Tilburg University

Cooperation and communication restrictions

Borm, P.E.M.; van den Nouweland, C.G.A.M.; Tijs, S.H.

Publication date:
1991

[Link to publication in Tilburg University Research Portal](#)

Citation for published version (APA):

Borm, P. E. M., van den Nouweland, C. G. A. M., & Tijs, S. H. (1991). *Cooperation and communication restrictions: A survey*. (Research memorandum / Tilburg University, Department of Economics; Vol. FEW 507). Unknown Publisher.

General rights

Copyright and moral rights for the publications made accessible in the public portal are retained by the authors and/or other copyright owners and it is a condition of accessing publications that users recognise and abide by the legal requirements associated with these rights.

- Users may download and print one copy of any publication from the public portal for the purpose of private study or research.
- You may not further distribute the material or use it for any profit-making activity or commercial gain
- You may freely distribute the URL identifying the publication in the public portal

Take down policy

If you believe that this document breaches copyright please contact us providing details, and we will remove access to the work immediately and investigate your claim.

COOPERATION AND COMMUNICATION RESTRICTIONS: A SURVEY

Peter Borm, Anne van den Nouweland and Stef Tijs

Department of Econometrics

Tilburg University

The Netherlands

Abstract

A survey is provided on the line of research on graph-restricted cooperative games as initiated by *Myerson* (1977). Special attention is paid to the results with respect to the Myerson value and the position value.

1 Introduction

In a *cooperative TU* (= Transferable Utility) game (N, v) , as modelled by *von Neumann* and *Morgenstern* (1944), N is a finite set of *players* and the *characteristic function* v assigns to each subgroup of players a real number which is to be interpreted as the maximal gains (or minimal cost) this coalition can achieve by cooperating, regardless of the actions of the other players. In developing the characteristic function it is generally assumed that there are no constraints on communication possibilities. This leads to the question how to modify the *TU*-model in order to capture the fact that in various economic situations there is a given network of communication channels among the agents/players.

The simplest model which incorporates communication restrictions in the theory of cooperative games is that of a *coalition structure*, defined as a partition of the player set N into disjoint coalitions. Once the coalition structure is determined, negotiations can take place only within each of the coalitions that constitute the structure. Coalition structures were introduced in *Aumann* and *Maschler* (1964) and elaborated further in *Aumann* and *Dreze* (1974) and *Hart* and *Kurz* (1983). However, coalition structures do not capture the fact that communication relations need not be transitive. To model this richer kind of framework, *Myerson* (1977) introduced the notion of a *communication graph*, in which the players are the vertices. One possible interpretation is that a link between two players (an edge in the graph) exists if it is possible for these players to communicate/negotiate directly and in a meaningful way. In particular, coalition structures may be modelled within this framework by defining two players to be linked if and only if they belong to the same partition element.

This paper surveys the line of research on *graph restricted games* (*communication games*) as initiated by *Myerson* (1977), and is based on the results in *Myerson* (1980), *Owen* (1986), *Aumann* and *Myerson* (1988), *Borm*, *Owen* and *Tijs* (1991), *Borm* (1990), *van den Nouweland* and *Borm* (1991) and *van den Nouweland* et al. (1990, 1991).

Moreover, possibilities for future research are suggested by indicating various open problems, extensions and modifications. Related work that is not discussed here can be found

in *Grofman and Owen (1982)*, *Kalai, Postlewaite and Roberts (1978)*, *Rosenthal (1988)* and *Faigle and Kern (1990)*.

To conclude the introduction, we briefly discuss two other approaches described in the literature to incorporate communicative and social constraints into economics or game theory.

For trade economies *Vind (1983, 1986)* introduced *institutions* as (possibly intersecting) subgroups of agents in which each of the participating agents has veto-power on the actions of the group. The concept of exchange equilibrium was modified accordingly.

For cooperative games *Gilles, Owen and van den Brink (1990)*, *Gilles and Owen (1990)* and *van den Brink and Gilles (1991)* introduced a *permission structure* as a directed graph on the set of players. Here it is assumed that for each player an incoming arc determines a ‘superior’ and that a player can only become ‘effective’ within a coalition if one (or, alternatively, all) of his superiors cooperate.

Preliminaries and notations. Let $N := \{1, 2, \dots, n\}$ and $2^N := \{S \mid S \subset N\}$. Let TU^N denote the class of all transferable utility games with player set N . A TU -game is often identified with its characteristic function $v : 2^N \rightarrow \mathbf{R}$ where $v(\emptyset) = 0$.

For each $S \in 2^N \setminus \{\emptyset\}$, the *unanimity game* $u_S \in TU^N$ is defined by

$$u_S(T) = \begin{cases} 1 & \text{if } S \subset T \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, $\{u_S \mid S \in 2^N \setminus \{\emptyset\}\}$ is a basis of TU^N and for each $v \in TU^N$ we have that

$$v = \sum_{S \in 2^N \setminus \{\emptyset\}} \Delta_v(S) u_S \tag{1.1}$$

where the *dividends* $\Delta_v(S)$ (cf. *Harsanyi (1959)*) are given by

$$\Delta_v(S) := \sum_{T \subset S} (-1)^{|S|-|T|} v(T) \tag{1.2}$$

for all $S \in 2^N \setminus \{\emptyset\}$.

The *Shapley value* $\Phi : TU^N \rightarrow \mathbf{R}^N$ (cf. *Shapley* (1953)) is a linear function with

$$\Phi(u_S) = \frac{1}{|S|} e^S,$$

where $e^S \in \mathbf{R}^N$ ($S \in 2^N \setminus \{\emptyset\}$) is such that

$$e_j^S = \begin{cases} 1 & \text{if } j \in S \\ 0 & \text{if } j \in N \setminus S \end{cases}$$

Hence, for all $v \in TU^N$,

$$\Phi(v) = \sum_{S \in 2^N \setminus \{\emptyset\}} \frac{\Delta_v(S)}{|S|} \quad (1.3)$$

Alternatively, the Shapley value can be characterized by

$$\Phi_i(v) = \frac{1}{n!} \sum_{\sigma \in P(N)} [v(PR_\sigma(i) \cup \{i\}) - v(PR_\sigma(i))] \quad (1.4)$$

for all $i \in N$, where $P(N)$ denotes the set of permutations of N and

$$PR_\sigma(i) := \{j \in N \mid \sigma(j) < \sigma(i)\}$$

denotes the set of predecessors of i according to σ .

2 Communication games and solution concepts

A *communication situation* is a triple (N, v, A) where $N := \{1, 2, \dots, n\}$ is the set of agents/players, (N, v) is a zero-normalized *TU*-game, i.e. $v(\{i\}) = 0$ for all $i \in N$, and (N, A) is an undirected communication graph without loops or parallel arcs.

In a communication situation the economic capabilities of a coalition, as given by the game (N, v) , might be restricted because of a possible lack of communication channels between the players. It is assumed that the players in a coalition $S \subset N$ can only effect communication through the links in $A(S) := \{\{i, j\} \in A \mid i \in S, j \in S\}$. Consequently, each coalition S is partitioned into communication components in the following way: $C \subset S$ is a component within S if and only if $(C, A(C))$ is a maximally connected subgraph of $(S, A(S))$. The resulting partition of S is denoted by S/A .

Correspondingly, we define the reward $r^v(S, A)$ of a coalition S in the communication situation (N, v, A) as the sum of the gains of its communication components, i.e.

$$r^v(S, A) := \sum_{C \in S/A} v(C).$$

It is straightforward to check that since v is zero-normalized $r^v(S, A) = r^v(N, A(S))$ and $r^v(\emptyset, A) = r^v(S, \emptyset) = 0$ for all S and A .

For communication situations two allocation rules were considered: the Myerson value (*Myerson* (1977)) and the position value (*Borm, Owen and Tijs* (1991)). These rules are based on the Shapley value of two different types of corresponding communication games: point games and arc games. While a point game focuses on the role of a node of the communication graph (a player) in establishing communication between the various coalitions, an arc game reflects a dual point of view and concentrates on the role of a communication link. Then, assuming each player to have veto power of the use of any arc he is an end point of, it seems reasonable to divide the (Shapley) value of an arc equally between the two players at its end points.

Definition. For a communication situation (N, v, A) the *point game* (N, r_A^v) and the *arc game* (A, r_N^v) are defined by

$$r_A^v(S) := r^v(S, A) \quad \text{and} \quad r_N^v(L) := r^v(N, L)$$

for all $S \subset N$ and $L \subset A$. Further, the *Myerson value* $\mu(N, v, A) \in \mathbb{R}^N$, the *link value* $\lambda(N, v, A) \in \mathbb{R}^A$ and the *position value* $\pi(N, v, A) \in \mathbb{R}^N$ are given by

$$\begin{aligned} \mu(N, v, A) &:= \Phi(N, r_A^v), \\ \lambda(N, v, A) &:= \Phi(A, r_N^v) \quad \text{and} \quad \pi_i(N, v, A) := \sum_{a \in A_i} \frac{1}{2} \lambda_a(N, v, A) \end{aligned}$$

for all $i \in N$, where $A_i := \{\{i, j\} \in A \mid j \in N\}$ denotes the subset of arcs of which player i is an end point.

Example 1. (*Unanimity games*).

Consider a communication situation (N, u_S, A) , where u_S is the unanimity game on $S \subset N$ with $|S| \geq 2$ and the communication graph (N, A) is a *tree*, i.e. (N, A) is connected and contains no cycles. Defining the *connected hull* $H(S) \subset N$ of S in (N, A) by

$$H(S) := \bigcap \{T \subset N \mid S \subset T, (T, A(T)) \text{ is connected}\}, \quad (2.1)$$

it readily follows that the point game and the arc game corresponding to (N, u_S, A) are given by

$$r_A^{u_S} = u_{H(S)} \quad \text{and} \quad r_N^{u_S} = u_{A(H(S))}.$$

Consequently,

$$\mu(N, u_S, A) = |H(S)|^{-1} e^{H(S)} \quad \text{and} \quad \lambda(N, v, A) = |A(H(S))|^{-1} e^{A(H(S))}.$$

Hence, with $d_i(N, A) := |A_i|$ denoting the *degree* of node i in the graph (N, A) , we have

$$\pi_i(N, v, A) = \frac{|A_i \cap A(H(S))|}{2 |A(H(S))|} = \frac{d_i(N, A(H(S)))}{\sum_{j \in N} d_j(N, A(H(S)))}$$

for all $i \in N$.

Let CS^N denote the class of all communication situations with player set N . For an *allocation rule* $\gamma : CS^N \rightarrow \mathbf{R}^N$ we consider the following properties:

- *component efficiency* (CE): for all $(N, v, A) \in CS^N$ and $C \in N/A$

$$\sum_{i \in C} \gamma_i(N, v, A) = v(C).$$

- *fairness* (F): for all $(N, v, A) \in CS^N$ and $\{i, j\} \in A$

$$\gamma_i(N, v, A) - \gamma_i(N, v, A \setminus \{i, j\}) = \gamma_j(N, v, A) - \gamma_j(N, v, A \setminus \{i, j\}).$$

- *additivity* (ADD): for all $(N, v, A) \in CS^N$ and $(N, w, A) \in CS^N$

$$\gamma(N, v + w, A) = \gamma(N, v, A) + \gamma(N, w, A).$$

- *superfluous arc property* (SAP): for all $(N, v, A) \in CS^N$ and $a \in A$ such that $r^v(N, L) = r^v(N, L \cup \{a\})$ for all $L \subset A$, it holds that

$$\gamma(N, v, A) = \gamma(N, v, A \setminus \{a\}).$$

- *arc anonymity* (AA): for all $(N, v, A) \in CS^N$ such that there exists a function $f : \{0, 1, \dots, |A|\} \rightarrow \mathbf{R}$ with $r^v(N, L) = f(|L|)$ for all $L \subset A$, it holds that

$$\gamma(N, v, A) = \alpha d(N, A) \text{ for some } \alpha \in \mathbf{R}.$$

- *point anonymity* (PA): for all $(N, v, A) \in CS^N$ such that there exists a function $f : \{0, 1, \dots, |D|\} \rightarrow \mathbf{R}$ with $r^v(S, A) = f(|S \cap D|)$ for all $S \subset N$, where $D := \{i \in N \mid d_i(N, A) \geq 1\}$, it holds that

$$\gamma(N, v, A) = \alpha e^D \text{ for some } \alpha \in \mathbf{R}.$$

Axiomatic characterizations of the Myerson value and the position value are provided in

Theorem 1.

- (i) (*Myerson* (1977)). The Myerson value is the unique allocation rule on CS^N that satisfies (CE) and (F).
- (ii) (*Borm, Owen and Tijs* (1991)). Let CS_*^N denote the subclass of communication situations for which the communication graphs do not contain any cycles. Then the following two assertions hold:
 - (a) The Myerson value is the unique allocation rule on CS_*^N that satisfies (CE), (ADD), (SAP) and (PA).
 - (b) The position value is the unique allocation rule on CS_*^N that satisfies (CE), (ADD), (SAP) and (AA).

The irredundancy of the axioms in theorem 1, part (ii), was shown in *van den Nouweland, Borm and Tijs* (1990). It is still an open problem how to characterize the position value for the class CS^N of all communication situations. Further, it would be interesting to provide an alternative characterization (on CS_*^N) à la *Young* (1985) or to find an axiomatic characterization of the position value for the class of all communication situations (N, v, A) where there are no communication restrictions at all, i.e. for which $A = \{\{i, j\} \in 2^N \mid i \neq j\}$, because this class corresponds to the class of all (zero-normalized) *TU*-games. For these problems, the following relation which formally describes the dual character of the link value and the Myerson value might be helpful.

Proposition 2. Let $(N, v, A) \in CS^N$ and denote by (A, M) the *dual graph* of (N, A) , i.e. for $a, b \in A$ with $a \neq b$ we have $\{a, b\} \in M$ if and only if there is a player $i \in N$ such that $\{a, b\} \subset A_i$. Then

$$\lambda(N, v, A) = \mu(A, r_N^v, M).$$

Proof. Clearly, it suffices to prove that the arc game r_N^v corresponding to (N, v, A) and the point game p corresponding to (A, r_N^v, M) are identical. Let $\emptyset \neq L \subset A$. Then, with $N(L) := \{i \in N \mid A_i \cap L \neq \emptyset\}$,

$$\begin{aligned} p(L) &= \sum_{K \in L/M} r_N^v(K) = \sum_{K \in L/M} \sum_{C \in N/K} v(C) \\ &= \sum_{K \in L/M} v(N(K)) \stackrel{(*)}{=} \sum_{C \in N/L} v(T) = r_N^v(L), \end{aligned}$$

where $(*)$ follows from the fact that the function $f : L/M \rightarrow \{C \in N/L \mid |C| \geq 2\}$, defined by $f(K) := N(K)$ for all $K \in L/M$, is a bijection. $\square$

Summarizing, to evaluate the effect of communication restrictions in cooperative TU -games we concentrated on the Shapley value of two different types of communication games. Of course one could also apply other game theoretic solutions concepts such as the *nucleolus* (*Schmeidler* (1969)) and the τ -value (*Tijs* (1981)), and compare results.

Further, one might consider a third type of communication game in which the communicative strength of players and arcs are evaluated simultaneously. This leads to the following definitions.

Definition. For $(N, v, A) \in CS^N$, define the TU -game $(N \cup A, w)$ by

$$w(S \cup L) := \sum_{C \in S/L} v(C)$$

for all $S \subset N$ and $L \subset A$. Correspondingly, let $\rho(N, v, A) \in \mathbf{R}^N$ be given by

$$\rho_i(N, v, A) = \Phi_i(N \cup A, w) + \sum_{a \in A_i} \frac{1}{2} \Phi_a(N \cup A, w)$$

for all $i \in N$.

It is not difficult to provide an axiomatic characterization of the allocation rule ρ restricted to the class CS_*^N à la theorem 1(ii): component efficiency, additivity, the superfluous arc property and some ‘combination’ of arc anonymity and point anonymity will do.

Further, if the underlying game is a unanimity game and the underlying communication graph is a tree, ρ is a convex combination of the Myerson value μ and the position value π .

Proposition 3. If (N, A) is a tree, then it holds for each $S \subset N$ with $|S| \geq 2$ that

$$\rho(N, u_S, A) = \alpha_S \pi(N, u_S, A) + (1 - \alpha_S) \mu(N, u_S, A),$$

where

$$\alpha_S = (2 |A(H(S))| + 1)^{-1} |A(H(S))|$$

Proof. Noting that $|H(S)| = |A(H(S))| + 1$, it readily follows that

$$\Phi(N \cup A, w) = (2 |A(H(S))| + 1)^{-1} e^{H(S) \cup A(H(S))} \in \mathbb{R}^{N \cup A}.$$

Consequently,

$$\rho_i(N, u_S, A) = \begin{cases} \left(1 + \frac{1}{2} |A_i \cap A(H(S))|\right) (2 |A(H(S))| + 1)^{-1} & \text{if } i \in H(S) \\ 0 & \text{if } i \in N \setminus H(S) \end{cases}$$

and the proposition follows from the expressions for $\mu(N, u_S, A)$ and $\pi(N, u_S, A)$ given in example 1. $\square$

Clearly, since ρ satisfies additivity and the weight α_S heavily depends on S , proposition 3 cannot be generalized towards general TU -games. Moreover, the following example shows that the result cannot be extended to communication graphs with cycles either.

Example 2. Let $N = \{1, 2, 3\}$ and consider the unanimity game $u_{\{1,2\}}$. If there are no communication restrictions, then $A = \{\{1, 2\}, \{1, 3\}, \{2, 3\}\}$ and

$$\begin{aligned}\mu(N, u_{\{1,2\}}, A) &= \Phi(N, u_{\{1,2\}}) = (\frac{1}{2}, \frac{1}{2}, 0) \\ \pi(N, u_{\{1,2\}}, A) &= (\frac{5}{12}, \frac{5}{12}, \frac{2}{12})\end{aligned}$$

and

$$\rho(N, u_{\{1,2\}}, A) = (\frac{7}{15}, \frac{7}{15}, \frac{1}{15})$$

Note that ρ is not a convex combination of μ and π .

3 Dividends and calculation methods

The calculation of the Myerson value and the position value can be rather complex and tedious. For some special classes of communication situations, however, elegant calculation methods are available. These methods strongly depend on relations between the dividends of the underlying game and the dividends of the corresponding point game and arc game.

Let (N, v, A) be a communication situation. For the definition of the dividends $\Delta_v(S)$ and $\Delta_{r_A^v}(S)$ we refer to (1.1) and (1.2). Note that the arc game can be written as a linear combination of the (arc) unanimity games (A, u_L) , $L \in 2^A \setminus \{\emptyset\}$, as follows:

$$r_N^v = \sum_{L \in 2^A \setminus \{\emptyset\}} \Delta_{r_N^v}(L),$$

with

$$\Delta_{r_N^v}(L) = \sum_{K \subset L} (-1)^{|L|-|K|} r_N^v(K)$$

for all $L \in 2^A \setminus \{\emptyset\}$.

Theorem 4. Let $(N, v, A) \in CS_*^N$. Then the following two assertions hold

- (i) (*Owen* (1986)). With respect to the dividends of the point game it holds for all $S \in 2^N \setminus \{\emptyset\}$ that

$$\Delta_{r_A^v}(S) = \sum_{T \in \Sigma(S)} \Delta_v(T),$$

where (cf. (5))

$$\Sigma(S) := \{T \in 2^N \setminus \{\emptyset\} \mid H(T) = S\}.$$

- (ii) (*Borm, Owen and Tijs* (1991)). With respect to the dividends of the arc game it holds for all $L \in 2^A \setminus \{\emptyset\}$ that

$$\Delta_{r_N^v}(L) = \sum_{S \in \Lambda(L)} \Delta_v(S),$$

where

$$\Lambda(L) := \{S \in 2^N \setminus \{\emptyset\} \mid L = A(H(S))\}.$$

The sets $\Sigma(S)$ and $\Lambda(L)$ can be characterized by means of extreme points. Here, for a connected graph (N, A) the set $Ext(N, A)$ of *extreme points* of (N, A) is defined by

$$Ext(N, A) := \{i \in N \mid d_i(N, A) = 1\}.$$

Lemma 5. Let (N, A) be a cycle-free graph. Then the following two assertions hold

(i) For all $S \in 2^N \setminus \{\emptyset\}$

$$\Sigma(S) = \begin{cases} \{T \subset S \mid Ext(S, A(S)) \subset T\} & \text{if } (S, A(S)) \text{ is connected} \\ \emptyset & \text{else.} \end{cases}$$

(ii) For all $L \in 2^A \setminus \{\emptyset\}$

$$\Lambda(L) = \begin{cases} \{T \subset N(L) \mid Ext(N(L), L) \subset T\} & \text{if } (N(L), L) \text{ is connected} \\ \emptyset & \text{else.} \end{cases}$$

where $N(L) := \{i \in N \mid A_i \cap L \neq \emptyset\}$ is the set of all players that correspond to an end point of an arc in L .

The formulas in theorem 4 and lemma 5 can be used to obtain the results for unanimity games as presented in example 1. We provide two more special classes of games for which the formulas lead to nice expressions for the Myerson value and the position value. For convenience we will restrict ourselves to connected graphs.

Quadratic measure games. Let $\omega = (\omega_1, \dots, \omega_n)$ be a non-negative (weight) vector. The *quadratic measure game* q_ω corresponding to ω (cf. Owen (1986)) is defined by

$$q_\omega(S) := \left(\sum_{i \in S} \omega_i\right)^2 - \sum_{i \in S} \omega_i^2 = \sum_{\{i,j\} \subset S, i \neq j} 2\omega_i \omega_j$$

for all $S \in 2^N \setminus \{\emptyset\}$. Hence, in a quadratic measure game the worth of a coalition is determined by the worths of its two-person subcoalitions. Using theorem 4, lemma 5 and (1.3), one can show

Proposition 6. Let $(N, q_\omega, A) \in CS_*^N$. Then the following two assertions hold

(i) (*Owen* (1986)). For all $i \in N$

$$\mu_i(N, q_\omega, A) = 2 \sum \frac{\omega_j \omega_k}{|H(\{j, k\})|},$$

where the sum is taken over all pairs $\{j, k\} \subset N, j \neq k$, such that i lies on the path from j to k . Note that $|H(\{j, k\})|$ is the number of nodes on the path from j to k .

(ii) (*Borm, Owen and Tijs* (1991)). For the link value of an arc $a \in A$ it holds that

$$\lambda_a(N, q_\omega, A) = 2 \sum \frac{\omega_j \omega_k}{|A(H(\{j, k\}))|},$$

where the sum is taken over all pairs $\{j, k\} \subset N, j \neq k$ such that a is an arc in the path from j to k . Note that $|A(H(\{j, k\}))|$ is the number of arcs in the path from j to k .

The computation of the formulas provided in proposition 6 can be readily carried out using *generating functions*, cf. *Owen* (1986) and *Borm, Owen and Tijs* (1991).

Pure overhead games. Let T be a subset of N . The (zero-normalization of the) *pure overhead game* p_T on T is defined by

$$p_T(S) := \begin{cases} -1 + |S \cap T| & \text{if } S \cap T \neq \emptyset \\ 0 & \text{else.} \end{cases}$$

It may be noted that p_T is the cost savings game corresponding to the cost game u_T^* with

$$u_T^*(S) := \begin{cases} 1 & \text{if } S \cap T \neq \emptyset \\ 0 & \text{else.} \end{cases}$$

Theorem 7. Let $(N, p_T, A) \in CS_*^N$. Then the following two assertions hold

(i) (*Owen* (1986)). For all $i \in N$

$$\mu_i(N, p_T, A) = \sum \frac{(-1)^{|S \cap T|}}{|S|},$$

where the sum is taken over all coalitions S with $i \in S$ and $|S| \geq 2$ such that $(S, A(S))$ is a connected (sub)graph with $Ext(S, A(S)) = S \cap T$.

(ii) (*Borm, Owen and Tijs* (1991)). For the link value of an arc $a \in A$ it holds that

$$\lambda_a(N, p_T, A) = \sum \frac{(-1)^{|N(L) \cap T|}}{|L|},$$

where the sum is taken over all $L \subset A$ with $a \in L$ such that $(N(L), L)$ is a connected (sub)graph with $Ext(N(L), L) = N(L) \cap T$.

Van den Nouweland et al. (1991) derived integral formulas for the computation of the formulas provided in theorem 7. We demonstrate the computation of the position value in

Example 3. Let $N = \{1, \dots, 10\}$ and $T = \{1, 3, 4, 7, 10\}$. Further, let (N, A) be the

graph as represented in figure 1.

Figure 1

We compute the position value of player 6 in the communication situation (N, p_T, A) . Hence, we have to compute the link value of the links $a = \{2, 6\}$, $b = \{6, 7\}$ and $c = \{6, 8\}$. For the link a we have to find all subsets $L \subset A$ containing a with the property that $(N(L), L)$ is connected and $Ext(N(L), L) = N(L) \cap T$. If we were to delete the arc a from the graph, then N would fall apart into two components,

$$C_1 = \{1, \dots, 5\} \quad \text{and} \quad C_2 = \{6, \dots, 10\}.$$

For each component we compute the probability $P_1(x)$ ($P_2(x)$) that at least one of the players in $T \cap C_1$ ($T \cap C_2$) can interact with player 2 (6), assuming that each arc has a probability x of being operational.

We obtain

$$P_1(x) = 2x - x^2 \quad \text{and} \quad P_2(x) = x + x^2 - x^3.$$

The probability that arc a connects elements of T is given by

$$\Theta_a(x) = P_1(x) \cdot P_2(x) = 2x^2 + x^3 - 3x^4 + x^5.$$

Now (cf. theorem 3 in *van den Nouweland* et al. (1991))

$$\lambda_a(N, p_T, A) = \int_0^1 \Theta_a(x) dx = \frac{29}{60}.$$

In a similar way we compute

$$\lambda_b(N, p_T, A) = \frac{31}{60} \quad \text{and} \quad \lambda_c(N, p_T, A) = \frac{24}{60}.$$

This gives

$$\pi_6(N, p_T, A) = \frac{1}{2}\lambda_a(N, p_T, A) + \frac{1}{2}\lambda_b(N, p_T, A) + \frac{1}{2}\lambda_c(N, p_T, A) = \frac{7}{10}.$$

4 Properties of communication games

This section concentrates on relations between properties of the two types of communication games and the underlying games. In particular, for a communication situation (N, v, A) this section provides conditions on the communication graph (N, A) such that convexity (balancedness, superadditivity) of v is inherited by the corresponding point game or arc game.

A TU -game (N, v) is called *superadditive* if

$$v(S \cup T) \geq v(S) + v(T)$$

for all disjoint coalitions $S, T \in 2^N$.

Theorem 8. Let (N, v, A) be a communication situation. Then the following two assertions hold

- (i) (*Owen* (1986)). If the game (N, v) is superadditive, then the point game (N, r_A^v) is superadditive.
- (ii) (*van den Nouweland* and *Borm* (1991)). If the arc game (A, r_N^v) is non-negative and superadditive, then the point game (N, r_A^v) is superadditive.

The following example shows that there is no non-trivial class of communication graphs for which superadditivity of the underlying game is inherited by the corresponding arc game.

Example 4. Let (N, v, A) be the communication situation where $N = \{1, 2, 3\}$, $A = \{\{1, 2\}, \{2, 3\}\}$ and $v(S) = \begin{cases} 0 & \text{if } |S| \leq 1 \\ |S| & \text{if } |S| \geq 2 \end{cases}$. The game (N, v) is superadditive, but, with $a := \{1, 2\}$ and $b := \{2, 3\}$, we have $r_N^v(\{a\}) + r_N^v(\{b\}) = 4 < 3 = r_N^v(\{a, b\})$. Hence, the game (A, r_N^v) is not superadditive.

Recall that the *core* $C(N, v)$ of a TU -game (N, v) is given by

$$C(N, v) := \{x \in \mathbf{R}^N \mid \sum_{i \in N} x_i = v(N) \text{ and } \sum_{i \in S} x_i \geq v(S) \text{ for all } S \in 2^N \setminus \{\emptyset\}\}$$

and that (N, v) is *balanced* if and only if it has a non-empty core. Further, (N, v) is *totally balanced* if for every $S \subset N$ the subgame $(S, v|_S)$ is balanced, where $v|_S(T) := v(T)$ for all $T \subset S$.

Theorem 9. (*van den Nouweland and Borm (1991)*). Let (N, v, A) be a communication situation. Then the following three assertions hold

- (i) If the communication graph (N, A) is connected and (N, v) is balanced, then the point game (N, r_A^v) is balanced.
- (ii) If (N, v) is totally balanced, then the point game (N, r_A^v) is totally balanced.
- (iii) If the arc game (A, r_N^v) is non-negative and balanced, then the point game (N, r_A^v) is balanced.

Example 4 shows that even if the underlying game in a communication situation is totally balanced, then the corresponding arc game need not be balanced.

A TU -game (N, v) is called *convex* if

$$v(S \cup \{i\}) - v(S) \leq v(T \cup \{i\}) - v(T)$$

for all $i \in N$ and all $S \subset T \subset N \setminus \{i\}$. It is well known that a convex game is totally balanced and that its Shapley value is the barycentre of the core.

We consider two communication situations where the underlying game is convex in

Example 5. Let (N, v, A) and (N, v, B) be communication situations with $N = \{1, \dots, 5\}$, $v(S) = |S| - 1$ for all $S \neq \emptyset$ and where (N, A) and (N, B) are the communication graphs represented in figure 2.

Figure 2

The game (N, v) is convex, since for all $i \in N$ and all $S \subset N \setminus \{i\}$ it holds that $v(S \cup \{i\}) - v(S) = \begin{cases} 1 & \text{if } S \neq \emptyset \\ 0 & \text{if } S = \emptyset \end{cases}$. Further, since

$$r_A^v(S) = \begin{cases} 0 & \text{if } |S| \leq 1 \\ |S| - 2 & \text{if } |S| \geq 2, 5 \in S \text{ and } 4 \notin S, \\ |S| - 1 & \text{else} \end{cases}$$

(N, r_A^v) is convex too.

However, (N, r_B^v) is not convex, because

$$r_B^v(\{1, 4\} \cup \{3\}) - r_B^v(\{1, 4\}) = 2 > 1 = r_B^v(\{1, 2, 4\} \cup \{3\}) - r_B^v(\{1, 2, 4\}).$$

A property that the graph (N, A) satisfies but (N, B) does not is given in the following

Definition. A graph (N, A) is called *cycle-complete* if it holds that

if there is a cycle $(x_1, x_2, \dots, x_t, x_1)$ in (N, A) where $x_1, \dots, x_t$ are all distinct elements of N , then the complete graph on $\{x_1, \dots, x_t\}$ is a subgraph of (N, A) .

In particular, both graphs without cycles and complete graphs are cycle-complete.

Theorem 10. (*van den Nouweland and Borm (1991)*). Let (N, v, A) be a communication situation. Then the following three assertions hold

- (i) If the communication graph (N, A) is cycle-complete and (N, v) is convex, then the point game (N, r_A^v) is convex and, consequently, $\mu(N, v, A) \in C(N, r_A^v)$.
- (ii) If the communication graph (N, A) is cycle-free and the point game (N, r_A^v) is convex, then the arc game (A, r_N^v) is convex and $\pi(N, v, A) \in C(N, r_A^v)$.
- (iii) If the arc game (A, r_N^v) is non-negative and convex, then the point game (N, r_A^v) is convex.

It immediately follows from theorem 10(i) and (ii) that if (N, v) is convex and (N, A) is cycle-free, then the corresponding arc game (A, r_N^v) is convex.

Van den Nouweland and Borm (1991) showed that the condition of cycle-completeness in theorem 10(i) is necessary in the sense that for each communication graph (N, A) that is not cycle-complete there exists a convex game (N, v) such that the corresponding point game (N, r_A^v) is not convex.

5 Strategic behaviour and endogenous formation of communication links

So far we only considered *exogenously* given communication graphs. The players were assumed to be anchored within a fixed ‘outside’ network of communication links. However, following *Aumann* and *Myerson* (1988), one could raise the following question. Given a game in coalitional form, what communication links may be expected to form between the players? In analyzing this question about *endogenous* formation of communication graphs, strategic considerations play a role. Not only the immediate consequences of ‘building’ a link (or not) have to be taken into account but one also has to consider where this decision will lead to at the end. To evaluate any given communication graph for any player, *Aumann* and *Myerson* (1988) used the Myerson value. In this section we use the position value instead and reconsider two of *Aumann* and *Myerson*’s examples. In the context of coalition structures, an explicit treatment of endogenous coalition formation can be found e.g. in *Hart* and *Kurz* (1983).

Let us first describe the (strategic) model for endogenous formation of communication links as suggested by *Aumann* and *Myerson* (1988).

Let (N, v) be a zero-normalized *TU*-game and let $\gamma : CS^N \rightarrow \mathbb{R}^N$ be an allocation rule for communication situations. Further, let $K := \{\{i, j\} \in 2^N \mid i \neq j\}$ denote the set of all possible communication links. Then the *linking game* $\Gamma_\gamma(v)$ in extensive form, corresponding to v with respect to γ , is defined as follows. At the beginning of the play there are no links between the players. Then, according to some specific *rule of order*, the players can propose to build links. The rule of order is common knowledge and each player $i \in N$, in his turn, may or may not propose to form some link $\{i, j\} \in K$.

If player i proposes to form the link $\{i, j\}$, then player j has to decide whether or not to accept this proposal. If player j accepts, then the link $\{i, j\}$ is formed. In all other cases, no link is formed at this stage.

After such a (propose-reject/accept) *round*, the next player in line (with respect to the rule of order) may take the initiative to build some new link etc. Note that, once formed,

a link cannot be destroyed. The game stops if in $|N|$ subsequent rounds no new link is formed. Assume that at the end of this period of link formation the communication graph (N, A) has been established. Then the payoff for player $i \in N$ is defined to be $\gamma_i(N, v, A)$.

Note that the linking game $\Gamma_\gamma(v)$ is a game in extensive form with *perfect information* in which at any time (at each decision node), each player has full knowledge of the entire history of proposals and rejections. A *subgame* of $\Gamma_\gamma(v)$ is defined as a game that starts out from an arbitrary decision node and that continues as $\Gamma_\gamma(v)$ itself. A *subgame perfect equilibrium* for $\Gamma_\gamma(v)$ is defined to be a Nash equilibrium (cf. *Nash* (1951)) for $\Gamma_\gamma(v)$ that induces a Nash equilibrium for each subgame. Since $\Gamma_\gamma(v)$ is a game of perfect information, a subgame perfect equilibrium in pure strategies exists (cf. *Selten* (1975)). Any communication graph (N, A) that is reached at the end of a play corresponding to such a pure subgame perfect equilibrium (for some rule of order) is called a *stable communication graph* for v with respect to γ .

To illustrate this model, we compare the stable communication graphs with respect to the Myerson value μ and the position value π in two examples.

(i) Let the three-person zero-normalized *TU*-game (N, v) be given by $N = \{1, 2, 3\}$ and

$$v(S) = \begin{cases} 0 & \text{if } |S| \leq 1 \\ 60 & \text{if } |S| = 2 \\ 72 & \text{if } |S| = 3 \end{cases} \quad (5.1)$$

Because of symmetry table 1 below completely determines the Myerson value and the position value for each possible communication graph.

A	$\pi(N, v, A)$	$\mu(N, v, A)$
$\emptyset$	(0,0,0)	(0,0,0)
$\{\{1, 2\}\}$	(30,30,0)	(30,30,0)
$\{\{1, 2\}, \{1, 3\}\}$	(36,18,18)	(44,14,14)
$\{\{1, 2\}, \{1, 3\}, \{2, 3\}\}$	(24,24,24)	(24,24,24)

Table 1

To determine all subgame perfect equilibria for the linking game $\Gamma_\pi(v)$ we use ‘backward induction’. Suppose we are already in the situation that two links have been formed. Then, the two players who cannot communicate directly, both will profit from establishing a direct communication link, because their payoffs will go up from 18 to 24. Hence, whenever a decision node is reached in which such a situation occurs, we will inevitably (i.e. according to each equilibrium path for the corresponding subgame) end up in a full communication graph with three links.

Let us go back one step further and suppose that we are in the situation that precisely one link has been formed, say the link between player 1 and player 2. Then, on each (subgame perfect) equilibrium path for the corresponding subgame, neither player 1 nor player 2 will propose or accept to form a link with player 3 because at the end, using the arguments above, this will lead to a payoff of 24 instead of 30.

Now we come to analyze the initial situation in which there are no links formed yet. Let us assume for the analysis (not for the argument) that the rule of order is (1,2,3). Suppose we are in the situation that player 3 can make a proposal and that no links have been formed yet. Clearly, a (subgame perfect) equilibrium path for the corresponding subgame will prescribe player 3 either to propose the link $\{1,3\}$ or the link $\{2,3\}$ and prescribe the player to whom the proposal is made, to accept. This is the case since this, at the end, will lead to a payoff of 30 for both players, which is better than the payoff of zero they would get otherwise.

Continuing our backward induction argument we now arrive at the situation that player 2 can make a proposal and that no links have been formed yet. Note that, in the corresponding subgame, the action of player 2 not to propose any link at all supports a

subgame perfect equilibrium play because player 3 still could offer the link $\{2,3\}$ in the next round. Similarly, also the action of player 2 to propose a link, together with either a subsequent rejection or acception, supports a subgame perfect equilibrium play. Finally, applying similar arguments to the first round of the game, one may conclude that there are three stable communication graphs for v with respect to π , each consisting of a single link between two of the players.

Note that exactly the same analysis can be applied for the linking game $\Gamma_\mu(v)$. Consequently, for this example, the stable communication graphs with respect to π and μ coincide. In general this need not be the case, as is seen in the following example.

(ii) Let the five-person zero-normalized TU -game (N, v) be determined by $N = \{1, 2, 3, 4, 5\}$ and

$$v(S) = \begin{cases} 1 & \text{if } S = \{2, 3, 4, 5\}, \text{ or, if both } |S| \geq 2 \text{ and } 1 \in S \\ 0 & \text{otherwise.} \end{cases} \quad (5.2)$$

Let $K_S := \{\{i, j\} \in 2^S \mid i \neq j\}$ correspond to the complete communication graph on a coalition S . For complete communication graphs the position value and the Myerson value are represented in table 2.

A	$\pi(N, v, A)$	$\mu(N, v, A)$
$K_{\{1,2\}}$	$(\frac{1}{2}, \frac{1}{2}, 0, 0, 0)$	$(\frac{1}{2}, \frac{1}{2}, 0, 0, 0)$
$K_{\{1,2,3\}}$	$(\frac{1}{2}, \frac{1}{4}, \frac{1}{4}, 0, 0)$	$(\frac{2}{3}, \frac{1}{6}, \frac{1}{6}, 0, 0)$
$K_{\{1,2,3,4\}}$	$(\frac{1}{2}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, 0)$	$(\frac{3}{4}, \frac{1}{12}, \frac{1}{12}, \frac{1}{12}, 0)$
$K_{\{2,3,4,5\}}$	$(0, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$	$(0, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$
$K_{\{1,2,3,4,5\}}$	$\frac{1}{1680}(716, 241, 241, 241, 241)$	$(\frac{3}{5}, \frac{1}{10}, \frac{1}{10}, \frac{1}{10}, \frac{1}{10})$

Table 2

As argued in *Aumann and Myerson* (1988), using a similar line of thought as in the first example, it is found that $K_{\{2,3,4,5\}}$ is the unique stable communication graph for v with

respect to the Myerson value μ . Here, we only provide a brief argumentation. Each subgame perfect equilibrium for $\Gamma_\mu(v)$ will convert $K_{\{1,2,3,4\}}$ into $K_{\{1,2,3,4,5\}}$ and $K_{\{1,2\}}$ into $K_{\{1,2,3\}}$. Therefore, the problem the players face at the beginning of the play boils down to a choice between $K_{\{1,2,3\}}$ and $K_{\{2,3,4,5\}}$. Clearly, $K_{\{2,3,4,5\}}$ results.

However, as can be seen in table 2, the same arguments do not apply to the linking game $\Gamma_\pi(v)$. It is not difficult to verify that the first four complete graphs represented in table 2 are stable communication graphs for v with respect to the position value π . So with respect to the position value there are many stable graphs for v .

As a possible alternative for the extensive form game $\Gamma_\gamma(v)$ to model endogenous formation of communication links, one might consider a *strategic form* game $G_\gamma(v)$, in which the players move simultaneously and each player proposes to ‘build’ some links of which he is an end point. Assuming a link to be formed if and only if it is proposed by both players at its end points, each such strategy combination leads to a communication graph. The corresponding payoffs to the players are determined by applying the allocation rule γ to the resulting communication situation. Concentrating on undominated Nash equilibria, this approach is elaborated in *Borm (1991)*.

6 Generalizations

Communication situations can be generalized in several directions. For example, the communication possibilities of the players can be modelled by means of hypergraphs or directed graphs instead of undirected graphs. Further, the economic possibilities of the players could be modelled by means of non-transferable utility games (*NTU*-games) instead of *TU*-games. In this section we will briefly address these three generalizations and at the end we introduce a more general framework for cooperation and communication involving control games on points and communication arcs.

Hypergraphs. A *hypergraph communication situation* (cf. Myerson (1980)) is a triple $(N, v, \mathcal{H})$, where (N, v) is a zero-normalized *TU*-game describing the economic possibilities of the players and $(N, \mathcal{H})$, with $\mathcal{H} \subset \{H \in 2^N \mid |H| \geq 2\}$, is a (*communication*) *hypergraph* describing the communicative possibilities of the players. It is assumed that communication is only possible within a *conference* $H \in \mathcal{H}$ and that all players in H have to be present before communication within H can take place.

In a hypergraph communication situation $(N, v, \mathcal{H})$ the players of a coalition $S \subset N$ can effect communication in the conferences in $\mathcal{H}(S) := \{H \in \mathcal{H} \mid H \subset S\}$. *Interaction sets* of $(S, \mathcal{H}(S))$ are defined recursively in the following way:

1. each $\{i\} \subset S$ is an interaction set
2. each $H \in \mathcal{H}(S)$ is an interaction set
3. if T^1 and T^2 are interaction sets such that $T^1 \cap T^2 \neq \emptyset$, then $T^1 \cup T^2$ is an interaction set.

The interaction sets of $(S, \mathcal{H}(S))$ are the subgroups of players in S who can interact with respect to communication in $(S, \mathcal{H}(S))$. Consequently, each coalition S is partitioned into maximal interaction sets. The resulting partition is denoted by $S/\mathcal{H}$. Correspondingly, the reward $r^v(S, \mathcal{H})$ of a coalition S in the hypergraph communication situation $(N, v, \mathcal{H})$ is defined by

$$r^v(S, \mathcal{H}) := \sum_{C \in S/\mathcal{H}} v(C).$$

The Myerson value and the position value can be defined analogously to the definitions in section 2.

Definition. For a hypergraph communication situation $(N, v, \mathcal{H})$ the *(extended) Myerson value* $\mu(N, v, \mathcal{H}) \in \mathbb{R}^N$ is defined by

$$\mu(N, v, \mathcal{H}) := \Phi(N, r_{\mathcal{H}}^v),$$

where the game $(N, r_{\mathcal{H}}^v)$ is defined by $r_{\mathcal{H}}^v(S) := r^v(S, \mathcal{H})$ for all $S \subset N$.

Further, the *(extended) position value* $\pi(N, v, \mathcal{H}) \in \mathbb{R}^N$ is defined by

$$\pi_i(N, v, \mathcal{H}) := \sum_{H \in \mathcal{H}_i} |H|^{-1} \Phi_H(\mathcal{H}, r_N^v)$$

for all $i \in N$, where the game $(\mathcal{H}, r_N^v)$ is defined by $r_N^v(\mathcal{A}) := r^v(N, \mathcal{A})$ for all $\mathcal{A} \subset \mathcal{H}$, and $\mathcal{H}_i := \{H \in \mathcal{H} \mid i \in H\}$ denotes the set of all conferences of which i is a member.

Example 6. Consider the hypergraph communication situation $(N, v, \mathcal{H})$, where $N = \{1, \dots, 6\}$, v is the unanimity game $u_{\{1,2,3\}}$ and $\mathcal{H} = \{\{1, 4\}, \{2, 5\}, \{3, 6\}, \{4, 5, 6\}\}$.

Then $r_{\mathcal{H}}^v(S) = \begin{cases} 1 & \text{if } S = N \\ 0 & \text{else} \end{cases}$ and, consequently, $\mu(N, v, \mathcal{H}) = (\frac{1}{6}, \dots, \frac{1}{6})$. Further, $r_N^v(\mathcal{A}) = \begin{cases} 1 & \text{if } \mathcal{A} = \mathcal{H} \\ 0 & \text{else.} \end{cases}$ Therefore, $\Phi_H(\mathcal{H}, r_N^v) = \frac{1}{4}$ for all $H \in \mathcal{H}$ and

$$\begin{aligned} \pi_1(N, v, \mathcal{H}) &= \pi_2(N, v, \mathcal{H}) = \pi_3(N, v, \mathcal{H}) = \frac{1}{2} \cdot \frac{1}{4} = \frac{1}{8}, \\ \pi_4(N, v, \mathcal{H}) &= \pi_5(N, v, \mathcal{H}) = \pi_6(N, v, \mathcal{H}) = \frac{1}{2} \cdot \frac{1}{4} + \frac{1}{3} \cdot \frac{1}{4} = \frac{5}{24}. \end{aligned}$$

Let HCS^N denote the class of all hypergraph communication situations with player set N . Van den Nouweland, Borm and Tijs (1990) provided axiomatic characterizations of

the Myerson value and the position value à la theorem 1 by extending the properties provided in section 2 to allocation rules $\gamma : HCS^N \rightarrow \mathbb{R}^N$.

Directed graphs. A *directed communication situation* is a triple (N, v, A) , where (N, v) is a zero-normalized *TU*-game and (N, A) is a directed graph.

In a directed communication situation (N, v, A) the players in a coalition S can effect communication in (i.e. send information via) $A(S) := \{(i, j) \in A \mid i \in S, j \in S\}$. Let us call a coalition S *strongly connected* if for all $i, j \in S, i \neq j$, there is a path from i to j and a path from j to i in the (sub)graph $(S, A(S))$. Clearly, each coalition S is partitioned into maximal strongly connected components. This partition of S is denoted by S/A . Again, we assume that the reward $r^v(S, A)$ of a coalition S is defined by the rewards of its components, i.e. $r^v(S, A) := \sum_{C \in S/A} v(C)$.

In this setting the point game (N, r_A^v) , the arc game (A, r_N^v) , the Myerson value $\mu(N, v, A)$ and the position value $\pi(N, v, A)$ can be defined in exactly the same way as in section 2. It may be noted that with respect to the position value one might want to distinguish between the roles of the begin point (information sender) and the end point (information receiver) of a directed arc by putting a weight vector $(\alpha, 1 - \alpha)$ with $0 \leq \alpha \leq 1$ on all arcs $(i, j) \in A$. The link value of each arc is then divided proportionally to this weight vector. In this way we obtain an α -position value.

Example 7. Consider the directed communication situation (N, v, A) , where $N = \{1, 2, 3\}$, v is the unanimity game $u_{\{1,2\}}$ and (N, A) is the directed graph represented in figure 3.

Figure 3

Then $r_A^v(S) = \begin{cases} 1 & \text{if } \{1, 2\} \subset S \\ 0 & \text{else} \end{cases}$ and, consequently, $\mu(N, v, A) = (\frac{1}{2}, \frac{1}{2}, 0)$. Further, it holds that

$$r_N^v(L) = \begin{cases} 1 & \text{if } \{a, b\} \subset L \text{ or } \{b, c, d\} \subset L \\ 0 & \text{else.} \end{cases}$$

Hence, $\Phi_a(A, r_N^v) = \frac{1}{4}$, $\Phi_b(A, r_N^v) = \frac{7}{12}$ and $\Phi_c(A, r_N^v) = \Phi_d(A, r_N^v) = \frac{1}{12}$. Consequently, with $\alpha \in [0, 1]$ the α -position value for player 1 is given by $\frac{1}{4}\alpha + \frac{1}{12}\alpha + \frac{7}{12}(1-\alpha) = \frac{7}{12} - \frac{1}{4}\alpha$, for player 2 by $\frac{1}{3} + \frac{1}{4}\alpha$ and for player 3 by $\frac{1}{12}$.

Adapting the axioms (AA) and (PA) introduced in section 2, *Koster* (1990) gave axiomatic characterizations à la theorem 1(ii) of the restrictions of the Myerson value and the α -position values to a special class of directed communication situations.

The conditions *Koster* (1990) imposed on the directed (communication) graphs seem rather strong. It would be interesting to see if the values can be characterized on a larger class of directed communication situations.

Obviously, an undirected graph (N, A) can be identified with a directed graph $(N, \vec{A})$ in a natural way: replace each undirected arc $\{i, j\} \in A$ by two directed arcs, (i, j) and (j, i) . It is easily seen that for all communication situations $(N, v, A) \in CS^N$ the Myerson value $\mu(N, v, A)$ equals the Myerson value $\mu(N, v, \vec{A})$ of the directed communication situation associated with (N, v, A) . However, such a relation need not exist for the α -position values. This is shown in

Example 8. Consider the (undirected) communication situation (N, v, A) where $N = \{1, 2, 3\}$, v is the unanimity game $u_{\{1,2\}}$ and (N, A) is the complete graph on N . Then (cf. example 2) $\pi(N, v, A) = (\frac{5}{12}, \frac{5}{12}, \frac{2}{12})$.

In the directed communication situation $(N, v, \vec{A})$ associated with (N, v, A) the link value of the links $(1,2)$ and $(2,1)$ equals $\frac{3}{10}$ and for the four other links it equals $\frac{1}{10}$. Therefore, for each $\alpha \in [0, 1]$, the α -position value equals $(\frac{2}{5}, \frac{2}{5}, \frac{1}{5})$.

It may be noted that in the model described above communication is only valuable if it is two-way. Maybe this is too strong an assumption and interesting things can be done if one-way communication is also rewarded.

NTU-games. An *NTU-game* (cf. Aumann (1967)) is a pair (N, V) , where $N = \{1, \dots, n\}$ is the set of players and V is a map that assigns to each coalition $S \in 2^N \setminus \{\emptyset\}$ a subset $V(S)$ of $\mathbb{R}^S$ that is non-empty and comprehensive. $V(S)$ is to be interpreted as the set of attainable payoff vectors for S .

Let (N, V) be an *NTU-game*. A *weight vector* $\lambda = (\lambda_i)_{i \in N} \in \mathbb{R}^N$ with $\lambda \geq 0$ and $\sum_{i \in N} \lambda_i = 1$ is called *V-feasible* if for all $S \subset N$, $S \neq \emptyset$,

$$v_\lambda(S) := \sup \left\{ \sum_{i \in S} \lambda_i x_i \mid x \in V(S) \right\} < \infty.$$

Hence, a *V-feasible* weight vector λ generates a *TU-game* v_λ .

Definition. Let (N, V) be an *NTU-game*. The *Shapley correspondence* $Sh(N, V) \subset \mathbb{R}^N$ (cf. Shapley (1969)) is defined by

$$Sh(N, V) := \{x \in V(N) \mid \text{there is a } V\text{-feasible } \lambda \in \mathbb{R}^N \text{ such that} \\ (\lambda_i x_i)_{i \in N} = \Phi(v_\lambda)\},$$

A vector $x \in Sh(N, V)$ is called a *Shapley value* of V .

An *NTU-communication situation* is a triple (N, V, A) , where (N, V) is an *NTU-game* with $V(\{i\}) = (-\infty, 0]$ for all $i \in N$, and (N, A) is an undirected communication graph without loops or parallel arcs. The definition of the Myerson value can straightforwardly be generalized to *NTU* communication situations.

Definition. For an *NTU* communication situation (N, V, A) the *(NTU-) point game*

(N, V_A) is defined by

$$V_A(S) := \prod_{C \in S/A} V(C) \subset \mathbb{R}^S$$

for all $S \subset N$ and the *Myerson correspondence* $M(N, V, A) \subset \mathbb{R}^N$ is defined by

$$M(N, V, A) := Sh(N, V_A).$$

In trying to extend the position value one would like to consider the sets $\prod_{C \in N/L} V(C)$ for all $L \subset A$ but, since $\prod_{C \in N/L} V(C) \subset \mathbb{R}^N$ for all $L \subset A$, this does not generate an *NTU*-arc game. However, we can introduce an *NTU*-position value without explicitly introducing an *NTU*-arc game as follows.

Let (N, V, A) be an *NTU* communication situation. A weight vector $\lambda \in \mathbb{R}^N$ is called *arc-admissible* if for all $L \subset A$

$$v_{N,\lambda}(L) := \sup \left\{ \sum_{i \in N} \lambda_i x_i \mid x \in \prod_{C \in N/L} V(C) \right\} < \infty. \quad (6.1)$$

Hence, an arc-admissible weight vector λ generates a *TU*-arc game $(A, v_{N,\lambda})$.

Definition. For an *NTU* communication situation (N, V, A) the *position correspondence* $\Pi(N, V, A) \subset \mathbb{R}^N$ is defined by

$$\begin{aligned} \Pi(N, V, A) := \{ & x \in \prod_{C \in N/A} V(C) \mid \text{there is an arc-admissible } \lambda \in \mathbb{R}^N \\ & \text{such that } \lambda_i x_i = \sum_{a \in A_i} \frac{1}{2} \Phi_a(A, v_{N,\lambda}) \text{ for all } i \in N \}. \end{aligned}$$

Example 9. Let $\alpha \in [0, \frac{1}{2}]$. Consider the *NTU* communication situation (N, V_α, A) , where $N = \{1, 2, 3\}$, $A = \{\{1, 3\}, \{2, 3\}\}$ and (N, V_α) is the *NTU*-game introduced by Roth (1980), i.e.

$$\begin{aligned}
V_\alpha(\{i\}) &:= \{x_i \in \mathbb{R}^{\{i\}} \mid x_i \leq 0\}, \\
V_\alpha(\{1, 2\}) &:= \{(x_1, x_2) \in \mathbb{R}^{\{1, 2\}} \mid x_1 \leq \tfrac{1}{2}, x_2 \leq \tfrac{1}{2}\}, \\
V_\alpha(\{1, 3\}) &:= \{(x_1, x_3) \in \mathbb{R}^{\{1, 3\}} \mid x_1 \leq \alpha, x_3 \leq 1 - \alpha\}, \\
V_\alpha(\{2, 3\}) &:= \{(x_2, x_3) \in \mathbb{R}^{\{2, 3\}} \mid x_2 \leq \alpha, x_3 \leq 1 - \alpha\}
\end{aligned}$$

and

$$\begin{aligned}
V_\alpha(\{1, 2, 3\}) &:= \{(x_1, x_2, x_3) \in \mathbb{R}^{\{1, 2, 3\}} \mid (x_1, x_2, x_3) \leq y \text{ for some } y \text{ in} \\
&\quad \text{the convex hull of } (\tfrac{1}{2}, \tfrac{1}{2}, 0), (\alpha, 0, 1 - \alpha) \text{ and } (0, \alpha, 1 - \alpha)\},
\end{aligned}$$

where the indices correspond to the players.

Considering the weight vector $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, we see that $(\frac{1}{6}, \frac{1}{6}, \frac{2}{3}) \in M(N, V_\alpha, A)$ if $\alpha \in [0, \frac{1}{3}]$ and $(\frac{1}{4}, \frac{1}{4}, \frac{1}{2}) \in \Pi(N, V_\alpha, A)$ for all $\alpha \in [0, \frac{1}{2}]$.

Each TU -game (N, v) corresponds to an NTU -game (N, V) where for all $S \subset N$ $V(S) := \{x \in \mathbb{R}^S \mid \sum_{i \in S} x_i \leq v(S)\}$. Theorem 11 states that the Myerson correspondence and the position correspondence are generalizations of the Myerson value and the position value for TU communication situations.

Theorem 11. Let $(N, v, A) \in CS^N$ and let (N, V) be the NTU -game corresponding to (N, v) . Then the following two assertions hold

- (i) $M(N, V, A) = \{\mu(N, v, A)\}$
- (ii) $\Pi(N, V, A) = \{\pi(N, v, A)\}$

Proof. Let $\lambda \in \mathbb{R}^N$ be a weight vector. Suppose $C \in N/A$ and $i, j \in C$ are such that $\lambda_i \neq \lambda_j$. Then, obviously, $\sup\{\sum_{k \in C} \lambda_k x_k \mid x \in V(C)\} = \infty$. Hence, λ is neither V_A -feasible nor arc-admissible.

Let the weight vector $\lambda \in \mathbb{R}^N$ be such that for each component $C \in N/A$ there exists a real $\lambda(C)$ with $\lambda_i = \lambda(C)$ for all $i \in C$. Then it is not difficult to see that λ is both V_A -feasible and arc-admissible. For the TU -game $v_{A, \lambda}$ corresponding to V_A and λ and for all $S \subset N$ it holds that

$$\begin{aligned}
v_{A,\lambda}(S) &= \sup\{\sum_{i \in S} \lambda_i x_i \mid x \in V_A(S)\} \\
&= \sup\{\sum_{C \in N/A} \lambda(C) (\sum_{i \in S \cap C} x_i) \mid x \in \mathbb{R}^S \text{ and } \sum_{i \in T} x_i \leq v(T) \\
&\quad \text{for all } T \in S/A\} \\
&= \sum_{C \in N/A} \lambda(C) (\sum_{T \in S \cap C/A} v(T)) \\
&= \sum_{C \in N/A} \lambda(C) r_A^v(S \cap C).
\end{aligned}$$

where the third equality follows from the fact that every component of S is contained in a component of N .

Then, for each $i \in N$ and $D \in N/A$ such that $i \in D$ it holds that

$$\begin{aligned}
\Phi_i(N, v_{A,\lambda}) &= \lambda(D) \Phi_i(D, r_A^v|_D) \\
&= \lambda(D) \Phi_i(N, r_A^v) = \lambda(D) \mu_i(N, v, A),
\end{aligned}$$

where $r_A^v|_D$ denotes the restriction of r_A^v to D . Hence, $(\lambda_i \mu_i(N, v, A))_{i \in N} = \Phi(N, v_{A,\lambda})$. Since $\mu(N, v, A) \in \prod_{C \in N/A} V(C)$, the definition of $M(N, V, A)$ implies that $M(N, V, A) = \{\mu(N, V, A)\}$.

For the TU -game $v_{N,\lambda}$ defined by (6.1) and for all $L \subset A$ it holds that

$$\begin{aligned}
v_{N,\lambda}(L) &= \sup\{\sum_{i \in N} \lambda_i x_i \mid x \in \prod_{T \in N/L} V(T)\} \\
&= \sup\{\sum_{C \in N/A} \lambda(C) (\sum_{i \in C} x_i) \mid x \in \mathbb{R}^N \text{ and } \sum_{i \in T} x_i \leq v(T) \\
&\quad \text{for all } T \in N/L\} \\
&= \sum_{C \in N/A} \lambda(C) (\sum_{T \in C/L} v(T)) \\
&= \sum_{C \in N/A} \lambda(C) r_C^v(L \cap A(C)).
\end{aligned}$$

Then, obviously, for each $a \in A$ and $D \in N/A$ such that $a \in A(D)$ it holds that

$$\Phi_a(A, v_{N,\lambda}) = \lambda(D) \Phi_a(A, r_D^v) = \lambda(D) \lambda_a(N, v, A).$$

Since for all $C \in N/A$

$$\begin{aligned} \sum_{i \in C} \pi_i(N, v, A) &= \sum_{i \in C} \sum_{a \in A_i} \frac{1}{2} \lambda_a(N, v, A) \\ &= \sum_{a \in A(C)} \lambda_a(N, v, A) = v(C), \end{aligned}$$

and, correspondingly, $\pi(N, v, A) \in \prod_{C \in N/A} V(C)$, the definition of $\Pi(N, V, A)$ implies that $\Pi(N, V, A) = \{\pi(N, v, A)\}$. $\square$

It would be interesting to investigate if one of the axiomatic characterizations of the Shapley correspondence provided by *Aumann* (1985) and *Kern* (1985) can be adapted to find axiomatic characterizations of the Myerson correspondence and the position correspondence, possibly restricted to a subclass of *NTU* communication situations. Alternatively, the notion of fairness might be extended to *NTU* communication situations to provide an axiomatic characterization of the Myerson correspondence à la *Myerson* (1977).

Oortwijn, Borm, Keiding and Tijs (1990) defined two possible extensions of the τ -value to *NTU*-games, the *NTU τ -value* and the *compromise value*. One could also consider applying these solution concepts in *NTU* communication situations instead of the Shapley correspondence.

Controlled communication networks. Consider a finite undirected graph (P, A) without loops or parallel arcs. We assume that for each point $p \in P$ a balanced simple control game (N, c_p) is given and, similarly, for each arc $a \in A$ we have a balanced simple control game (N, c_a) . Here, a *TU*-game (N, v) is called *simple* if $v(S) \in \{0, 1\}$ for all $S \subset N$ and $v(N) = 1$. If c_p is the (simple) control game for point $p \in P$, then a coalition $S \subset N$ is allowed to use point p if and only if $c_p(S) = 1$. The control games c_a for arcs $a \in A$ have a similar interpretation.

Furthermore, we assume that there is a reward function r on subsets of points and arcs $r : 2^P \times 2^A \rightarrow \mathbf{R}$. In this setting it seems reasonable to assume that an arc is

useless without both of its end points, i.e. for all $Q \subset P$ and $L \subset A$ it holds that $r(Q, L) = r(Q, L \setminus \{\{p_1, p_2\}\})$ if $\{p_1, p_2\} \in L$ is such that $\{p_1, p_2\}$ is not a subset of Q . Further, we assume that r is zero-normalized, i.e. $r(\{p\}, \emptyset) = 0$ for all $p \in P$, and that r has an additive structure with respect to the components of (N, A) , i.e.

$$r(Q, L) = \sum_{C \in Q/L} r(C, L(C))$$

for all $Q \subset P$ and $L \subset A$ such that $N(L) \subset Q$.

A *controlled communication situation* is a 6-tuple $(N, P, A, \{c_p \mid p \in P\}, \{c_a \mid a \in A\}, r)$ as described above.

Note that for a communication situation $(N, v, A) \in CS^N$ the corresponding reward function r^v satisfies the three properties mentioned. Further, the points in the communication graph (N, A) are controlled by a single player, each player controlling exactly one point, and each arc is controlled by the two players at its end points.

Using the following theorem of *Curiel* (1988) on balanced simple games we can introduce a generalized Myerson value and position value.

Theorem 12. (*Curiel* (1988)). Let (N, v) be a balanced simple game. Then

$$\text{veto}(v) := \bigcap \{S \subset N \mid v(S) = 1\} \neq \emptyset$$

and, moreover,

$$C(N, v) = \{x \in \mathbb{R}^N \mid x \geq 0 \text{ and } \sum_{i \in \text{veto}(v)} x_i = 1\}.$$

Definition. Let $\mathcal{C} = (N, P, A, \{c_p \mid p \in P\}, \{c_a \mid a \in A\}, r)$ be a controlled communication situation. Then the Myerson value $\mu(\mathcal{C}) \in \mathbb{R}^N$ is defined by

$$\mu_i(\mathcal{C}) := \sum_{p \in P: i \in \text{veto}(c_p)} |\text{veto}(c_p)|^{-1} \Phi_p(P, r_A)$$

for all $i \in N$, where the game (P, r_A) is defined by $r_A(Q) := r(Q, A)$ for all $Q \subset P$.

Further, the position value $\pi(\mathcal{C}) \in \mathbb{R}^N$ is defined by

$$\pi_i(\mathcal{C}) := \sum_{a \in A: i \in \text{veto}(c_a)} |\text{veto}(c_a)|^{-1} \Phi_a(A, r_P)$$

for all $i \in N$, where the game (A, r_P) is defined by $r_P(L) := r(P, L)$ for all $L \subset A$.

Axiomatic characterizations à la theorem 1(ii) can be easily established for both the Myerson value and the position value for the class of controlled communication situations for which the underlying graphs contain no cycles.

References

- Aumann, R.J. (1967). A survey of cooperative games without side payments. In: Essays in Math. Economics (Ed. Shubik, M.), Princeton University Press, Princeton, 3-27.
- Aumann, R.J. (1985). An axiomatization of the non-transferable utility value. *Econometrica*, 53, 599-612.
- Aumann, R.J. and Dreze, J.H. (1974). Cooperative games with coalition structures. *Intern. J. Game Theory*, 3, 217-237.
- Aumann, R.J. and Maschler, M. (1964). The bargaining set for cooperative games. In: *Advances in Game Theory* (Eds. Dresher, M., Shapley, L.S. and Tucker, A.W.), Princeton University Press, Princeton, 443-471.
- Aumann, R.J. and Myerson, R.B. (1988). Endogenous formation of links between players and coalitions: an application of the Shapley value. In: *The Shapley Value* (Ed. A.E. Roth), 175-190.
- Borm, P.E.M. (1990). On game theoretic models and solution concepts. PhD Dissertation, University of Nijmegen, The Netherlands.
- Borm, P. (1991). On the strategic formation of communication links. Dept. of Econometrics, Tilburg University, The Netherlands. (In preparation).

- Borm, P., Owen, G. and Tijs, S.H. (1991). On the position value for communication situations. To appear in SIAM J. on Discr. Math.
- Brink, R. van den and Gilles, R.P. (1991). Axiomatizations of the conjunctive permission value for games with permission structures. Research Memorandum FEW 485, Dept. of Econometrics, Tilburg University, The Netherlands.
- Curiel, I.J. (1988). Cooperative game theory and applications. PhD Dissertation, University of Nijmegen, The Netherlands.
- Faigle, U. and Kern, W. (1990). The position value of general communication games. Mimeo, Dept. of Applied Mathematics, University of Twente, The Netherlands.
- Gilles, R.P. and Owen, G. (1990). Games with permission structures: the disjunctive approach. Research Memorandum, Dept. of Econometrics, Tilburg University, The Netherlands.
- Gilles, R.P., Owen, G. and Brink, R. van den (1991). Games with permission structures: the conjunctive approach. Research Memorandum FEW 473, Dept. of Econometrics, Tilburg University, The Netherlands.
- Grofman, B. and Owen, G. (1982). A game-theoretic approach to measuring centrality. *Social Networks*, 3, 213-224.
- Harsanyi, J.C. (1959). A bargaining model for the cooperative n-person game. In: *Contributions to the Theory of Games, IV* (Eds. Tucker, A.W. and Luce, R.D.), Princeton University Press, Princeton, 325-356.
- Hart, S. and Kurz, M. (1983). Endogenous formation of coalitions. *Econometrica*, 51, 1047-1064.
- Kalai, E., Postlewaite, A. and Roberts, J. (1978). Barriers to trade and disadvantageous middlemen: nonmonotonicity of the core. *Journal of Economic Theory*, 19, 200-209.

- Kern, R. (1985). The Shapley transfer value without zero weights. *Intern. J. Game Theory*, 14, 73-92.
- Koster, A.M. (1990). Cooperation and communication restrictions. Master's Thesis (In Dutch), Dept. of Mathematics, University of Nijmegen, The Netherlands.
- Myerson, R.B. (1977). Graphs and cooperation in games. *Math. of Oper. Research*, 2, 225-229.
- Myerson, R.B. (1980). Conference structures and fair allocation rules. *Intern. J. Game Theory*, 9, 169-182.
- Nouweland, A. van den and Borm, P. (1991). On the convexity of communication games. *Intern. J. Game Theory*, 19, 421-430.
- Nouweland, A. van den, Borm, P. and Tijs, S.H. (1990). Allocation rules for hypergraph communication situations. Report 9033, Dept. of Mathematics, University of Nijmegen, The Netherlands.
- Nouweland, A. van den, Borm, P., Owen, G. and Tijs, S.H. (1991). Cost allocation and communication. Research Memorandum FEW 469, Dept. of Econometrics, Tilburg University, The Netherlands.
- Neumann, J. von and Morgenstern, O. (1944). *Theory of games and economic behavior* (3th edition, 1953), Princeton University Press, Princeton.
- Oortwijn, S., Borm, P., Keiding, H. and Tijs, S.H. (1990). Extensions of the τ -value to *NTU*-games. Research Memorandum FEW 457, Dept. of Econometrics, Tilburg University, The Netherlands.
- Owen, G. (1986). Values of graph-restricted games. *SIAM J. Alg. Discr. Meth.*, 7, 210-220.
- Rosenthal, E.C. (1988). Communication networks and their role in cooperative games. *Social Networks*, 10, 255-263.

- Rosenthal, E.C. (1988). Communication and its cost in graph-restricted games. *Theory and Decision*, 25, 275-286.
- Shapley, L.S. (1953). A value for n -person games. *Contributions to the Theory of Games*, II (Eds. Tucker, A.W. and Kuhn, H.), 307-317.
- Shapley, L.S. (1969). Utility comparison and the theory of games. In: *La Decision: Aggregation et Dynamique des Ordres de Preference*, Paris: Editions du Centre National de la Recherche Scientifique, 251-263.
- Schmeidler, D. (1969). The nucleolus of a characteristic function game. *SIAM J. Appl. Math.*, 17, 1163-1170.
- Tijs, S.H. (1981). Bounds for the core and the τ -value. In: *Game Theory and Mathematical Economics* (Eds. Moeschlin, O. and Pallaschke, P.), North-Holland, Amsterdam, 123-132.
- Vind, K. (1983). Equilibrium with coordination. *Journal of Mathematical Economics*, 12, 275-285.
- Vind, K. (1986). Exchange equilibrium. In: *Contributions to Mathematical Economics* (Eds. Hildenbrand, W. and A. Mas-Colell), North-Holland, Amsterdam.
- Young, H.P. (1985). Monotonic solutions of cooperative games. *Intern. J. Game Theory*, 14, 65-72.