N)} s_{\theta}^\sigma(v).$$

Note that the consensus value corresponds to the case $\theta = \frac{1}{2}$.

As mentioned in Section 3, dependent on the degree to which individualism or collectivism is preferred by society, the dummy property can be generalized. Defining the $\theta$-dummy property of a one-point solution concept $f : TU^N \rightarrow \mathbb{R}^N$ by $f_i(v) = \theta v(\{i\}) + (1 - \theta) \left( v(\{i\}) + \frac{v(N) - \sum_{j \in N} v(\{j\})}{|N|} \right)$ for all $v \in TU^N$ and every dummy player $i \in N$ with respect to $v$, we obtain the following theorem.

**Theorem 4.2** (a) The generalized consensus value $\gamma_{\theta}$ is the unique one-point solution concept on $TU^N$ that satisfies efficiency, symmetry, the $\theta$-dummy property and additivity.

(b) For any $v \in TU^N$, it holds that

$$\gamma_{\theta}(v) = \theta \Phi(v) + (1 - \theta) E(v).$$

(c) The generalized consensus value $\gamma_{\theta}$ is the unique function that satisfies efficiency, symmetry, the $\theta$-dummy property and the transfer property over the class of TU games.

The expression of the generalized consensus value as provided in part (b) of Theorem 4.2 is in the same spirit as the so-called compound measures in the context of digraph competitions (cf. Borm, van den Brink and Slikker (2002)).

Finally, we want to note that, in particular, for $\theta = 1$, the generalized consensus value is actually the Shapley value: the average serial remainder value turns to be the average serial marginal contribution value. When $\theta = 0$, the generalized consensus value equals to the equal surplus solution.

Consequently, we have the following characterizations for the equal surplus solution.

**Corollary 4.3** (a) The equal surplus solution $E$ is the unique one-point solution concept that satisfies efficiency, symmetry, the 0-dummy property and additivity.

(a) The equal surplus solution $E$ is the unique one-point solution concept that satisfies efficiency, symmetry, the 0-dummy property and transfer property.
References


