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**CONGESTION NETWORK PROBLEMS AND
RELATED GAMES**

By M. Quant, P.E.M. Borm, J.H. Reijnierse

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Congestion network problems and related games

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Abstract

This paper analyzes network problems with congestion effects from a cooperative game theoretic perspective. It is shown that for network problems with convex congestion costs, the corresponding games have a non-empty core. If congestion costs are concave, then the corresponding game has not necessarily core elements, but it is derived that, contrary to the convex congestion situation, there always exist optimal tree networks. Extensions of these results to a class of relaxed network problems and associated games are derived.

Keywords: Congestion, network problems, cooperative games, core.

JEL Classification Number: C71.

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1 Introduction

Generally speaking, in economic congestion situations agents use facilities from a common pool. Typically the costs of a facility will depend on the number of users. Within game theoretic literature the first paper to consider congestion effects is probably Rosenthal (1973). Here congestion effects are analyzed in a strategic non-cooperative setting. This line of work has been continued by Monderer and Shapley (1996) establishing a connection between potential games and congestion situations. An excellent survey of the related literature can be found in Voorneveld (1999).

Rather surprisingly, in cooperative game theoretic literature congestion effects have been considered far less explicitly. One branch of cooperative literature especially suited by its very nature to accommodate considerations regarding congestion is the literature on Operations Research Games as surveyed by Borm, Hamers, and Hendrickx (2001). An example of this is Matsubayashi, Umezawa, Masuda, and Nishino (2003), where hub-spoke network systems with congestion effects are studied using cooperative games.

This paper will focus on a particular extension of a standard operations research problem: minimum cost spanning network problems where the total costs of a specific network depend on the actual number of users of the various parts of the network. In short, we consider congestion network problems. In the classical setting, without congestion, this type of problems is known as minimum cost spanning tree problems. The study on the associated cooperative games has been initiated by Bird (1976). For a survey we refer to Feltkamp (1995).

The aim of this paper is to analyze congestion network problems from a cooperative point of view by focussing not only on finding an optimal network for a specific set of users but also on the problem of how to allocate the associated jointly generated minimal costs in a fair way among the users.

The structure of the paper is as follows. In section 2 we formally introduce congestion network problems. As two specific examples the cases with constant and linear congestion costs respectively are considered. Section 3 analyzes convex congestion network problems and shows that the corresponding transferable utility games have a non-empty core. Concave congestion network problems are studied in section 4. It is seen that the corresponding games can have an empty core, but that there always exist optimal network structures without cycles. This is not the case for convex congestion network problems. Finally section 5 considers a type of relaxed congestion network problems. It is derived that the main results for both the convex and concave non-relaxed congestion situations carry forward to

this setting.

2 Congestion network problems and games

Formally, a *congestion network problem* is a triple $T = (N, *, (k_a)_{a \in A_{N*}})$, where $N = \{1, \dots, n\}$ is a set of agents/players, $*$ is the source and $N^* := N \cup \{*\}$. The set A_S denotes the set of all arcs between pairs of elements in $S \subset N^*$, i.e. (S, A_S) denotes the complete digraph on S . For each arc $a \in A_{N*}$ the function $k_a : \{0, 1, \dots, n\} \rightarrow \mathbb{R}_+$ is a nonnegative (weakly) increasing cost function which depends on the number of users of a . We assume that for all $a \in A_{N*}$ it holds that $k_a(0) = 0$. Elements of A_{N*} will be denoted by a or by (i, j) , where $i, j \in N^*$. The arc (i, j) denotes the connection between i and j in the direction from i to j . The node i is called the tail of (i, j) and j the head of (i, j) . The cost function of an arc (i, j) , $i, j \in N^*$ is denoted by k_{ij} . A congestion network problem is symmetric if $k_{ij} = k_{ji}$ for all $i, j \in N^*$.

Let $T = (N, *, (k_a)_{a \in A_{N*}})$ be a congestion network problem. An optimal network can be described by $f : A_{N*} \rightarrow \{0, \dots, n\}$. Let F be the set consisting of all such networks. A network f assigns to each arc a number of users of this arc. The indegree for a network $f \in F$ and $i \in N^*$ is defined by $\text{indegree}(i) = \sum_{j \in N^* \setminus \{i\}} f((j, i))$. Similarly the outdegree is defined by: $\text{outdegree}(i) = \sum_{j \in N^* \setminus \{i\}} f((i, j))$. For a coalition $S \in 2^N \setminus \{\emptyset\}$ the collection of all feasible networks connecting the members of S to the source is given by:

$$\begin{aligned} F_S = \{ f \in F \mid & \text{outdegree}(i) - \text{indegree}(i) = 1 \text{ for all } i \in S, \\ & \text{outdegree}(i) = \text{indegree}(i) = 0 \text{ for all } j \in N \setminus S, \\ & f(a) \in \{0, \dots, |S|\}, \forall a \in A_{N*} \}. \end{aligned}$$

The costs of a network $f \in F_S$ is naturally defined by:

$$k(f) = \sum_{a \in A_{N*}} k_a(f(a)).$$

The aim of S is to construct a feasible network such that all its members are connected to the source and total costs are minimized.

A transferable utility cost game consists of a pair (N, c) , in which $N = \{1, \dots, n\}$ is a set of players and $c : 2^N \rightarrow \mathbb{R}$ is a function assigning to each coalition $S \in 2^N$ a cost of $c(S)$. By definition $c(\emptyset) = 0$. With each

congestion network problem $T = (N, *, (k_a)_{a \in A_{N^*}})$ one can associate a congestion network game (N, c^T) , such that $c^T(S)$ denotes the minimum costs of a network connecting all players of S to the source:

$$c^T(S) = \min_{f \in F_S} k(f).$$

Example 2.1 Consider a congestion network problem $T = (N, *, (k_a)_{a \in A_{N^*}})$, such that cost functions are symmetric and constant. This means that for all $i, j \in N^*$ and for all $m \in \{1, \dots, n\}$ it holds that $k_{ij}(m) = k_{ji}(m) = k_{ij}(1)$ and $k_{ij}(0) = 0$. So for all $a \in A_{N^*}$ and for all $m \in \{1, \dots, n\}$, $k_a(m) = k_a(1)$. It is readily verified that this congestion network problem is equivalent to a minimum cost spanning tree problem.

Each network f induces a digraph (N^*, A^f) . A^f consists of all arcs used by the network f :

$$A^f = \{a \in A_{N^*} \mid f(a) > 0\}.$$

Let f be an optimal network for coalition N . We can assume that the digraph (N^*, A^f) does not contain a circuit¹. To see this, assume that (N^*, A^f) contains a circuit C . Change the network f in such a way that the numbers of users of each arc of C is decreased by 1 and the number of users of all other arcs stay the same. The resulting network is still a feasible network for N , since C is a circuit. Because the cost functions are increasing functions and f is an optimal network for N the costs of the resulting network should be the same as the costs of f . This means that one can change the network in a finite number of steps such that all circuits are deleted and the network left is still optimal for N .

For an arbitrary arc set A the set $E(A)$ is the set of all undirected edges induced by A : $E(A) = \{\{i, j\} \mid (i, j) \in A \text{ or } (j, i) \in A\}$. We say that the digraph (N, A) contains a cycle if the induced undirected graph $(N, E(A))$ contains a cycle. Similarly the digraph (N, A) will be called a tree if $(N, E(A))$ is a tree. The following example shows that optimal networks in a congestion network problem can contain a cycle.

Example 2.2 Consider a symmetric congestion network problem, in which there are three players. For the arcs, the costs of one, two and three users

¹A digraph (N^*, A) contains a circuit if there exists a sequence $(i_1, i_2), (i_2, i_3), \dots, (i_{p-1}, i_p)$ such that $i_1 = i_p$ and $(i_m, i_{m+1}) \in A$ for all $m \in \{1, \dots, p-1\}$.

respectively are given by:

$$\begin{aligned}
k_{1*} &= (1, 3, 6), \\
k_{2*} &= (5, 10, 15), \\
k_{3*} &= (3, 7, 11), \\
k_{12} &= (3, 6, 9), \\
k_{13} &= (1, 2, 3), \\
k_{23} &= (1, 2, 3).
\end{aligned}$$

The optimal network of N is drawn in Figure 1.

Figure 1: Optimal network of the problem given in Example 2.2.

The core of a TU-game (N, c) is given by:

$$\text{Core}(c) = \left\{ x \in \mathbb{R}^N \mid \sum_{i \in N} x_i = c(N), \sum_{i \in S} x_i \leq c(S), \forall S \in 2^N \setminus \{\emptyset\} \right\}.$$

The core of a game consists of those cost allocation vectors such that no coalition has an incentive to split off. If cost functions are constant and symmetric (see Example 2.1), then the congestion network game is a minimum cost spanning tree game and the core will be non-empty (cf. Bird (1976)). This does not hold for arbitrary cost functions, which is illustrated in the following example.

Example 2.3 Consider a congestion network problem $T = (N, *, (k_a)_{a \in A_{N*}})$,

with $N = \{1, 2, 3\}$ and symmetric functions k_a defined by:

$$\begin{aligned} k_{1*} &= (1, 1, 2), \\ k_{2*} &= (1, 1, 2), \\ k_{3*} &= (1, 1, 2), \\ k_{12} &= (0, 0, 0), \\ k_{13} &= (0, 0, 0), \\ k_{23} &= (0, 0, 0). \end{aligned}$$

The game (N, c^T) is given by:

S	1	2	3	12	13	23	N
c^T	1	1	1	1	1	1	2

For an element $x \in \text{Core}(c^T)$ the following equations should hold:

$$\begin{aligned} x_1 + x_2 + x_3 &= 2 \\ x_1 + x_2 &\leq 1 \\ x_1 + x_3 &\leq 1 \\ x_2 + x_3 &\leq 1 \end{aligned}$$

Adding the last three equations yields a contradiction with the first one. Hence $\text{Core}(c^T) = \emptyset$.

A cost function k_a is linear if for all $m \in \{0, \dots, n\}$ $k_a(m) = m \cdot k_a(1)$. Congestion network problems with linear costs are very similar to congestion network problems with constant costs.

A path from i to $*$ in the digraph (N^*, A_{N^*}) is a sequence of arcs $P = ((i_0, i_1), (i_1, i_2), \dots, (i_{p-1}, i_p))$, such that $i_0 = i$, $i_p = *$ and $i_k \neq i_l$ for all $k, l \in \{0, \dots, p\}$. It is intuitively clear that if each player $i \in N$ chooses a path P_i to the source such that the costs of this path (which equal $\sum_{a \in P_i} k_a(1)$) is minimal, the combination of these paths yields an optimal structure for the problem corresponding to coalition N .

Consider a relaxation of the problem in which all arcs are fully public. Now for each coalition $S \in 2^N$ one can consider the relaxed problem of finding a network with minimal costs, connecting all players of S to the source. Coalition S is allowed to use any arc to establish this. Hence the set of feasible networks for $S \in 2^N$ becomes:

$$\begin{aligned} \{f \in F \mid &\text{outdegree}(i) - \text{indegree}(i) = 1 \text{ for all } i \in S, \\ &\text{outdegree}(i) - \text{indegree}(i) = 0 \text{ for all } i \in N \setminus S, \\ &f(a) \in \{0, \dots, |S|\}, \forall a \in A_{N^*}\}. \end{aligned}$$

Denote the corresponding coalitional value by $d^T(S)$. Clearly, for each coalition an optimal network in this relaxed problem is given by the combination of the optimal paths of its members. Hence the game is additive and assigning to each player the costs of its optimal path will yield the unique core element. Since $c^T(N) = d^T(N)$ and $c^T(S) \geq d^T(S)$ for all $S \in 2^N$ this allocation also yields a core element of the original congestion network game c^T .

Example 2.4 Consider the symmetric congestion network problem T with linear cost functions of Figure 2 with $N = \{1, 2, 3\}$. The numbers at an arc represent $k_a(1)$. The game c^T is given by:

S	1	2	3	12	13	23	N
c^T	7	10	5	12	16	15	21

The optimal path of player 1 is $P_1 = ((1, *))$, for players 2 and 3 the optimal paths are: $P_2 = ((2, 1), (1, *))$ and $P_3 = ((3, *))$. Each player $i \in N$ pays the costs of P_i which yields the allocation vector $x = (7, 9, 5)$. It can be easily checked that this is indeed a core element of the game (N, c^T) .

Figure 2: Congestion network problem of Example 2.4 with linear cost functions.

3 Convex congestion

In this section we examine so called *convex congestion network problems* $T = (N, *, (k_a)_{a \in A_{N*}})$ in which all k_a are convex. It will be shown that for a corresponding convex congestion game there always exists a core allocation.

A cost function k_a , $a \in A_{N*}$ is convex if for all $m \in \{1, \dots, n-1\}$:

$$k_a(m+1) - k_a(m) \geq k_a(m) - k_a(m-1).$$

As illustrated in Example 2.2 convex congestion network problems can have an optimal structure containing a cycle. On the other hand all congestion network games with convex cost functions have a non-empty core, as is stated in the following theorem.

Theorem 3.1 *Let $T = (N, *, (k_a)_{a \in A_{N*}})$ be a convex congestion network problem, then $\text{Core}(c^T) \neq \emptyset$.*

To prove this theorem we will consider a specific relaxation, in which non integer networks are allowed. A network in this relaxed problem can be described by: $f : A_{N*} \rightarrow [0, n]$. Let $\mathcal{F}$ be the set consisting of all relaxed networks. The set of all feasible networks connecting a coalition $S \in 2^N$ to the source is extended to:

$$\mathcal{F}_S = \left\{ f \in \mathcal{F} \mid \begin{array}{l} \text{outdegree}(i) - \text{indegree}(i) = 1 \text{ for all } i \in S, \\ \text{outdegree}(i) = \text{indegree}(i) = 0 \text{ for all } j \in N \setminus S \end{array} \right\}. \quad (1)$$

One can extend the cost functions k_a , $a \in A_{N*}$ in a piecewise linear way to $\bar{k}_a : [0, n] \rightarrow \mathbb{R}^+$:²

$$\bar{k}_a(x) := k_a(\lfloor x \rfloor) + (x - \lfloor x \rfloor) \cdot (k_a(\lceil x \rceil) - k_a(\lfloor x \rfloor)).$$

Clearly $\bar{k}_a$ is convex on $[0, n]$. The cost of a network $f \in \mathcal{F}_S$ is now defined by:

$$\bar{k}(f) = \sum_{a \in A_{N*}} \bar{k}_a(f(a)).$$

Moreover, let the relaxed congestion network game $(N, \bar{c}^T)$ be defined by:

$$\bar{c}^T(S) := \min_{f \in \mathcal{F}_S} \bar{k}(f)$$

for all $S \in 2^N \setminus \{\emptyset\}$. Note that:

$$\bar{c}^T(S) \leq c^T(S),$$

since for all $S \in 2^N \setminus \{\emptyset\}$ it holds that $F_S \subset \mathcal{F}_S$ and $\bar{k}_a$ extends k_a . The following lemma proves that extending a congestion network problem in a linear way does not change the values of the corresponding game.

²For $x \in \mathbb{R}$ the lower entier function is defined by: $\lfloor x \rfloor = \max\{y \in \mathbb{Z} \mid y \leq x\}$, similarly the upper entier function is defined by: $\lceil x \rceil = \min\{y \in \mathbb{Z} \mid y \geq x\}$.

Lemma 3.1 *Let $T = (N, *, (k_a)_{a \in A_{N^*}})$ be a congestion network problem. Then $\bar{c}^T(S) = c^T(S)$ for all $S \subset N$.*

Proof: Take $S \in 2^N$. It is sufficient to prove that there exists a network $f^* \in \mathcal{F}_S$ such that $\bar{c}^T(S) = \bar{k}(f^*)$ and f^* is integer valued. Let $f \in \mathcal{F}_S$ such that $\bar{c}^T(S) = \bar{k}(f)$ and let f be such that for all other $g \in \mathcal{F}_S$ such that $\bar{c}^T(S) = \bar{k}(g)$ it holds that:

$$|\{a \in A_{S^*} \mid f(a) \notin \mathbb{N}\}| \leq |\{a \in A_{S^*} \mid g(a) \notin \mathbb{N}\}|$$

This means that f is chosen within the set of all optimal networks for S such that the number of arcs that have a non-integer f -value is minimal.

Let D be the set of arcs that have a non-integer f -value. If $D = \emptyset$ we are done, since then $f \in F_S$. So assume $D \neq \emptyset$. Because the difference between outdegree and indegree of a node in S^* is integer, each node is either adjacent to no arcs of D , or to two or more. So it is possible to find a cycle C inside D .

Choose one of the two orientations of C . Let C^+ be the set of arcs in C that are directed according to this orientation. Let $C^- = C \setminus C^+$. Perturb the network f in two opposite directions. First, consider the network f^+ , which is obtained network f by increasing the number of users through C^+ by ε and decreasing the number of users through C^- by ε .

$$f^+(a) := \begin{cases} f(a) & \text{if } a \in A_{N^*} \setminus C, \\ f(a) + \varepsilon & \text{if } a \in C^+, \\ f(a) - \varepsilon & \text{if } a \in C^-. \end{cases}$$

Similarly, define f^- by increasing the number of users through C^- by ε and decreasing the number of users through C^+ by ε . The value of ε is chosen maximal such that the following inequalities are true for all $a \in C$:

$$\begin{aligned} \lfloor f(a) \rfloor &\leq f^+(a) \leq \lceil f(a) \rceil, \\ \lfloor f(a) \rfloor &\leq f^-(a) \leq \lceil f(a) \rceil. \end{aligned}$$

We have to show that the network f^+ is feasible for S . By the definition of ε , it is non-negative. Let i be a node on the cycle C . Then there are exactly two arcs in C , say a_1 and a_2 , adjacent to i . If $\text{head}(a_1) = \text{head}(a_2) = i$, then one of the arcs a_1 or a_2 is an element of C^+ and the other one is an element of C^- . Hence, the perturbation of f causes an increase as well as a decrease of ε to the indegree of i . The outdegree is not affected. If $\text{tail}(a_1) = \text{tail}(a_2) = i$, the roles of in- and outdegree are switched. If

$\text{head}(a_1) = \text{tail}(a_2) = i$, then a_1 and a_2 are either both elements of C^+ or both elements of C^- . In both cases the indegree and the outdegree of i are both increased by either ε or $-\varepsilon$. The net indegree remains the same. We conclude that f^+ is a feasible network, in the same way it is proved that f^- is feasible for S .

We find two networks f^+ and f^- in $\mathcal{F}_S$ with:

$$\frac{1}{2}f^- + \frac{1}{2}f^+ = f.$$

Let a be any arc. Because of the definition of ε , there exists an integer ℓ such that $\{f^-(a), f(a), f^+(a)\} \subset [\ell, \ell + 1]$. Because $\bar{k}_a$ restricted to $[\ell, \ell + 1]$ is a linear function, we have:

$$\bar{k}_a(f) = \bar{k}_a(\frac{1}{2}f^- + \frac{1}{2}f^+) = \frac{1}{2}\bar{k}_a(f^-) + \frac{1}{2}\bar{k}_a(f^+).$$

Hence, $\bar{k}(f) = \frac{1}{2}\bar{k}(f^-) + \frac{1}{2}\bar{k}(f^+)$. Because f has minimal costs, this can only be the case if f^- and f^+ have the same costs as f .

By definition of ε there is at least one arc $\hat{a}$ such that $f^-(\hat{a})$ or $f^+(\hat{a})$ is integer valued, say $f^+(\hat{a})$. This gives that the number of non-integer f^+ -valued arcs is strictly less than the number of integer f -valued arcs, contradicting the assumption that f is minimal with respect to this feature. $\square$

The next lemma proves that the relaxed congestion network game $(N, \bar{c}^T)$ has a non-empty core if the cost functions are convex. A characterization of games with a non-empty core is given independently by Bondareva (1963) and Shapley (1967). This characterization uses the notion of balanced sets and maps. A map $\lambda : 2^N \setminus \{\emptyset\} \rightarrow \mathbb{R}_+$ is called balanced if:

$$\sum_{S \in 2^N \setminus \{\emptyset\}} \lambda(S) e^S = e^N.$$

The vector e^S denotes the characteristic vector for coalition S :

$$e_i^S = \begin{cases} 1 & \text{if } i \in S, \\ 0 & \text{if } i \in N \setminus S. \end{cases}$$

A collection B of coalitions is a balanced collection if there exists a balanced map λ such that:

$$B = \{S \in 2^N \setminus \{\emptyset\} \mid \lambda(S) > 0\}.$$

A cost game (N, c) is balanced if for each balanced map λ it holds that:

$$\sum_{S \in 2^N \setminus \{\emptyset\}} \lambda(S) c(S) \geq c(N).$$

Bondareva (1963) and Shapley (1967) proved that a game is balanced if and only if it has a non-empty core.

Lemma 3.2 *Let $T = (N, *, (k_a)_{a \in A_{N*}})$ be a convex congestion network problem. Then $(N, \bar{c}^T)$ is a balanced game.*

Proof: Let λ be a balanced map and B the balanced collection such that:

$$B = \{S \in 2^N \setminus \{\emptyset\} \mid \lambda(S) > 0\}.$$

For all $S \in B$ take a network $f^S \in \mathcal{F}_S$ such that $\bar{c}^T(S) = \bar{k}(f^S)$. Define the network $f \in \mathcal{F}$ as:

$$f(a) := \sum_{S \in B} \lambda(S) f^S(a).$$

Let $i \in N$. The difference between outdegree and indegree according to f equals:

$$\begin{aligned} \text{outdegree}^f(i) - \text{indegree}^f(i) &= \sum_{S \in B} \lambda(S) (\text{outdegree}^{f^S}(i) - \text{indegree}^{f^S}(i)) \\ &= \sum_{S \in B: i \in S} \lambda(S) (\text{outdegree}^{f^S}(i) - \text{indegree}^{f^S}(i)) \\ &= \sum_{S \in B: i \in S} \lambda(S) \cdot 1 \\ &= 1. \end{aligned}$$

In the last equality it is used that λ is a balanced map. Since f is also

nonnegative it holds that $f \in \mathcal{F}_N$. Moreover:

$$\begin{aligned}
\bar{k}(f) &= \sum_{a \in A_{N^*}} \bar{k}_a(f(a)) \\
&= \sum_{a \in A_{N^*}} \bar{k}_a\left(\sum_{S \in B} \lambda(S) f^S(a)\right) \\
&\leq \sum_{a \in A_{N^*}} \sum_{S \in B} \lambda(S) \bar{k}_a(f^S(a)) \\
&= \sum_{S \in B} \sum_{a \in A_{N^*}} \lambda(S) \bar{k}_a(f^S(a)) \\
&= \sum_{S \in B} \lambda(S) \bar{k}(f^S) \\
&= \sum_{S \in B} \lambda(S) \bar{c}^T(S).
\end{aligned}$$

Note that for an arc $a \in A_{N^*}$, $f(a)$ is a convex combination of $f^S(a)$ for $S \in B$ and 0. So the above inequality follows from the convexity of the functions and the fact that $\bar{k}_a(0) = 0$. Since $f \in \mathcal{F}_N$:

$$\begin{aligned}
\bar{c}^T(N) &\leq \bar{k}(f) \\
&\leq \sum_{S \in B} \lambda(S) c^T(S),
\end{aligned}$$

proving that $(N, \bar{c}^T)$ is a balanced game. $\square$

Theorem 3.1 is now a direct consequence of Lemma 3.1 and Lemma 3.2.

4 Concave congestion

In this section we examine so called *concave congestion network problems* $T = (N, *, (k_a)_{a \in A_{N^*}})$ in which all k_a are concave. A cost function k_a , $a \in A_{N^*}$ is concave if for all $m \in \{1, \dots, n-1\}$:

$$k_a(m+1) - k_a(m) \leq k_a(m) - k_a(m-1).$$

In a convex congestion network problem the corresponding digraph of an optimal network for the grand coalition could contain cycles. The following theorem proves that concave congestion network problems have optimal network which induce a tree.

Theorem 4.1 *Let $T = (N, *, (k_a)_{a \in A_{N^*}})$ be a concave congestion network problem. There exists a network $f \in F_N$ such that $k(f) = c^T(N)$ and (N^*, A^f) is a tree.*

Proof: Let $f \in F_N$ such that $c^T(N) = k(f)$, this means that f is an optimal network for N . As we have noted before we may assume that A^f does not contain any circuits. Now suppose that A^f is not a tree. Then (N^*, A^f) contains a cycle. Hence there exist two nodes $i, j \in N^*$ such that there are two disjoint paths from i to j . Define P_1 as the set of arcs of the first path and P_2 as the set of arcs of the second path, so $P_1 \cap P_2 = \emptyset$. Define:

$$\varepsilon := \min_{a \in P_1 \cup P_2} f(a).$$

Now we perturb the network f in two different ways. First consider the network f^1 , which is obtained from f by increasing the number of users through P_1 by ε and decreasing the number of users through P_2 by ε .

$$f^1(a) = \begin{cases} f(a) & \text{if } a \in A_{N^*} \setminus (P_1 \cup P_2), \\ f(a) + \varepsilon & \text{if } a \in P_1, \\ f(a) - \varepsilon & \text{if } a \in P_2. \end{cases}$$

Note that by the definition of ε , f^1 is nonnegative and integer-valued. The difference between indegree and outdegree does not change for a node. It follows that $f^1 \in F_N$. Similarly $f^2 \in F_N$ arises from f by decreasing the number of users through P_1 by ε and increasing the number of users through P_2 by ε . Hence:

$$f^2(a) = \begin{cases} f(a) & \text{if } a \in A_{N^*} \setminus (P_1 \cup P_2), \\ f(a) - \varepsilon & \text{if } a \in P_1, \\ f(a) + \varepsilon & \text{if } a \in P_2. \end{cases}$$

Because f is an optimal network it holds that:

$$\begin{aligned} 0 &\leq k(f^1) - k(f) \\ &= \sum_{a \in P_1} (k_a(f(a) + \varepsilon) - k_a(f(a))) - \sum_{a \in P_2} (k_a(f(a)) - k_a(f(a) - \varepsilon)) \\ &\leq \sum_{a \in P_1} (k_a(f(a)) - k_a(f(a) - \varepsilon)) - \sum_{a \in P_2} (k_a(f(a) + \varepsilon) - k_a(f(a))) \\ &= k(f) - k(f^2) \\ &\leq 0. \end{aligned}$$

The second inequality follows from the concavity of the functions k_a . We can conclude that the costs of the networks f , f^1 and f^2 are all the same. This means that f^1 and f^2 are both optimal networks for N .

By the definition of ε in either f^1 and f^2 the number of users of at least one arc in A^f becomes zero. Assume without loss of generality that this is f^1 . Then $|A^{f^1}| < |A^f|$. If f^1 is a tree, then the proof is finished. If not there is again a cycle in f^1 and we can repeat the above reasoning. Note that this process will end in a finite number of steps, because the number of arcs with a positive number of users is decreased by at least one in each step. Hence we will find a network $\hat{f} \in F_N$ such that $k(\hat{f}) = k(f)$ and $(N^*, A^{\hat{f}})$ is a tree. $\square$

In the following example it is illustrated that a congestion network problem with concave cost functions need not to be balanced.

Example 4.1 *Consider a symmetric congestion network problem with 6 players. Assume that the cost function of arcs which are not drawn in Figure 3, are too expensive to use in any optimal network. The cost functions for arcs towards the source equal:*

$$k_{i*} = (10, 20, 20, 20, 20, 20), \quad i \in \{1, 2, 3\}.$$

For all other edges drawn in Figure 3, the cost function is linear with coefficient 10: $k_a(m) = 10m$, $m \in \{0, 1, \dots, 6\}$. The value of a coalition can be easily calculated by using Theorem 4.1 and just checking all trees connecting S to the source. This yields that $c^T(\{1245\}) = c^T(\{2346\}) = c^T(\{1356\}) = 50$ and $c^T(N) = 80$. The collection $B = \{\{1245\}, \{1356\}, \{2346\}\}$ is a balanced collection with $\lambda(S) = \frac{1}{2}$ for all $S \in B$. It follows that:

$$\frac{1}{2}(c^T(\{1245\}) + c^T(\{1356\}) + c^T(\{2346\})) \leq c^T(N),$$

which proves that (N, c^T) is not balanced.

5 Relaxed congestion network games

A specific type of relaxed congestion network games (using a piecewise linear extension) has been used to prove that convex congestion network games are balanced. The main feature of a relaxed congestion network problem is that the feasible networks need not to be integer valued. Relaxed congestion

Figure 3: Sketch of the situation of Example 4.1.

network problems can be the appropriate model if e.g. it is possible that nodes use different connections to the source for certain periods of time (as e.g. in computer networks). This feature can be easily modelled by dropping the restriction that networks should be integer valued.

In this section we extend the results for convex and concave congestion network problems earlier developed to the relaxed congestion network problems.

A relaxed congestion network problem is given by $\mathcal{T} = (N, *, (k_a)_{a \in A_{N*}})$ in which $k_a : [0, n] \rightarrow \mathbb{R}_+$ is a (weakly) increasing cost function for all $a \in A_{N*}$. The corresponding relaxed congestion network game is denoted by $(N, c^{\mathcal{T}})$. The set of all networks is denoted by $\mathcal{F}$. For $S \in 2^N$ all feasible networks are denoted by $\mathcal{F}_S$ (see formula (1)).

A relaxed congestion network problem is convex if all functions k_a are convex. Similarly a relaxed congestion network problem is concave if all functions k_a are concave.

If a relaxed congestion network problem $\mathcal{T} = (N, *, (k_a)_{a \in A_{N*}})$ is given, one can easily find a related congestion network problem by restricting the function k_a to the domain $\{0, \dots, n\}$. The congestion network problem achieved in this way will be denoted by $T(\mathcal{T})$. In the following example it is shown that a relaxed congestion network game can differ from a congestion network game with the restricted cost functions.

Example 5.1 Consider a symmetric two person congestion network problem as depicted in the first graph of Figure 4, with player set $N = \{1, 2\}$. The cost functions are given by: $k_{*1}(x) = 2x^2$, $k_{*2}(x) = 3x$ and $k_{12}(x) = 0$, $x \in [0, 2]$. Note that cost functions are convex in this example. The op-

timial networks of the relaxed problem and the non-relaxed problem can be found in the second and third picture of Figure 4. It can be calculated that $c^T(N) = 4\frac{1}{2}$ and $c^{T(T)}(N) = 5$.

Figure 4: A two person congestion network problem.

Convex congestion network games are balanced games, this result is also true for relaxed convex congestion network games.

Theorem 5.1 *Let $\mathcal{T} = (N, *, (k_a)_{a \in A_{N*}})$ be a relaxed convex congestion network problem. Then $\text{Core}(c^T) \neq \emptyset$.*

The proof follows the proof of Lemma 3.2, since the latter does not use the assumption of piecewise linearity of cost functions. Only convexity has been used.

Concave congestion network problems always have an optimal network for N which induces a tree. This remains true for relaxed concave congestion network problems.

Theorem 5.2 *Let $\mathcal{T} = (N, *, (k_a)_{a \in A_{N*}})$ be a relaxed concave congestion network problem. Then there is at least one network $f \in \mathcal{F}_N$ such that $\bar{k}(f) = c^T(N)$ and (N^*, A^f) is a tree.*

The proof of this theorem is similar to the proof of Theorem 4.1 and is therefore omitted. As a result of Theorem 5.2 it can be proved that restricting a concave cost function does not change the corresponding congestion network game.

Corollary 5.1 *Let $\mathcal{T} = (N, *, (k_a)_{a \in A_{N*}})$ be a relaxed concave congestion network problem. Then $c^T(S) = c^{T(T)}(S)$ for all $S \in 2^N$.*

Proof: Let $S \in 2^N$. Clearly Theorem 5.2 implies that there exists a network $f^S \in \mathcal{F}_S$ such that $c^T(S) = k(f^S)$ and (S^*, A^{f^S}) is a tree. This

implies that $f^S \in F_S$. Thus it holds that:

$$\begin{aligned} c^{T(T)}(S) &\geq k(f^S) \\ &= c^T(S). \end{aligned}$$

Since it always holds that $c^{T(T)}(S) \leq c^T(S)$ it follows that $c^{T(T)}(S) = c^T(S)$.
 $\square$

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