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Learning, Inflation Reduction and Optimal Monetary Policy

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Publication date:
2003

[Link to publication in Tilburg University Research Portal](#)

Citation for published version (APA):

Schaling, E. (2003). *Learning, Inflation Reduction and Optimal Monetary Policy*. (CentER Discussion Paper; Vol. 2003-74). Macroeconomics.

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No. 2003–74

**LEARNING, INFLATION REDUCTION AND OPTIMAL
MONETARY POLICY**

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June 2003

ISSN 0924-7815

Learning, Inflation Reduction and Optimal Monetary Policy

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This Version 30 June 2003

Abstract

In this paper we analyze disinflation in two environments. One in which the central bank has perfect knowledge, in the sense that it understands and observes the process by which private sector inflation expectations are generated, and one in which the central bank has to learn the private sector inflation forecasting rule. Here, the learning scheme we investigate is that of least-squares learning (recursive OLS) using the Kalman filter. With imperfect knowledge, results depend on the learning scheme that is employed. A novel feature of the passive learning policy - compared to the central bank's disinflation policy under perfect knowledge - is that the degree of monetary accommodation (the extent to which the central bank accommodates private sector inflation expectations) is no longer constant across the disinflation, but becomes state-dependent. This means that the central bank's behaviour changes during the disinflation as it collects more information.

Keywords: Learning, Rational Expectations, Separation Principle, Kalman Filter, Time-Varying Parameters, Optimal Control

JEL Codes: C53, E43, E52, F33

1 INTRODUCTION ¹

As pointed out by Bullard (1991), in the three decades since the publication of the seminal work on rational expectations (RE) in the early 1960s, a steely paradigm was forged in the economics profession regarding acceptable modelling procedures. Simply stated, the paradigm was that economic actors do not persist in making foolish mistakes in forecasting over time.

Since the late 1980s researchers have challenged this paradigm by examining the idea that *how* systematic forecast errors are eliminated may have important implications for macroeconomic policy. Researchers who have focused on this question have been studying what is called ‘learning’, because any method of expectations formation is known as a learning mechanism. Thus, since the late 1980s a learning literature, or learning paradigm, developed.² An excellent introduction to – and survey of – this paradigm is presented in Evans and Honkapohja (2001).

A different strand of literature in the economics profession has been dealing with optimal control or dynamic optimisation. The method of dynamic programming advanced by Bellman has been a main tool for optimisation over time under uncertainty.

In general there are few papers in the literature that combine the themes of *learning* and (optimal) *control*. An exception is recent and important work by Wieland (2000a,b). Wieland (2000a) analyses the situation where a central bank has limited information concerning the transmission channel of monetary policy. Then, the CB is faced with the difficult task of simultaneously *controlling* the policy target and estimating (*learning*) the impact of policy actions. Thus, the so-called separation principle does not hold, and a trade-off between estimation and control arises because policy actions influence estimation (learning) and provide information that may improve future performance. Wieland analyses this trade-off in a simple model with parameter uncertainty and conducts dynamic simulations of the central bank’s decision problem.

In this paper we apply the themes of learning and control to the problem of how a central bank should organize a disinflation process, i.e. *how to reduce inflation*. Thus, our approach follows recent relevant work by Sargent (1999).

Central banks throughout the world are moving to adopt long-run price stability as their primary goal. Thus, there is agreement among central bankers, academics and financial market representatives that low or zero inflation is the appropriate long-run goal of monetary policy. However, there is less agreement on what strategies should be adopted to achieve price stability. For example, on the one hand we have the view that a major cause of rising unemployment in the 1980s in OECD countries was the tight monetary policy that those countries pursued to reduce inflation. On the other we have the view that a sharp disinflation may be preferable to gradualism because the latter invites speculation about future reversals or U-turns in policy.

The received view in the literature - as expressed by King (1996) at the Kansas City Fed symposium on Achieving Price Stability at Jackson Hole - seems to be for a gradual timetable, with inflation targets consistently set below the public’s inflation expectations.

¹ An earlier version of this paper is forthcoming as a Bank of Finland Discussion Paper. It was written while I was a visiting scholar at the Research Department of the Bank of Finland, which I thank for their hospitality. The views expressed are those of the author and do not necessarily reflect the views of the Bank of Finland. I am grateful for helpful comments by Seppo Honkapohja, Juha Tarkka, Marco Hoeberichts, David Mayes, Jouko Vilmunen, Martin Ellison, Alain Kabundi, Justin Prentice, Karim Sadrieh and seminar participants at the Bank of Finland, Tilburg University and RAU.

² Important papers are Lucas (1987) and Marcet and Sargent (1988, 1989a,b,c).

King shows how the optimal speed of disinflation depends crucially on whether the private sector immediately believes in the new low inflation regime or not. If they do, the best strategy is to disinflate quickly, since the output costs are zero. Of course, if expectations are slower to adapt, disinflation should be more gradual as well.

A central bank may try to convince the private sector of its commitment to price stability by choosing to reduce its inflation target towards the inflation target quickly. King calls this ‘teaching by doing’. Then the choice of a particular inflation rate influences the speed at which expectations adjust to price stability.

Teaching by doing effects have recently been analyzed by Hoyerichs and Schaling (2000) for simple macro models with both linear and nonlinear (convex) Phillips curves. They also allow the central bank’s ‘doing’ to affect private sector forecasting. Of course, if the CB recognizes its potential for active ‘teaching’ its incentive structure changes. More specific, it should realize that by disinflating faster, it can reduce the associated output costs by ‘teaching’ the private sector that it means business. Thus, the dependence of private sector expectations on the actual inflation rate should be part of its optimization problem. This is in fact what they find: allowing for ‘teaching by doing’ effects always speeds up the disinflation vis-à-vis the case where this effect is absent. So, their result is that ‘speed’ in the disinflation process does not necessarily ‘kill’ in the sense of creating large output losses.

In this paper we analyze disinflation in two environments. One in which the central bank has *perfect knowledge*, in the sense that it understands and observes the process by which private sector inflation expectations are generated, and one in which the central bank has to *learn* the private sector inflation forecasting rule. Here following Evans and Honkapohja (2001), the learning scheme we investigate is that of *least-squares learning* (recursive OLS) using the Kalman filter.

For the case of perfect knowledge we find that the optimal disinflation is faster under commitment than discretion. Next, in the commitment case the disinflation is less gradual, the higher the central bank’s rate of time preference and – interestingly – the higher the degree of persistence in inflation expectations.

With imperfect knowledge results depend on the learning scheme that is employed. A novel feature of the passive learning policy - compared to the central bank’s optimal disinflation policy under perfect knowledge - is that the degree of monetary accommodation (the extent to which the central bank accommodates private sector inflation expectations) is no longer constant across the disinflation, but becomes state-dependent. This means that the central bank’s behaviour changes during the disinflation as it collects more information.

The remainder of this paper is organized as follows. Section 2 discusses private sector behaviour regarding the credibility of the central bank’s inflation target. In section 3 we present the benchmark case of perfect knowledge and contrast discretion and commitment. Imperfect knowledge and the Kalman filter are introduced in section 4. Section 5 analyzes disinflation policies under imperfect knowledge. We conclude in section 6. The appendices contain the derivation of steady state relationships, the commitment solution, and the derivation of the Kalman filter equations used in the main text.

2 THE ENVIRONMENT

In this section we examine the speed of disinflation that would be chosen by a central bank in a world in which monetary policy affects real output in the short run but not in the long run. We use a simple macroeconomic model that combines nominal wage and price stickiness and slow adjustment of expectations to a new monetary policy regime. The model has three key equations – for monetary policy preferences, aggregate supply, and inflation expectations.

Suppose monetary policy is conducted by a central bank with inflation and output targets $\mathbf{p}^*$ and z^* . Interpret monetary policy as implying that the central bank's objective in period t is to choose a sequence of current and future inflation rates $\{\mathbf{p}_t\}_{t=t}^{\infty}$ so as to minimize

$$E_t \sum_{t=t}^{\infty} \mathbf{d}^{t-t} L(\mathbf{p}_t, z_t) \quad (2.1)$$

where E_t denotes expectations conditional upon (the central bank's) information available in year t , the discount factor is defined as $\mathbf{d} = (1 + \mathbf{r})^{-1}$ and fulfils $0 < \mathbf{d} < 1$ (where $\mathbf{r}$ denotes the discount rate), and the period loss function $L(\mathbf{p}_t, z_t)$ is

$$\frac{1}{2} [a(\mathbf{p}_t - \mathbf{p}^*)^2 + (z_t - z^*)^2]$$

where $\mathbf{p}$ is the inflation rate in year t , z is the natural logarithm of output, while $0 < a < \infty$ represents the central bank's relative weight on inflation stabilization. In what follows we set z^* equal to the natural rate of output (which in turn is normalized to zero).

The model is simple. Aggregate supply exceeds the natural rate of output when inflation is higher than was expected by agents when nominal contracts were set. This is captured by a simple short-run Phillips curve.

$$z_t = \mathbf{p}_t - E_{t-1}^{\wedge} \mathbf{p}_t \quad (2.2)$$

Where z is the natural logarithm of the output gap.³ Here the superscript $\wedge$ indicates that the expectation of inflation is the subjective expectation (belief) of private agents. This belief does not necessarily coincide with a rational expectation.

Private agents believe that inflation will be reduced from its initial level towards the inflation target, but are not sure by how much. More specific, the public's inflation beliefs are given by

$$\mathbf{p}_t = \mathbf{p}^* + u_t \quad (2.3)$$

³ It would be straightforward to extend the Phillips curve with an aggregate supply shock. Standard assumptions on nominal rigidities would then imply that inflation expectations are set before the shock is observed, while monetary policy would be set in full knowledge of the shock to output.

where u is a shock to the inflation rate. So, we assume that equation (2.3) is the *perceived law of motion* of private agents.

In order to study changes in inflation expectations, we extend this system with a stochastic process for u . As in Bullard and Schaling (2001) we use a two-state process defined by

$$u_t = \mathbf{p}^b \cdot s_t = \begin{cases} \mathbf{p}^b & \text{if } s_t = 1 \\ 0 & \text{if } s_t = 0 \end{cases} \quad \text{where } 0 < \mathbf{p}^b < \mathbf{p}_0 - \mathbf{p}^* \quad (2.4)$$

If $\mathbf{p}^b \rightarrow 0$ there is no difference between the two regimes, and so we can think of $\mathbf{p}^b$ as scaling the effect of the difference in inflation beliefs in the two regimes. In the case where the parameter $\mathbf{p}^b \rightarrow 0$, there is a completely credible regime switch. Thus, $\mathbf{p}^b$ is a measure for the extent to which the public's beliefs (and consequently expectations) about inflation are uncoupled from the intended policy objective.⁴ Figure 2.1 illustrates.

Figure 2.1 The Perceived Law of Motion for Inflation

$\mathbf{p}$ is the private sector's perceived law of motion of the inflation rate, $\mathbf{p}_0$ is the initial inflation rate and $\mathbf{p}^b$ is the difference in inflation beliefs in the two regimes.

The unobserved state of the system s_t takes on a value of zero or one, and follows a two-state Markov process.⁵ There is an associated transition probability matrix $T = \begin{bmatrix} p & 1-p \\ 1-q & q \end{bmatrix}$, where

⁴ I have borrowed this terminology from Orphanides and Williams (2002). Or, using a term from the older time-consistency literature, it can be seen as a measure for the lack of 'credibility' of the CB's inflation target.

⁵ We adopt the usual convention that for discrete-valued variables, capital letters denote the random variable and small letters a particular realization. If both interpretations apply we will use small letters.

$$\begin{aligned}
\Pr ob[S_{t+1} = 1 | S_t = 1] &= p, \\
\Pr ob[S_{t+1} = 0 | S_t = 1] &= 1 - p, \\
\Pr ob[S_{t+1} = 0 | S_t = 0] &= q, \\
\Pr ob[S_{t+1} = 1 | S_t = 0] &= 1 - q
\end{aligned} \tag{2.5}$$

So the probability of remaining in the high (low) state conditional on being in the high (low) state in the previous period is p (q), and the probability of switching from the high (low) to the low (high) state is $1 - q$ ($1 - p$).

As suggested by Hamilton (1989), the stochastic process for equation (2.5) admits the following AR(1) representation:

$$s_{t+1} = (1 - q) + \mathbf{g}_t + v_{t+1} \tag{2.6}$$

where $\mathbf{g} \equiv p + q - 1$, and v_t is a (discrete) white noise process with mean zero and variance $\mathbf{S}_v^2$.⁶

We want to study the dynamics of the system following a switch to a new regime, which in our model will constitute a switch from one state to the other. We are particularly interested in the effects of this switch on the dynamics of private sector inflation beliefs.

3 OPTIMAL MONETARY POLICY UNDER PERFECT KNOWLEDGE

To get some straightforward results, we assume that the central bank understands and observes the process by which private sector inflation expectations are generated. This is the benchmark case of *perfect knowledge*. We model least-squares learning by the central bank in section 4.

Consider a switch from a monetary policy regime in which inflation has averaged $\mathbf{p}_0$ to a regime of price stability in which inflation equals the inflation target $\mathbf{p}^*$. What is the optimal transition path? That will depend upon how quickly private sector inflation expectations adjust to the new regime. Following King (1996), it is useful to consider two cases: (1) a completely credible policy regime switch: private sector expectations adjust immediately to the new policy reaction function – this is the case of rational or model consistent expectations; (2) ‘endogenous forecasting’: the private sector’s forecasting rule depends on the policy choices made in the new regime.

With a completely credible regime change, private sector inflation expectations are consistent with the new inflation target. From equation (2.4) it can be seen that this is the case where $\mathbf{p}^b = 0$, so that we have $\hat{E}_{t-1} \mathbf{p}_t = \mathbf{p}^*$. Hence output and inflation are given by $z_t = 0$ and $\mathbf{p}_t = \mathbf{p}^*$. Since the level of output is independent of the inflation rate, policy can aim at price stability without any expected output loss. The optimal policy is to move immediately to the inflation target.

⁶ For the technical details see Appendix A.

The more general case is that of ‘endogenous forecasting’ by the private sector. We define endogenous forecasting as the case where the private sector’s forecasting rule depends on the policy choices made in the new regime.

To see that this in fact the relevant situation in our model, we apply $(1 - \mathbf{g}L)$ where L is the lag operator ($L^j x_t = x_{t-j}$) to (2.3) and take account of (2.6),

$$(1 - \mathbf{g}L)\mathbf{p}_t = \mathbf{p}^b.(1 - \mathbf{g}L)s_t + (1 - \mathbf{g})\mathbf{p}^* \quad (3.1)$$

Substituting for $(1 - \mathbf{g}L)s_t$ from (2.6), we may rewrite equation (3.1) as

$$\mathbf{p}_t = \mathbf{g}\mathbf{p}_{t-1} + (1 - \mathbf{g})\mathbf{p}^* + \mathbf{p}^b.(1 - q) + \mathbf{p}^b.v_t \quad (3.2)$$

Hence, the combination of the private sector’s perceived law of motion (equation (2.3)) and the AR(1) representation of the inflation state (equation (2.6)) gives rise to a first-order stochastic difference equation for inflation.

Taking expectations at time $t - 1$ of equation (3.2), where the expectations operator $\hat{E}$ refers to agents’ subjective expectations, we obtain

$$\hat{E}_{t-1}\mathbf{p}_t = \mathbf{g}\mathbf{p}_{t-1} + (1 - \mathbf{g})\mathbf{p}^* + \mathbf{p}^b.(1 - q) \quad (3.3)$$

Our main finding is that the private sector’s optimal inflation forecast - in an environment where the perceived law of motion is one with unobserved regime shifts - involves a lagged inflation term. More precise, at time $t - 1$ private agents’ inflation expectations for period t are a linear function of the inflation target, the lagged inflation rate and a constant, where the coefficients are functions of the structural parameters of the Markov switching process, as shown in equation (3.3). Thus, indeed the private sector’s forecasting rule depends on the policy choices made in the new monetary policy regime; this can be seen from the presence of the $\mathbf{p}_{t-1}$ term.

An important limiting case of equation (3.3) is when $\mathbf{g} = \mathbf{p}^b = 0$. In this case the shock to inflation becomes serially uncorrelated and the private sector’s optimal inflation forecast is the inflation target.

3.1 Discretion

Now we solve the model for the case where the central bank does not internalize the constraint (3.3). This is equivalent to the case where $\mathbf{d} \rightarrow 0$. In this discretionary case the minimization problem of the central bank reduces to the static problem

$$\text{Min}_{\{\mathbf{p}_t\}} E_t \frac{1}{2} [a(\mathbf{p}_t - \mathbf{p}^*)^2 + (z_t)^2]$$

The associated optimal policy is

$$\mathbf{p}_t = \frac{1}{1+a} x_t + \frac{a}{1+a} \mathbf{p}^* \quad (3.5.1)$$

Where $x_t = E_{t-1} \mathbf{p}_t$. The optimal transition to price stability is to allow inflation to fall gradually. The inflation rate should decline as a constant proportion of the exogenously expected inflation rate. That proportion depends on the weight a attached to the importance of keeping inflation close to the inflation target relative to keeping output close to its natural rate. The inflation rate moves gradually to the level of the inflation target, but is always below expected inflation. Figure 3.2 shows an example in which expectations decline steadily. Note that inflation adjusts to its long-run value gradually over time.

By substituting (3.3) into (3.5.1) we can derive the solution for the inflation process under discretion

$$\mathbf{p}_t = \frac{\mathbf{g}}{1+a} \mathbf{p}_{t-1} + \frac{(1-\mathbf{g}+a)\mathbf{p}^* + \mathbf{p}^b \cdot (1-q)}{1+a} \quad (3.5.2)$$

It can be easily seen that the greater the central bank's weight on inflation stabilization, the smaller the first-order autocorrelation in inflation. The persistence of inflation has a limiting value approaching $\mathbf{g}$ (where $\mathbf{g}$ is a function of the structural parameters of the Markov switching process, i.e. $\mathbf{g} \equiv p + q - 1$) when a approaches zero, and zero when a approaches infinity. That is, if the central bank cares only about output stabilization – that is engages in a policy of full accommodation – the inflation rate becomes highly persistent. Conversely, if the central bank cares only about inflation stabilization – that is follows a ‘cold turkey’ strategy – the inflation rate displays no serial correlation. It is clear that in the first extreme all output loss is eliminated, but at the cost of inflation falling only at the exogenous rate of decline of private sector inflation expectations, while in the second case price stability is achieved even during the transition period, but only at the cost of an expected cumulative output loss of $-\sum_{t=t}^{\infty} (x_t - \mathbf{p}^*)$.

3.2 Commitment

In his discussion of what we term *endogenous forecasting*, King (1996, p. 68) says that there are good reasons for the private sector to suppose that in trying to learn about the future inflation rate many of the relevant factors are exogenous to the path of inflation itself. But a central bank may try to convince the private sector of its commitment to price stability by choosing to reduce its inflation target towards the inflation target quickly. King calls this ‘teaching by doing’. Then the choice of a particular inflation rate influences the speed at which expectations adjust to price stability.

In this section we allow the central bank's ‘doing’ to affect private sector forecasting. Of course, if the CB recognises its potential for active ‘teaching’ its incentive structure changes. More specific, it should realise that by disinflating faster, it can reduce the associated output costs by ‘teaching’ the private sector that it means business. Thus, the dependence of private sector expectations on the actual inflation rate - equation (3.3) above - should be part of its optimisation problem. In what follows we refer to this as the case of ‘commitment’.

Now, the central bank's problem is to choose $\{\mathbf{p}_t\}_{t=t}^{\infty}$ so as to maximize

$$E_t \sum_{t=l}^{\infty} d^{t-l} \left[-a(\mathbf{p}_t - \mathbf{p}^*)^2 - (z_t - z^*)^2 \right] \quad (2.1)$$

subject to

$$z_t = \mathbf{p}_t - \hat{E}_{t-1} \mathbf{p}_t \quad (2.2)$$

$$\hat{E}_{t-1} \mathbf{p}_t = \mathbf{g} \mathbf{p}_{t-1} + (1 - \mathbf{g}) \mathbf{p}^* + \mathbf{p}^b (1 - q) \quad (3.3)$$

It is convenient to define $x_t = \hat{E}_{t-1} \mathbf{p}_t$ as the state variable and $u_t = \mathbf{p}_t$ as the control. We solve this problem by the method of Lagrange multipliers.⁷ Introduce the Lagrange multiplier $\mathbf{m}$, and set to zero the derivatives of the Lagrangean expression:

$$L = E_t \left[\sum_{t=l}^{\infty} \left\{ \frac{d^{t-l}}{2} \left[-a(u_t - u^*)^2 - (u_t - x_t)^2 \right] - d^{t-l+1} \mathbf{m}_{t+1} [x_{t+1} - \mathbf{g} u_t - (1 - \mathbf{g}) u^* - \mathbf{p}^b (1 - q)] \right\} \right] \quad (\text{B.1})$$

In Appendix B, it is shown that the first-order condition for this problem can be written as⁸

$$\mathbf{p}_t = C_1 \hat{E}_{t-1} \mathbf{p}_t + C_2 \quad (3.6)$$

The coefficients $\frac{1}{(1+a) + d\mathbf{g}^2} < C_1 < \frac{d\mathbf{g}^2 + 1}{(1+a) + d\mathbf{g}^2}$ and $0 \leq C_2 (C_1) \leq \mathbf{p}^*$, are given by

$$C_1 = \frac{1}{2} \left\{ \left[\frac{(1+a) + d\mathbf{g}^2}{d\mathbf{g}^2} \right] - \sqrt{\frac{[(1+a)^2 + d\mathbf{g}^2]^2 - 4d\mathbf{g}^2}{d^2 \mathbf{g}^4}} \right\} \quad (\text{B.15})$$

$$C_2 = \frac{d\mathbf{g}[(C_1 - 1)\mathbf{p}^b(1 - q)] + [d\mathbf{g}(C_1 - 1)(1 - \mathbf{g}) + a]\mathbf{p}^*}{(1+a) - d\mathbf{g}[(C_1 - 1)\mathbf{g} + 1]} \quad (\text{B.16})$$

We now turn to a calibrated case to illustrate our results. Table 3.1 summarizes the parameter values used in our calibrated economy.

⁷ For a discussion of the relative merits of the methods of dynamic programming and Lagrange, see Schaling (2001).

⁸ See Bullard and Schaling (2001) and Schaling (2002) for examples of the method of solving for the optimal policy.

Table 3.1 Parameter Configuration¹

Parameter	Controls	Value
p	Probability of remaining in the high inflation state conditional on being in the high inflation state	0.95
q	Probability of remaining in the low inflation state conditional on being in the low inflation state	0.95
g	Persistence in PS inflation expectations	0.9
$p^b = p^b$	Difference in PS inflation beliefs	13
p_0	Initial inflation rate	20
p^*	Central bank's inflation target	2.5
a	Central bank's inflation aversion	0.2
d	Discount factor	0.9
C_1	Optimal degree of monetary accommodation	0.71
C_2	Reduced form (equilibrium) constant	0.48

¹We illustrate our analytical findings using these calibrations.

Using the above parameter values, Figures 3.2 and 3.3 show the discretionary and commitment disinflation policy respectively.

Figure 3.2 Disinflation under Discretion **Figure 3.3 Disinflation under Commitment**

From Figure 3.3 it can be seen that disinflation under commitment is always faster than under discretion.

The reason is that the choice of a particular inflation rate influences the speed at which expectations adjust. In fact, a quicker disinflation policy 'pays for itself' by speeding up the adjustment of expectations. Of course, the central bank takes this fact into account when deciding on its disinflation program. Another way

to think about this, is that central bank credibility - a crucial variable in defining the output loss of the disinflation - here is endogenous. In fact, the central bank's credibility can be increased by the CB by starting off the disinflation by putting its money where its mouth is.

From equations (B.15) and (B.16) it can be seen that the optimal values of the coefficients are nonlinear functions of the central bank's weight on inflation stabilization, the discount rate and the degree of persistence in inflation expectations.

More specific, we now show:⁹

PROPOSITION B.1: The higher a the lower the optimal value of the feedback parameter C_1 .

For the proof, see Appendix B. The argument is as follows. A central bank that is more concerned with inflation will be less concerned with output, and hence will accommodate inflation expectations to a lesser extent. To give a numerical example, for our basic parameter configuration (see Table 3.1), $C_1 \approx 0.71$. If we increase a to 0.5, say, C_1 decreases to 0.55.

We can also derive a result in terms of the central bank's degree of time preference. In Appendix B we verify:

PROPOSITION B.2: If C_1 is smaller than an upper bound $\bar{C}_1$, the higher d the lower the optimal value of the feedback parameter C_1 .

The intuition is that the higher d , the more concerned the central bank is about the future, i.e. the longer is its policy horizon (conversely if this parameter is zero, the central bank only 'lives for today'). Under a live for today policy, the central bank is not interested how monetary accommodation today effects inflation expectations for tomorrow. If it becomes more concerned about the future (higher d) however, it will start paying attention to expected future 'expectations invoices', and accommodate current inflation expectations to a lesser extent, hence the monetary accommodation coefficient C_1 falls. To give a numerical example for our basic parameter configuration (see Table 3.1) and $d = 0.225$, $C_1 \approx 0.81$. If we increase d to 0.9, say, C_1 decreases to 0.71 (see above)

Let us now look how the central bank responds to less faith in its inflation target, as proxied by a higher weight placed on past inflation by private agents in forecasting future inflation. This is the case of more persistence in inflation expectations. It is easy to show:

PROPOSITION B3: If C_1 is smaller than an upper bound $\bar{C}_1$, the higher g the lower the optimal value of the feedback parameter C_1 .

The argument is that the higher g , the better agents 'remember' past inflation rates, and use those to forecast future inflation. If the central bank cares about the future ($d \neq 0$), it will try to offset this 'memory effect' by less monetary accommodation. In this way it lets the lower inflation outcomes influence the *level* of expectations to try to offset the higher *persistence* of those expectations. For example, for our basic

⁹ Additional results (propositions) are presented in Appendix B.

parameter configuration (see Table 3.1) and $\mathbf{g} = 0.2$, $C_1 \approx 0.83$. If we increase $\mathbf{g}$ to 0.9, C_1 decreases to 0.71.

By substituting (3.3) into (3.6) we can derive the solution for the inflation process under commitment

$$\mathbf{p}_t = C_1 \mathbf{g} \mathbf{p}_{t-1} + C_1 [(1 - \mathbf{g}) \mathbf{p}^* + \mathbf{p}^b (1 - q)] + C_2 \quad (3.7)$$

It can be easily seen that the greater the parameter C_1 , the greater the first-order autocorrelation in inflation. Since this parameter is decreasing in the central bank's weight on inflation stabilization a (see proposition B.1), the greater the central bank's weight on inflation stabilization, the smaller the first-order autocorrelation in inflation (a similar result as under discretion). Similarly, according to proposition B.2, the higher the central bank's discount factor $\mathbf{d}$, the lower the optimal value of the parameter C_1 . Thus, if the central bank becomes more concerned about the future (the longer its policy horizon and the higher $\mathbf{d}$), the lower the persistence of inflation.

Finally, from proposition B.3 we know that the feedback parameter C_1 is decreasing in the parameter $\mathbf{g}$.

Therefore, the dependence of the degree of inflation persistence on $\mathbf{g}$ is given by

$\partial(C_1 \mathbf{g}) / \partial \mathbf{g} = (\partial C_1 / \partial \mathbf{g}) \mathbf{g} + C_1$, where the sign is ambiguous.

4 IMPERFECT KNOWLEDGE, FILTERING AND PREDICTION

The case of perfect knowledge can be represented as follows. First, at time $t - 1$ the central bank sets its expectation (forecast) for private sector inflation expectations. Next, also at time $t - 1$ the private sector sets its forecast, x_t , of inflation for period t . Then, the CB sets inflation at time t based on its own forecast, where - importantly - the forecast turns out to be correct. Figure 4.1 summarizes.

Figure 4.1 Perfect Knowledge: Timing of Events

Time $t - 1$		Time t
<u>Stage 1:</u>	<u>Stage 2:</u>	<u>Stage 3:</u>
<ul style="list-style-type: none"> CB forecasts PS inflation expectations; i.e. sets $E_{t-1}[x_t]$. 	<ul style="list-style-type: none"> PS forecasts inflation using equation (3.3), i.e. sets x_t. 	<ul style="list-style-type: none"> CB decides on monetary policy, i.e. sets $\mathbf{p}_t(E_{t-1}[x_t], \mathbf{p}^*) = \mathbf{p}_t(x_t, \mathbf{p}^*)$, on the basis of either discretion or 'commitment'.

Of course, the idea that the CB can forecast x_t without error is hardly realistic. This assumption will now be relaxed.

4.1 The Kalman Filter

Suppose the CB can no longer forecast private agents' inflation expectations x_t without error. Assume that the CB has a forecast $E_{t-1}y_t$ at time $t - 1$ of x_t which it subsequently uses to set the inflation rate $\mathbf{p}_t$ at time t .

More specifically, let y_t be the CB's noisy signal on x_t

$$y_t = x_t + \mathbf{e}_t \quad (4.1)$$

where y_t is the central bank's signal of x_t , and $\mathbf{e}_t$ is its measurement error. The only information available to the CB when it sets policy at time t is its forecast of y which is conditional on past values of y ; i.e. $E[y_t|t-1] = E[y_t|y_{t-n}, n=1,2,\dots]$. Even ex post the CB cannot observe separately the two components of y , x and $\mathbf{e}$. We assume the measurement error is normally distributed with mean zero and variance $\mathbf{S}_e^2$. So, the central bank's signal is unbiased, but not without error. An important limiting case of (4.1) is when $\mathbf{S}_e^2 \rightarrow 0$ and we are back to the previous case of perfect knowledge, i.e. $y_t = x_t$.

To make the problem more tractable we set $\mathbf{p}^b = [\mathbf{g} - (1 - \mathbf{g})\mathbf{p}^*]/1 - q \equiv \mathbf{p}^b$. Then, $\mathbf{p}^b$ is no longer a free parameter (on those occasions the symbol $\mathbf{p}^b$ is used). This assumption has the advantage of reducing the dimension of the state space in the central bank's optimal filtering problem. In this way we avoid what Ljungqvist and Sargent (2000) call the 'curse of dimensionality'.¹⁰

Setting $\mathbf{p}^b = \mathbf{p}^b$ and defining $x_t \equiv E_{t-1} \mathbf{p}_t$, equation (3.3) simplifies to

$$x_t = \mathbf{g}(\mathbf{p}_{t-1} + 1) = \mathbf{g}v_{t-1} \quad \text{where } w_{t-1} \equiv (\mathbf{p}_{t-1} + 1) \quad (3.4)$$

Note that the situation above can be represented as the case where the CB believes that private sector inflation expectations follow the stochastic process

$$y_t = cw_{t-1} + \mathbf{e}_t \quad (4.2)$$

corresponding to the true (actual) law of motion of PS inflation expectations, but that $\mathbf{g}$ is unknown to them (this can be seen by substituting the expression for private sector inflation expectations (3.4) into equation (4.1)). Thus, here we assume that the central bank employs a reduced form of the expectations formation process that is correctly specified.¹¹

So, we assume that equation (4.2) is the *perceived law of motion* of the central bank and that the policymaker attempts to estimate $\mathbf{g}$. Following Evans and Honkapohja (2001), this is our key bounded rationality assumption: we back away from the rational expectations assumption, replacing it with the assumption that, in forecasting private sector inflation expectations, the central bank acts like an econometrician.

The central bank's estimates will be updated over time as more information is collected. Letting c_{t-1} denote its estimate through time $t-1$, the central bank's one-step-ahead forecast at $t-1$, is given by

¹⁰ For the technical details see Appendix D. In addition, we then choose the parameter $\mathbf{p}_0$ in such a way that the inequality $0 < \mathbf{p}^b < \mathbf{p}_0 - \mathbf{p}^*$ (see equation (2.4)) remains satisfied. Note that Wieland (2000a,b) studies the problem of a single decision maker, who attempts to control a linear stochastic process with two unknown parameters. Using the notation of this paper Wieland considers the stochastic process $y_t = \mathbf{a} + \mathbf{b}\mathbf{p}_t + \mathbf{e}_t$. Thus, in his framework there are no lagged dependent variables. As a consequence all dynamics in Wieland (2000a,b) are due to learning.

¹¹ Instead - as pointed out by Orphanides and Williams (2002) - the learner may be uncertain of the correct form and estimate a more general specification, for example, in our case a linear regression with additional lags of expected inflation which nests (4.2).

$$E_{t-1}[y_t] = c_{t-1} w_{t-1} \quad (4.3)$$

Under this assumption we have the following model of the evolution of the economy. Define $z_t \equiv w_{t-1}$, and imagine that we have already calculated the ordinary least squares estimate $c_{t-1} \sum_{i=1}^{T-1} (z'_i)^2 = \sum_{i=1}^{T-1} z'_i y_i$ of $\mathbf{g}$ in the model $(Y_{t-1}; Z'_{t-1} \mathbf{g}, \mathbf{S}_e^2)$, where $Y_{t-1} = [y_1, \dots, y_{t-1}]$ is a vector of $t-1$ scalar observations and $Z'_{t-1} = [z'_1, \dots, z'_{t-1}]$ is a vector compressing $t-1$ successive observations of the explanatory variable.. Given the new information, which is provided by the observations y_t, z_t , we wish to form a revised or updated estimate of $\mathbf{g}$. Using data through period t , the least squares regression parameter for equation (4.2) can be written in recursive form (see Appendix D for details)

$$c_t = c_{t-1} + \mathbf{k}_t (y_t - w_{t-1} c_{t-1}) \quad (4.4)$$

$$p_t = p_{t-1} - \mathbf{k}_t w_{t-1} p_{t-1} \quad (4.5)$$

$$\mathbf{k}_t = p_t w_{t-1} (\mathbf{S}_e^2)^{-1} \quad (\text{D.10})$$

The method by which the revised estimate of $\mathbf{g}$ is obtained may be described as a filtering process, which maps the sequence of prediction errors into a sequence of revisions; and $\mathbf{k}_t = p_t w_{t-1} (\mathbf{S}_e^2)^{-1}$ may be described as the gain of the filter, i.e. the Kalman gain.

Equations (4.4) and (4.5) are known as the updating, or smoothing equations. These updating equations represent the *learning channel*, through which the current policy action $\mathbf{p}_t$ ¹² affects next period's estimate or *beliefs* $\mathbf{q}_{t+1}$, where $\mathbf{q}$ is a 1 x 2 row vector of state variables containing the mean and variance of the estimate, i.e. $\mathbf{q} = [c_t \quad p_t]'$.

5 CENTRAL BANK LEARNING AND OPTIMAL MONETARY POLICY

We now examine how the nature of monetary policy is affected by learning considerations. Under imperfect knowledge the central bank maximizes

$$E_t \sum_{t=t}^{\infty} \mathbf{d}^{t-t} [-a(\mathbf{p}_t - \mathbf{p}^*)^2 - (E_t z_t)^2] \quad (\text{C.1})^{13}$$

subject to

$$E_t z_t = \mathbf{p}_t - E_{t-1} y_t \quad (\text{C.2})$$

and the Kalman filter equations (4.3)-(4.5) and (D.10):

¹² Note that $\mathbf{p}_t = w_t - 1$.

¹³ Our one-period objective function differs from Wieland (2000b, p. 506) who – using the notation of this paper – considers $U(y, \mathbf{p}) = -(y - y^*)^2 - a\mathbf{p}^2$. Thus, in his set-up both the control $\mathbf{p}$ and the signal y , affect the agent's pay-off. For $a = 0$ his objective function coincides with the input-target model that is often used in studies of learning by doing [e.g. Jovanovich and Nyarko (1996) and Foster and Rosenzweig (1995)].

$$E_{t-1}[y_t] = c_{t-1}(1 + \mathbf{p}_{t-1}) \quad (4.3)$$

$$c_t = c_{t-1} + \mathbf{k}_t(y_t - E_{t-1}y_t) \quad (4.4')^{14}$$

$$p_t = p_{t-1} - \mathbf{k}_t(1 + \mathbf{p}_{t-1})p_{t-1} \quad (4.5)$$

$$\mathbf{k}_t = \frac{1}{1 + \mathbf{p}_{t-1}} \left[1 + \mathbf{s}_e^2 (p_{t-1}(1 + \mathbf{p}_{t-1})^2)^{-1} \right]^{-1} = \frac{1}{1 + \mathbf{p}_{t-1}} [1 + \mathbf{s}]^{-1} \quad (\mathbf{D.10})$$

Now the timing of events is as follows. First, at time $t - 1$ the central bank sets its expectation (forecast) for private sector inflation expectations according to equation (4.3). Here the ordinary least squares estimate c_{t-1} of $\mathbf{g}$ in the model $(y_{t-1}; w_{t-2}, \mathbf{g}, \mathbf{s}_e^2)$ has been calculated on the basis of (4.4) and (4.5); that is using values for $c_{t-2}, \mathbf{k}_{t-1}, y_{t-1}, w_{t-2}$ (where I have used w_t as shorthand for $(1 + \mathbf{p}_{t-1})$). Next, also at time $t - 1$ private sector inflation expectations for period t , x_t , are determined by w_{t-1} and the true parameter $\mathbf{g}$ according to equation (3.4).

Then, the CB sets inflation at time t based on its own forecast. Also, in period t given the new information provided by y_t, w_{t-1} , the central bank forms a revised or updated estimate of $\mathbf{g}$.

Note that the Kalman gain ($\mathbf{D.10}$), is a nonlinear function of the central bank's control variable. Hence, the updating equations (4.4) and (4.5) are also nonlinear in the inflation rate. These updating equations represent the *learning channel*, through which the current policy choice $\mathbf{p}_t$ affects next period's parameter estimate c_{t+1} and the associated prediction $E_t y_{t+1}$.

To obtain some intuition, we now take a closer look at the term $\mathbf{s} \equiv \mathbf{s}_e^2 [p_{t-1}(1 + \mathbf{p}_{t-1})^2]^{-1}$ in the Kalman gain. Here $p_{t-1}(1 + \mathbf{p}_{t-1})^2$ is the portion of the prediction error variance due to uncertainty in c_{t-1} and $\mathbf{s}_e^2$ is the portion of the prediction error variance due to the random shock $\mathbf{e}_t$. Thus, it is nothing else than the inverse of the *signal to noise ratio*. We can easily see that $|\partial \mathbf{k}_t / \partial \mathbf{s}| < 0$, suggesting that as the amount of noise in the signal y_t increases, relatively less weight is given to new information in the prediction error, $(y_t - E_{t-1}y_t)$. This is quite intuitive, since an increase in the noise may be interpreted as a deterioration of the information content of $(y_t - E_{t-1}y_t)$ relative to c_{t-1} . Similarly, if the amount of noise that is contaminating the signal diminishes, more weight will be given to new information relative to the previous estimate c_{t-1} .

The model has two important limiting cases. One limiting case is the one where the noise to signal ratio goes to infinity. Then, new observations are so noisy that they are essentially useless, and play no role in updating the previous estimate. This is the case where the Kalman gain goes to zero, or

$$\lim_{\mathbf{s} \rightarrow \infty} \frac{1}{1 + \mathbf{p}_{t-1}} \left[\frac{1}{1 + \mathbf{s}} \right] = \lim_{\mathbf{s} \rightarrow \infty} \mathbf{k}_t = 0$$

¹⁴ Where I have substituted $E_{t-1}y_t$ for $c_{t-1}(1 + \mathbf{p}_{t-1})$ using equation (4.3).

Substituting this expression into the updating equation (4.4) gives $c_t = c_{t-1} = c_0$. Hence, this is the case where the central bank engages in forecasting or *prediction*,

$$E_{t-1}[y_t] = c_0(1 + \mathbf{p}_{t-1}) \quad (4.3)$$

but not in updating. Note that in this case the *separation principle* holds, as the central bank's optimal estimation of the state $\mathbf{g}$, no longer depends on policy outcomes. Note that the model can then be solved for either discretion or commitment.

The mirror image of the previous situation is the case where the new observations are not polluted by any noise. Then, the central bank should just set policy based on its most recent observation y_t . In this case the relevant limit of the Kalman gain is given by

$$\lim_{s \rightarrow 0} \frac{1}{1 + \mathbf{p}_{t-1}} \left[\frac{1}{1 + \mathbf{s}} \right] = \lim_{s \rightarrow 0} \mathbf{k}_t = \frac{1}{1 + \mathbf{p}_{t-1}}$$

Substituting this expression into the updating equation (4.4) gives

$$c_t = \frac{y_t}{1 + \mathbf{p}_{t-1}} = \frac{x_t}{1 + \mathbf{p}_{t-1}} = \frac{\mathbf{g}(1 + \mathbf{p}_{t-1})}{1 + \mathbf{p}_{t-1}} = \mathbf{g} \quad (5.1)$$

In this case at time $t - 1$ the central bank would be able to forecast private sector inflation expectations perfectly, and then at time t set policy based on its forecast. So, this case is nothing else but the case of perfect knowledge analyzed in section 3. Note that in this case - unlike Wieland (2000b) - the optimization problem is still *dynamic*, the optimal policy would be the commitment policy of section 3.2. Only in the case where $\mathbf{s} = \mathbf{d} = 0$ does the optimization problem reduce to the static discretionary problem of section 3.1.

5.1 Optimal Learning and the Value of Experimentation

We now turn to the case where estimation and control are not separated. Of course, estimation and control cannot be separated because parameter updates and forecasts depend on past monetary policy choices. The effect of policy on future estimates and forecasts is also apparent from the Bellman equation associated with this dynamic programming problem:

$$V_t(x'_t) = \max_{\mathbf{p}_t} \left\{ r(\mathbf{p}_t, x'_t) + \mathbf{d} E_t V_{t+1}(x'_t) \right\} \quad (5.2)$$

Where the vector of state variables is $x'_t = (E_t y_{t+1}, c_t, \mathbf{k}_t, y_t, p_t)$, $\mathbf{p}$ is the control and r is the one-period return function.

Following Wieland (2000b), it can easily be seen that the two terms on the right hand side characterize the tradeoff between current control and estimation (which here is used as an umbrella term to include prediction). The first term is current expected reward, while the second term is the expected continuation value in the next period, which reflects the expected improvement in future payoffs due to better information

about the unknown parameter.

Note that if $\mathbf{d} \rightarrow 0$ the central bank only ‘lives for today’, and is not interested to consider the effects of its policy actions on future payoffs. Then there is no horse race, and the optimal policy is simply to maximize the one-period return function. In that case the optimal policy is simply the discretionary solution, which under imperfect knowledge is presented in section 5.3.1 below.

As shown by Easley and Kiefer (1988) and Kiefer and Nyarko (1989) an optimal feedback rule exists and the value function is continuous and satisfies the Bellman equation.¹⁵ Policy and value functions can be obtained using an iterative algorithm based on the Bellman equation starting with an initial guess about $V(\cdot)$.¹⁶ However, analytical solutions are not feasible because the dynamic constraint of the optimization problem associated with the Kalman filter is highly nonlinear. As pointed out by Wieland (2000b), there are many examples, including Wieland (2000a,b), Ellison and Valla (2001) for which no analytical solutions have been found even though the unknown stochastic process is linear and the return function is quadratic.

5.2 The Case of Passive Learning

In order to get some analytical results, we now consider the case of *passive learning*. This is the case where the central bank disregards the effect of current policy actions on future estimation and prediction. In this case the policy maker treats control and estimation separately. As pointed out by Bertocchi and Spagat (1993) in this case learning is *passive* in the sense that there is no experimentation.

The central bank will first choose $\mathbf{p}_t$ to minimise the expected loss based on its current parameter estimate (its belief about $\mathbf{g}$). Then, a white noise shock $\mathbf{e}_t$ occurs and a new realization y_t can be observed. Before choosing next period’s control $\mathbf{p}_{t+1}$ the central bank will proceed by updating its estimate (belief) using the new information $(\mathbf{p}_t, y_t)$. Figure 5.1 summarizes.

¹⁵ As pointed out by Wieland (2000b), one can use standard dynamic programming methods and show that Blackwell’s sufficiency condition – monotonicity and discounting – are satisfied. Thus, equation (5.2) has a fixed point in the space of continuous functions, which is the value function $V(x')$.

¹⁶ A typical algorithm is described in Ljungqvist and Sargent (2000), Chapter 3.

Figure 5.1 Passive Learning: Timing of Events

Time $t - 1$		Time t	
<u>Stage 1:</u>	<u>Stage 2:</u>	<u>Stage 3:</u>	<u>Stage 4:</u>
<ul style="list-style-type: none"> CB forecasts PS inflation expectations using c_{t-1} and w_{t-1}, i.e. sets $E_{t-1}[x_t] = E_{t-1}[y_t]$. 	<ul style="list-style-type: none"> PS forecasts inflation using w_{t-1} and $\mathbf{g}$, i.e. sets x_t. 	<ul style="list-style-type: none"> 3a) CB decides on monetary policy, i.e. sets $\mathbf{p}_t(E_{t-1}[y_t], \mathbf{p}^*)$, on the basis of either discretion or ‘commitment’ 3b) Nature chooses $\mathbf{e}_t$, and $y_t = x_t + \mathbf{e}_t$ realizes. 3c) CB observes y_t and forms a revised estimate c_t. 	<ul style="list-style-type: none"> Back to stage 1, for time $t = t + 1$ etc.

As pointed out by Wieland (2000b) this behaviour is myopic since it disregards the effect of current policy actions on future predictions and estimates.

5.2.1 Discretion

As before first we consider the case where the central bank does not internalize its ‘teaching by doing’, that is the case where $\mathbf{d} \rightarrow 0$. In this discretionary case the central bank will choose $\mathbf{p}_t$ to minimise the expected one-period loss

$$E_t \frac{1}{2} [-a(\mathbf{p}_t - \mathbf{p}^*)^2 - (E_t z_t)^2] \quad (5.3)$$

The first order condition of this problem is

$$\mathbf{p}_t = \frac{1}{1+a} E_{t-1} y_t + \frac{a}{1+a} \mathbf{p}^* \quad (5.4)$$

So, we find that the optimal policy rule is the one where the central bank should partially accommodate its *forecast* of private sector inflation expectations. Using equation (4.3) in the expression above, this policy can also be expressed in terms of a response to the determinants of this forecast, namely past inflation and the central bank’s initial estimate of the degree of persistence of inflation expectations.

$$\mathbf{p}_t = \frac{c_{t-1}}{1+a} (\mathbf{p}_{t-1} + 1) + \frac{a}{1+a} \mathbf{p}^* \quad (5.5)$$

Compared to the decision rule in (3.5.2) the parameter $\mathbf{g}$ is replaced by its point estimate c_{t-1} .

A novel feature of the passive learning policy - compared to the central bank's discretionary disinflation policy under perfect knowledge - is that the degree of monetary accommodation (the extent to which the central bank accommodates private sector inflation expectations) is no longer constant across the disinflation, but becomes state-dependent. This means that the central bank's behaviour changes during the disinflation as it collects more information. This can be easily seen from equation (5.5) where the monetary accommodation coefficient is now time-varying.

5.2.2 Commitment

Now we turn to the case where the central bank internalizes the effects of today's monetary accommodation on tomorrow's inflation expectations. To solve this problem, first we derive the central bank's policy rule $\mathbf{p}_t(E_{t-1}y_t, \mathbf{p}^*)$, which selects an action based on the current state $E_{t-1}y_t$. Given a specification of the central bank's forecast, given by the Kalman filter equations this policy rule can be derived analytically. Formally, the central bank's control problem is now to maximize

$$E_t \sum_{t=t}^{\infty} \mathbf{d}^{t-t} \left[-a(\mathbf{p}_t - \mathbf{p}^*)^2 - (E_t z_t)^2 \right] \quad (\text{C.1})$$

subject to

$$E_t z_t = \mathbf{p}_t - E_{t-1} y_t \quad (\text{C.2})$$

$$E_{t-1}[y_t] = c_{t-1}(1 + \mathbf{p}_{t-1}) \quad (\text{4.3})$$

It is convenient to define $x_t = E_{t-1}[y_t]$ as the state variable and $u_t = \mathbf{p}_t$ as the control. We solve this problem by the method of Lagrange multipliers. Introduce the Lagrange multiplier $\mathbf{m}_t$, and set to zero the derivatives of the Lagrangean expression:

$$L = E_t \left[\sum_{t=t}^{\infty} \left\{ \frac{\mathbf{d}^{t-t}}{2} \left[-a(u_t - u^*)^2 - (u_t - x_t)^2 \right] - \mathbf{d}^{t-t+1} \mathbf{m}_{t+1} [x_{t+1} - c_t(1 + u_t)] \right\} \right] \quad (\text{C.3})$$

Following a line of reasoning similar to the corresponding case of perfect knowledge (for details see Appendix C.1), it can easily be shown that the first-order condition can be written as

$$\begin{aligned} \mathbf{p}_t &= C_{1,t-1} E_{t-1}[y_t] + C_{2,t-1} \\ &= C_{1,t-1} c_{t-1} (1 + \mathbf{p}_{t-1}) + C_{2,t-1} \end{aligned} \quad (\text{5.6})$$

Note that the optimal disinflation policy under passive learning is identical to the commitment policy under perfect knowledge, except that the unknown structural persistence parameter $\mathbf{g}$ has been replaced by its point estimate c_{t-1} .

The coefficients $\frac{1}{(1+a)+\mathbf{d}_{t-1}^2} < C_{1,t-1} < \frac{\mathbf{d}_{t-1}^2+1}{(1+a)+\mathbf{d}_{t-1}^2}$ and $0 \leq C_{2,t-1}(C_{1,t-1}) \leq \mathbf{p}^*$, are given by

$$C_1 = \frac{1}{2} \left\{ \left[\frac{(1+a)+\mathbf{d}_{t-1}^2}{\mathbf{d}_{t-1}^2} \right] - \sqrt{\frac{[(1+a)^2+\mathbf{d}_{t-1}^2]^2 - 4\mathbf{d}_{t-1}^2}{\mathbf{d}_{t-1}^4}} \right\} \quad (\text{C.13})$$

$$C_2 = \frac{\mathbf{d}_{t-1}^2[(C_1-1)] + a\mathbf{p}^*}{(1+a)-\mathbf{d}_{t-1}[(C_1-1)c_{t-1}+1]} \quad (\text{C.14})$$

Again, a novel feature of the passive learning policy - compared to the central bank's optimal disinflation policy under perfect knowledge - is that the degree of monetary accommodation (the extent to which the central bank accommodates private sector inflation expectations) is no longer constant across the disinflation, but becomes state-dependent. This means that the central bank's behaviour changes during the disinflation as it collects more information. This can be easily seen from equation (5.6) where the coefficients are now time-varying.

6 CONCLUSIONS AND SUGGESTIONS FOR FURTHER RESEARCH

In this paper we have analyze disinflation in two environments. One in which the central bank has *perfect knowledge*, in the sense that it understands and observes the process by which private sector inflation expectations are generated, and one in which the central bank has to *learn* the private sector inflation forecasting rule.

For the case of perfect knowledge we found that the optimal disinflation is faster under commitment than discretion. Next, in the commitment case the disinflation is less gradual, the higher the central bank's rate of time preference and the higher the degree of persistence in inflation expectations.

With imperfect knowledge results depend on the learning scheme that is employed. A novel feature of the passive learning policy - compared to the central bank's optimal disinflation policy under perfect knowledge - is that the degree of monetary accommodation (the extent to which the central bank accommodates private sector inflation expectations) is no longer constant across the disinflation, but becomes state-dependent. This means that the central bank's behaviour changes during the disinflation as it collects more information.

There are a number of ways the paper can be extended. One limitation of the present analysis is that there is no rational learning of private agents about the monetary policy regime. It would be more plausible if agents also update their beliefs about the evolution of inflation following observations about actual monetary policy choices.¹⁷

An example of a paper that looks at the case where the private sector is learning about central bank behavior is Andolfatto, Hendry and Moran (2002). Using a standard monetary dynamic stochastic general equilibrium model, they embed a learning mechanism regarding the interest-rate-targeting rule that the monetary authorities follow. There the learning mechanism enables optimizing economic agents to distinguish

¹⁷ This is modelled by Hoerichts and Schaling (2000), using Bayesian learning.

between transitory shocks to the policy rule and occasional shifts in the inflation target of the monetary policy authorities. We see this as one potential avenue for further work.

REFERENCES

- Andolfatto, D., Hendry, S. and K. Moran (2002).** 'Inflation Expectations and Learning about Monetary Policy', Bank of Canada Working Paper 2002-30, October.
- Bertocchi, G. and Spagat M. (1993).** 'Learning, Experimentation and Monetary Policy', *Journal of Monetary Economics*, 32, pp. 169-183.
- Bullard, James (1991).** 'Learning, Rational Expectations and Policy: A Summary of Recent Research', *Federal Reserve Bank of St. Louis Review*, January/February, pp. 50-60.
- Bullard, James and Eric Schaling (2001).** 'New Economy – New Policy Rules?', *Federal Reserve Bank of St. Louis Review*, Vol. 83, No. 5, September/October, pp. 57-66.
- Easley, D. and Kiefer, N. (1988).** 'Controlling a Stochastic Process with Unknown Parameters', *Econometrica*, 56, 1045-1064.
- Ellison, M. and N. Valla (2001).** 'Learning, Uncertainty and Central Bank Activism in an Economy with Strategic Interactions', *Journal of Monetary Economics*, Vol. 48, Issue 1, August, pp. 153-171.
- Evans, George W. and Seppo Honkapohja (2001).** *Learning and Expectations in Macroeconomics*, Princeton University Press.
- Foster, A. and Rosenzweig, M. (1995).** 'Learning by Doing and Learning from Others: Human Capital and Technical Change in Agriculture', *Journal of Political Economy*, 103, pp. 1176-1209.
- Hamilton, James (1989).** 'A New Approach to the Economic Analysis of Nonstationary Time Series and the Business Cycle', *Econometrica*, Vol. 57, No. 2, March, pp. 357-384.
- Hoeberichts, Marco and Eric Schaling. (2000).** 'Why Speed Doesn't Kill: Learning to Believe in Disinflation', Research Paper Department of Economics RAU, No. 0009, October.
- Jovanovich, B. and Nyarko, Y. (1996).** 'Learning by Doing and the Choice of Technology', *Econometrica*, 64, pp. 1299-1310.
- Kiefer, N. and Nyarko, Y. (1989).** 'Optimal Control of an Unknown Linear Process with Learning', *International Economic Review*, 30, 571-586.
- King, Mervyn (1996),** 'How Should Central Banks Reduce Inflation? - Conceptual Issues', Economic Review Federal Reserve Bank of Kansas City, Vol. 81, No 4, Fourth Quarter, pp. 25-52.
- Lucas, Robert E. Jr. (1987).** 'Adaptive Behaviour and Economic Theory', in Robin M. Hogarth and Melvin W. Reder, eds, *Rational Choice: The Contrast Between Economics and Psychology*, University of Chicago Press.
- Ljungqvist, L. and T. Sargent (2000).** *Recursive Macroeconomic Theory* MIT Press.

Marcet, Albert and Thomas Sargent (1988), 'The Fate of Systems with Adaptive Expectations', *American Economic Review*, pp. 168-71.

Marcet, Albert and Thomas Sargent (1989a), 'Convergence of Least-Squares Learning Mechanisms in Self Referential, Linear Stochastic Models', *Journal of Economic Theory*, pp. 337-68.

Marcet, Albert and Thomas Sargent (1989b), 'Least-Squares Learning and the Dynamics of Hyper Inflation', in William A. Barnett, John Geweke, and Karl Shell, eds, *Economic Complexity: Chaos, Sunspots, Bubbles, and Nonlinearity*, Cambridge University Press, pp. 119-40.

Marcet, Albert and Thomas Sargent (1989c), 'Convergence of Least Squares Learning in Environments with Hidden State Variables and Private Information', *Journal of Political Economy*, pp. 1306-22.

Orphanides, A. and John Williams (2002), 'Imperfect Knowledge, Inflation Expectations, and Monetary Policy', Federal Reserve Board FEDS Papers, May.

Pollock, D.S.G (1999), 'A Handbook of Time-Series Analysis, Signal Processing and Dynamics', Academic Press.

Sargent, Thomas (1999). '*The Conquest of American Inflation*', Princeton University Press.

Schaling, Eric (2001). 'Dynamic Programming versus Lagrange: Is there a Horse Race in Dynamic Optimization ?', in Klok, H., T. van Schaik and S. Smulders (eds), *Economologues – Liber Amicorum for Theo van de Klundert*, Tilburg University Press, March, pp. 334-349.

Schaling, Eric (2002). 'The Non-Linear Phillips Curve and Inflation Forecast Targeting: Symmetric vs. Asymmetric Monetary Policy Rules', *Journal of Money, Credit and Banking*, Forthcoming.

Wieland, Volker (2000a). 'Monetary Policy, Parameter Uncertainty and Optimal Learning' , *Journal of Monetary Economics*, pp. 199-228.

Wieland, Volker (2000b). 'Learning by Doing and the Value of Optimal Experimentation', *Journal of Economic Dynamics and Control*, 24, pp. 501-534.

APPENDIX A: STEADY STATE EQUILIBRIUM

The innovation sequence $\{V_t\}$ in equation (2.6) satisfies

$$\begin{aligned} \Pr ob[V_{t+1} = (1-p)|S_t = 1] &= p, \\ \Pr ob[V_{t+1} = -p|S_t = 1] &= 1-p, \\ \Pr ob[V_{t+1} = -(1-q)|S_t = 0] &= q, \\ \Pr ob[V_{t+1} = q|S_t = 0] &= 1-q \end{aligned} \tag{A.1}$$

with $\hat{E}_t V_{t+1} = 0$ and $\mathbf{s}_v^2 = \hat{E}(V_t^2) = p(1-p)\bar{p} + q(1-q)(1-\bar{p})$.

(where I have used that $\bar{p} \equiv (1-q)/(1-p+1-q)$)¹⁸

From (A.1) we see that $\hat{E}_0 V_t = 0$ for all $t > 0$. Using this fact, and iterating (2.6) into the future, we can write

$$\hat{E}_0 S_t = \mathbf{g}' \hat{E}_0 S_0 + \frac{(1-q)(1-\mathbf{g}')}{(1-\mathbf{g})} \tag{A.2}$$

where $\hat{E}_0$ denotes the private sector expectation conditional on information available at date zero (which need not include observation of s_0). Observing that $\hat{E}_0 S_t$ can be interpreted as the probability that $S_t = 1$ given information at time zero (denoted $P_0[S_t = 1]$), (A.2) can be rewritten

$$P_0[S_t = 1] = \bar{p} + \mathbf{g}'(\bar{p}_0 - \bar{p}) \tag{A.3}$$

where $p_0 \equiv P_0[S_0 = 1]$.

From equation (A.3) we can see that for large t the economy is expected to be in the high inflation state (state 1) with probability $\bar{p}$, in which case u would be $\mathbf{p}^b$. Similarly, the economy will be in the low inflation state (state 0) with probability $1 - \bar{p}$, in which case u would be zero. Hence, the *expected* long-run level of u (denoted as $\bar{u}$) is

¹⁸ For more details see Hamilton (1989, pp. 360-363).

$$\bar{u} = \bar{p} \cdot \mathbf{p}^b \quad (\text{A.4})$$

From equation (2.3) it then follows that the (unconditional mean) steady state level of inflation ($\bar{p}$), is

$$\bar{p} = \mathbf{p}^* + \bar{p} \cdot \mathbf{p}^b \quad (\text{A.5})$$

APPENDIX B THE COMMITMENT SOLUTION

Now, the central bank's problem is to choose $\{\mathbf{p}_t\}_{t=t}^{\infty}$ so as to maximize

$$E_t \sum_{t=t}^{\infty} \mathbf{d}^{t-t} \left[-a(\mathbf{p}_t - \mathbf{p}^*)^2 - (z_t - z^*)^2 \right] \quad (\text{2.1})$$

subject to

$$z_t = \mathbf{p}_t - \hat{E}_{t-1} \mathbf{p}_t \quad (\text{2.2})$$

and

$$\hat{E}_{t-1} \mathbf{p}_t = \mathbf{g} \mathbf{p}_{t-1} + (1 - \mathbf{g}) \mathbf{p}^* + \mathbf{p}^b \cdot (1 - q) \quad (\text{3.3})$$

It is convenient to define $x_t = \hat{E}_{t-1} \mathbf{p}_t$ as the state variable and $u_t = \mathbf{p}_t$ as the control. We solve this problem by the method of Lagrange multiplier. Introduce the Lagrange multiplier $\mathbf{m}$, and set to zero the derivatives of the Lagrangean expression:

$$L = E_t \left[\sum_{t=t}^{\infty} \left\{ \frac{\mathbf{d}^{t-t}}{2} \left[-a(u_t - u^*)^2 - (u_t - x_t)^2 \right] - \mathbf{d}^{t-t+1} \mathbf{m}_{t+1} [x_{t+1} - \mathbf{g} u_t - (1 - \mathbf{g}) u^* - \mathbf{p}^b (1 - q)] \right\} \right] \quad (\text{B.1})^{19}$$

The central bank's first-order conditions take the form

$$-a(u_t - u^*) - (u_t - x_t) + \mathbf{d} \mathbf{g} E_t \mathbf{m}_{t+1} = 0 \quad (\text{B.4})$$

$$\mathbf{m} = (u_t - x_t) \quad (\text{B.5})$$

First, we find an expression for $E_t \mathbf{m}_{t+1}$. Leading (B.5) by one period and taking expectations we get:

$$E_t \mathbf{m}_{t+1} = (E_t u_{t+1} - E_t x_{t+1}) \quad (\text{B.6})$$

¹⁹ It is easy to convert the Lagrangean (B.1) into the standard form used by Schaling (2002) by setting $t = 0$ in (B.1). Then the central bank chooses the sequence $\{\mathbf{p}_t\}_{t=0}^{\infty}$ rather than $\{\mathbf{p}_t\}_{t=t}^{\infty}$.

Substituting (B.6) into (B.4), we can derive the Euler equation

$$-a(u_t - u^*) - (u_t - x_t) + \mathbf{d}\mathbf{g}(E_t u_{t+1} - E_t x_{t+1}) = 0 \quad (\text{B.7})$$

In the case of a policy of strict inflation reduction, the rule would be

$$u_t = u^* \quad (\text{B.8})$$

Similarly, in the case of full accommodation of expectations, the rule would be

$$u_t = x_t \quad (\text{B.9})$$

Thus, it appears that in case of flexible inflation targeting the rule will be a linear combination of (B.8) and (B.9), that is $u_t = cx_t + (1-c)u^*$, where $0 \leq c \leq 1$. Or alternatively,

$$u_t = C_1 x_t + C_2 \quad (\text{B.10})$$

which is equation (3.6) in the main text (where I have substituted $x_t = \hat{E}_{t-1} \mathbf{p}_t$ and $u_t = \mathbf{p}_t$).

Here the coefficients C_1 and C_2 remain to be determined, and the prior is that $0 \leq C_1 \leq 1$ and $0 \leq C_2(C_1) \leq u^*$. Now we identify the coefficients C_1 and C_2 .

Expectations for the state at period $t+1$ follow from the constraint in (B.1), combining the latter with the decision rule for u , we can write:

$$E_t x_{t+1} = \mathbf{g}C_1 x_t + \mathbf{g}C_2 + (1-\mathbf{g})u^* + \mathbf{p}^b(1-q) \quad (\text{B.11})$$

From (B.10) it follows that

$$E_t u_{t+1} = C_1 E_t x_{t+1} + C_2 = C_1 [\mathbf{g}C_1 x_t + \mathbf{g}C_2 + (1-\mathbf{g})u^* + \mathbf{p}^b(1-q)] + C_2 \quad (\text{B.12})$$

Substituting (B.11) and (B.12) into the Euler equation (B.7) above, and equating constant terms and coefficients on the state variables yields the following expressions for C_1 and C_2 in terms of the structural parameters of the model

$$C_1 = \frac{1}{(1+a) + \mathbf{d}\mathbf{g}^2} [\mathbf{d}\mathbf{g}^2 C_1^2 + 1] \quad (\text{B.13})$$

$$C_2 = \frac{\mathbf{d}\mathbf{g}[(C_1-1)\mathbf{g}+1]}{1+a} C_2 + \frac{\mathbf{d}\mathbf{g}\mathbf{p}^b(1-q) + (1-\mathbf{g})u^* [(C_1-1) + au^*]}{1+a} \quad (\text{B.14})$$

Equation (B.13) implicitly defines the value of C_1 . It can be written as $C_1 = F(C_1)$. Note that the function $F(C_1)$ on the right hand side with domain $\langle 0,1 \rangle$ is monotonically increasing in C_1 , that

$\lim_{C_1 \rightarrow 0} F(C_1) = \frac{1}{(1+a)+dg^2}$, $\lim_{C_1 \rightarrow 1} F(C_1) = \frac{dg^2+1}{(1+a)+dg^2}$. We realize that there is a unique positive

solution C_1 , which fulfills $\frac{1}{(1+a)+dg^2} < C_1 < \frac{dg^2+1}{(1+a)+dg^2}$. It can be solved analytically:

$$C_1 = \frac{1}{2} \left\{ \left[\frac{(1+a)+dg^2}{dg^2} \right] - \sqrt{\frac{[(1+a)^2+dg^2]^2-4dg^2}{d^2g^4}} \right\} \quad (\text{B.15})$$

Similarly, Equation (B.14) implicitly defines the value of C_2 . It can be written as $C_2 = G(C_2)$. Note that the function $G(C_2)$ on the right hand side with domain $\langle 0, u^* \rangle$ is monotonically increasing in C_2 . Again, we realize that there is a unique positive solution C_2^* . It can be solved analytically:

$$C_2 = \frac{dg[(C_1-1)p^b(1-q)] + [dg(C_1-1)(1-g)+a]u^*}{(1+a)-dg[(C_1-1)g+1]} \quad (\text{B.16})$$

Moreover, it can be easily established that $\lim_{a \rightarrow 0} C_2(C_1) = 0$ and that $\lim_{a \rightarrow \infty} C_2 = u^*$.

We are now ready to prove:

PROPOSITION B.1: The higher a the lower the optimal value of the feedback parameter C_1 .

Proof: $\frac{\partial F}{\partial a} = -\frac{(dg^2C_1^2+1)}{[(1+a)+dg^2]^2} < 0$, this implies that when a goes up, the function $F(C_1)$ shifts downward.

As a consequence, the equilibrium value of C_1 decreases.

PROPOSITION B.2: If C_1 is smaller than an upper bound $\bar{C}_1$, the higher d the lower the optimal value of the feedback parameter C_1 .

Proof: $\frac{\partial F}{\partial d} = \frac{g^2[C_1^2(1+a)-1]}{[(1+a)+dg^2]^2}$. Note that the nominator of this expression is negative if the above condition

is satisfied. It can be written as $C_1 < \bar{C}_1$, where $\bar{C}_1 = \sqrt{\frac{1}{1+a}}$.

Numerical results indicate that for our basic parameter configuration (see Table 3.1), the above condition is satisfied for the entire range of inflation aversion preferences ($0 < a < \infty$).

PROPOSITION B3: If C_1 is smaller than an upper bound $\bar{C}_1$, the higher g the lower the optimal value of the feedback parameter C_1 .

Proof: $\frac{\partial F}{\partial \mathbf{g}} = \frac{2\mathbf{d}\mathbf{g}[C_1^2(1+a)-1]}{[(1+a)+\mathbf{d}\mathbf{g}^2]^2}$. Note that the nominator of this expression is negative if the above condition is satisfied. For more details see Proposition B.2 above.

Numerical results indicate that for our basic parameter configuration (see Table 3.1), the above condition is satisfied for the entire range of inflation aversion preferences ($0 < a < \infty$).

PROPOSITION B4: If $a > -\mathbf{d}\mathbf{g}(C_1 - 1)(1 - \mathbf{g})$, the higher u^* the higher the value of the constant C_2 .

Proof: $\frac{\partial C_2}{\partial u^*} = \frac{[\mathbf{d}\mathbf{g}(C_1 - 1)(1 - \mathbf{g}) + a]}{(1 + a) - \mathbf{d}\mathbf{g}[(C_1 - 1)\mathbf{g} + 1]}$. The nominator of this expression is positive if the above condition is satisfied.

PROPOSITION B5: The higher a the higher the value of the constant C_2 .

Proof: $\frac{\partial C_2}{\partial a} = \frac{(1 - \mathbf{d}\mathbf{g}C_1)u^* - \mathbf{d}\mathbf{g}[(C_1 - 1)\mathbf{p}^b(1 - q)]}{\{(1 + a) - \mathbf{d}\mathbf{g}[(C_1 - 1)\mathbf{g} + 1]\}^2} > 0$.

To give a numerical example, for our basic parameter configuration (see Table 3.1), $C_2 \approx 0.48$. If we increase a to 0.5, say, C_2 increases to 0.90.

PROPOSITION B6: The higher $\mathbf{p}^b$ the lower the value of the constant C_2 .

Proof: $\frac{\partial C_2}{\partial \mathbf{p}^b} = \frac{\mathbf{d}\mathbf{g}(C_1 - 1)(1 - q)}{(1 + a) - \mathbf{d}\mathbf{g}[(C_1 - 1)\mathbf{g} + 1]} < 0$.

PROPOSITION B7: If C_2 is greater than a lower bound $\underline{C_2}$, the higher $\mathbf{d}$ the higher the value of the constant C_2 .

Proof: $\frac{\partial G}{\partial \mathbf{d}} = \frac{\mathbf{g}[\mathbf{p}^b(1 - q) + (1 - \mathbf{g})u^* + \mathbf{g}C_2] + [(C_1 - 1) + \mathbf{d}(\partial C_1 / \partial \mathbf{d})] + C_2}{1 + a}$. The nominator of this

expression is positive if the above condition is satisfied. It can be written as $C_2 > \underline{C_2}$, where

$\underline{C_2} = \frac{-[\mathbf{p}^b(1 - q) + (1 - \mathbf{g})u^*] + [(C_1 - 1) + \mathbf{d}(\partial C_1 / \partial \mathbf{d})]}{\{\mathbf{g}[(C_1 - 1) + \mathbf{d}(\partial C_1 / \partial \mathbf{d})] + 1\}}$. For plausible parameter values this condition is

likely to be satisfied.

To give a numerical example for our basic parameter configuration (see Table 3.1) and $\mathbf{d} = 0.225$, $C_2 \approx 0.45$. If we increase $\mathbf{d}$ to 0.9, say, C_2 increases to 0.48.

PROPOSITION B8: If C_2 is greater than a lower bound $\underline{C_2}$, the higher $\mathbf{g}$ the higher the value of the constant C_2 .

Proof: $\frac{\partial G}{\partial \mathbf{g}} = \frac{\mathbf{d}[\mathbf{g}C_2 + \mathbf{p}^b(1-q) + u^*(1-\mathbf{g})]\{[(C_1-1) + \mathbf{g}(\partial C_1 / \partial \mathbf{g})] + \mathbf{d}\mathbf{g}(C_1-1)[C_2 - u^*] + \mathbf{d}C_2\}}{1+a}$. The nominator of this expression is positive if the above condition is satisfied. It can be written as $C_2 > \underline{C_2}$, where $\underline{C_2} = \frac{-\{[(C_1-1) + \mathbf{g}(\partial C_1 / \partial \mathbf{g})]\mathbf{d}[\mathbf{p}^b(1-q) + u^*(1-\mathbf{g})] + \mathbf{d}\mathbf{g}^*(C_1-1)\}}{\mathbf{d}\{2\mathbf{g}(C_1-1) + \mathbf{g}^2(\partial C_1 / \partial \mathbf{g}) + 1\}}$. For plausible parameter values this condition is likely to be satisfied.

For example, for our basic parameter configuration (see Table 3.1) and $\mathbf{g} = 0.2$, $C_2 \approx 0.27$. If we increase $\mathbf{g}$ to 0.9, C_2 increases to 0.48.

APPENDIX C OPTIMAL LEARNING AND THE VALUE OF EXPERIMENTATION

C.1 PASSIVE LEARNING

In the case of passive learning the central bank ignores the dependence of the current policy action $\mathbf{p}_t$ on next period's estimate or *beliefs* $\mathbf{q}_{t+1}$, where $\mathbf{q}$ is a 1 x 2 row vector of state variables containing the mean and variance of the estimate, i.e. $\mathbf{q}_t = [c_t \quad p_t]'$. That is, the central bank ignores the constraints (4.4), (4.5) and (D.10).

Then, the central bank's problem is to choose $\{\mathbf{p}_t\}_{t=t}^{\infty}$ so as to maximize

$$E_t \sum_{t=t}^{\infty} \mathbf{d}^{t-t} \left[-a(\mathbf{p}_t - \mathbf{p}^*)^2 - (E_t z_t)^2 \right] \quad (\text{C.1})$$

subject to

$$E_t z_t = \mathbf{p}_t - E_{t-1} y_t \quad (\text{C.2})$$

and

$$E_{t-1}[y_t] = c_{t-1}(1 + \mathbf{p}_{t-1}) \quad (\text{4.3})$$

It is convenient to define $x_t = E_{t-1} y_t$ as the state variable and $u_t = \mathbf{p}_t$ as the control. We solve this problem by the method of Lagrange multipliers. Introduce the Lagrange multiplier $\mathbf{m}$, and set to zero the derivatives of the Lagrangean expression:

$$L = E_t \left[\sum_{t=t}^{\infty} \left\{ \frac{\mathbf{d}^{t-t}}{2} \left[-a(u_t - u^*)^2 - (u_t - x_t)^2 \right] - \mathbf{d}^{t-t+1} \mathbf{m}_{t+1} [x_{t+1} - c_t(1 + u_t)] \right\} \right] \quad (\text{C.3})$$

The central bank's first-order conditions take the form

$$-a(u_t - u^*) - (u_t - x_t) + \mathbf{d}E_t(c_t \mathbf{m}_{t+1}) = 0 \quad (\text{C.4})$$

$$\mathbf{m}_t = (u_t - x_t) \quad (\text{C.5})$$

First, we find an expression for $E_t(c_t \mathbf{m}_{t+1})$. We realize that this can be written as

$E_t(c_t \mathbf{m}_{t+1}) = (E_t c_t) \cdot (E_t \mathbf{m}_{t+1})$. When the central bank sets policy at time t , it doesn't observe c_t , the only piece of information that is available at that time is the existing estimate c_{t-1} , thus $E_t c_t = c_{t-1}$, and we have

$$E_t(c_t \mathbf{m}_{t+1}) = c_{t-1} \cdot (E_t \mathbf{m}_{t+1}) \quad (\text{C.6})$$

Leading (C.5) by one period and taking expectations we get:

$$E_t \mathbf{m}_{t+1} = (E_t u_{t+1} - E_t x_{t+1}) \quad (\text{C.7})$$

Substituting (C.6) and (C.7) into (C.4), we can derive the Euler equation

$$-a(u_t - u^*) - (u_t - x_t) + \mathbf{d}c_{t-1}(E_t u_{t+1} - E_t x_{t+1}) = 0 \quad (\text{C.8})$$

In the case of a policy of strict inflation reduction, the rule would be (B.8). Similarly, in the case of full accommodation of (the central bank's forecast of) private sector inflation expectations, the rule would be (B.9). Thus, it appears that in case of flexible inflation targeting the rule will be a linear combination of these special cases, that is $u_t = cx_t + (1-c)u^*$, where $0 \leq c \leq 1$. Or alternatively,

$$u_t = C_1 x_t + C_2 \quad (\text{B.10})$$

Here the coefficients C_1 and C_2 remain to be determined, and the prior is that $0 \leq C_1 \leq 1$ and $0 \leq C_2(C_1) \leq u^*$. Now we identify the coefficients C_1 and C_2 .

Expectations for the state at period $t+1$ follow from the constraint in (C.3), combining the latter with the decision rule for u , we can write:

$$E_t x_{t+1} = E_t [c_t (1 + u_t)] = c_{t-1} (u_t + 1) = c_{t-1} C_1 x_t + c_{t-1} C_2 + c_{t-1} \quad (\text{C.9})$$

From (B.10) it follows that

$$E_t u_{t+1} = C_1 E_t x_{t+1} + C_2 = C_1 [c_{t-1} C_1 x_t + c_{t-1} C_2 + c_{t-1}] + C_2 \quad (\text{C.10})$$

Substituting (C.9) and (C.10) into the Euler equation (C.8) above, and equating constant terms and coefficients on the state variables yields the following expressions for C_1 and C_2 in terms of the structural parameters of the model

$$C_1 = \frac{1}{(1+a) + \mathbf{d}_{t-1}^2} [\mathbf{d}_{t-1}^2 C_1^2 + 1] \quad (\text{C.11})$$

$$C_2 = \frac{\mathbf{d}_{t-1} [(C_1 - 1)c_{t-1} + 1]}{1+a} C_2 + \frac{\mathbf{d}_{t-1}^2 (C_1 - 1) + au^*}{1+a} \quad (\text{C.12})$$

Equation (C.11) implicitly defines the value of C_1 . It can be written as $C_1 = F(C_1)$. Note that the function $F(C_1)$ on the right hand side with domain $\langle 0, 1 \rangle$ is monotonically increasing in C_1 , that

$\lim_{C_1 \rightarrow 0} F(C_1) = \frac{1}{(1+a) + \mathbf{d}_{t-1}^2}$, $\lim_{C_1 \rightarrow 1} F(C_1) = \frac{\mathbf{d}_{t-1}^2 + 1}{(1+a) + \mathbf{d}_{t-1}^2}$. We realize that there is a unique positive solution C_1 , which fulfills $\frac{1}{(1+a) + \mathbf{d}_{t-1}^2} < C_1 < \frac{\mathbf{d}_{t-1}^2 + 1}{(1+a) + \mathbf{d}_{t-1}^2}$. It can be solved analytically:

$$C_1 = \frac{1}{2} \left\{ \left[\frac{(1+a) + \mathbf{d}_{t-1}^2}{\mathbf{d}_{t-1}^2} \right] - \sqrt{\frac{[(1+a)^2 + \mathbf{d}_{t-1}^2]^2 - 4\mathbf{d}_{t-1}^2}{\mathbf{d}_{t-1}^4}} \right\} \quad (\text{C.13})$$

Similarly, Equation (C.12) implicitly defines the value of C_2 . It can be written as $C_2 = G(C_2)$. Note that the function $G(C_2)$ on the right hand side with domain $\langle 0, u^* \rangle$ is monotonically increasing in C_2 . Again, we realize that there is a unique positive solution C_2^* . It can be solved analytically:

$$C_2 = \frac{\mathbf{d}_{t-1}^2 [(C_1 - 1)] + au^*}{(1+a) - \mathbf{d}_{t-1} [(C_1 - 1)c_{t-1} + 1]} \quad (\text{C.14})$$

Moreover, it can be easily established that $\lim_{a \rightarrow 0} C_2(C_1) = 0$ and that $\lim_{a \rightarrow \infty} C_2 = u^*$.

Finally, the Lagrange multiplier m can be solved by substituting (B.10) into (C.5) this yields

$$m = (C_1 - 1)x_t + C_2 \quad (\text{C.15})$$

APPENDIX D THE CENTRAL BANK'S OPTIMAL FILTERING PROBLEM

In this appendix we derive the central bank's optimal forecasting rule for private sector inflation expectations by applying the Kalman filter.

D.1 State Space Form

The policymaker's estimation problem can be put into state-space form by defining the state vector as the parameter $\mathbf{g}$. Then the (state) *transition equation* is

$$\mathbf{g}_t = \mathbf{g}_{t-1} \quad (\text{D.1})$$

However, the state is not observed directly. Instead the state of the system is conveyed by an observed variable (signal) y_t , which is subject to contamination by noise (measurement error) $\mathbf{e}_t$. Thus, the *measurement equation* is

$$y_t = \mathbf{g}_t' \mathbf{z}_t + \mathbf{e}_t \quad (\text{D.2})$$

where the scalar $\mathbf{z}_t' \equiv w_{t-1} = (1 + \mathbf{p}_{t-1})$, and $\mathbf{e}_t$ is a serially uncorrelated disturbance with mean zero and variance $\mathbf{s}_e^2$, that is $E(\mathbf{e}_t) = 0$ and $Var(\mathbf{e}_t) = \mathbf{s}_e^2$.²⁰

D.2 The Kalman Filter

The technique of the Kalman filter depends on the system that consists of (D.1) and (D.2) and its aim is to find unbiased estimates of the sequence of the state $\mathbf{g}_t$ via a recursive process of estimation.²¹

The process starts at time $t = 1$ say; and it is assumed that prior information on the previous state vector $\mathbf{g}_0$ is available in the form of an unbiased estimate c_0 , which has been drawn from a distribution with a mean of $\mathbf{g}_0$ and variance p_0 . Depending on the uncertainty surrounding the initial estimate, large (small) values should be attributed to p_0 to reflect the low (high) precision of the initial estimate. In the terminology of Bayesian statistics, this is a matter of attributing a diffuse prior distribution to $\mathbf{g}_0$.

The basic filter is described by four equations governing prediction, (namely equations (D.3), (D.5), (D.6.1) and (D.6.2)), and two for updating/smoothing (namely equations (D.9) and (D.12)). These equations are derived below.

In each time period, new information on the system is provided by the variable y_t ; and estimates of $\mathbf{g}_t$ may be formed both before and after the receipt of this information. The estimate of the state at time t formed without knowledge of y_t will be denoted by $c_{t|t-1}$; the estimate that incorporates the information of y_t will be denoted by c_t .

In the absence of information of y_t , the estimate $c_{t|t-1}$ of $\mathbf{g}_t$ comes directly from equation (D.1) where

²⁰ Note that (D.2) is in the form of an ordinary regression equation.

²¹ A process of estimation which keeps pace with the data by generating an estimate of the current state variable with each new observation y_t is described as *filtering*. The retrospective enhancement of a state estimate, using data - which has arisen subsequently - is described as *smoothing*. The estimation of a future state variable is described as *prediction*.

$\mathbf{g}_{t-1}$ is replaced by c_{t-1} . Thus

$$c_{t|t-1} = c_{t-1} \quad (\text{D.3})$$

Equation (D.3) is the *state prediction equation*.

The mean-square error of this estimator will be denoted by $p_{t|t-1} = E\{(\mathbf{g}_t - c_{t|t-1})^2\}$, whilst that of the updated estimator c_t will be denoted by $p_t = E\{(\mathbf{g}_t - c_t)^2\}$. To derive the expression for $p_{t|t-1}$ in terms of p_{t-1} , we subtract equation (D.3) from equation (D.1) to give

$$\mathbf{g}_t - c_{t|t-1} = \mathbf{g}_{t-1} - c_{t-1} \quad (\text{D.4})$$

It follows that the *prediction variance* is

$$p_{t|t-1} = E\{(\mathbf{g}_t - c_{t|t-1})^2\} = E\{(\mathbf{g}_{t-1} - c_{t-1})^2\} = p_{t-1} \quad (\text{D.5})$$

Before learning its value, we may predict y_t from equation (D.2) by replacing $\mathbf{g}_t$ by its estimate $c_{t|t-1}$ and replacing $\mathbf{e}_t$ by $E(\mathbf{e}_t) = 0$. This gives the *observation prediction equation*

$$E_{t-1} y_t = z_t' c_{t|t-1} \quad (\text{D.6.1})$$

The mean-square-error of this prediction is $f_{t|t-1} = E\{(y_t - E_{t-1} y_t)^2\}$. To express $f_{t|t-1}$ in terms of $p_{t|t-1}$, we subtract equation (D.6.1) from equation (D.2) to give the prediction error $e_t = y_t - E_{t-1} y_t = z_t'(\mathbf{g}_t - c_{t|t-1}) + \mathbf{e}_t$. Then, since $(\mathbf{g}_t - c_{t|t-1})$ and $\mathbf{e}_t$ are statistically independent, and since $\text{Var}(\mathbf{e}_t) = \mathbf{S}_e^2$, it follows that the *prediction variance* is

$$f_{t|t-1} = (z_t')^2 p_{t|t-1} + \mathbf{S}_e^2 \quad (\text{D.6.2})$$

The business of incorporating the new information provided by y_t into the estimate of the state variable may be regarded as a matter of estimating the parameter $\mathbf{g}_t$ in the system

$$\begin{bmatrix} c_{t|t-1} \\ y_t \end{bmatrix} = \begin{bmatrix} 1 \\ z_t' \end{bmatrix} \mathbf{g}_t + \begin{bmatrix} \mathbf{V}_t \\ \mathbf{e}_t \end{bmatrix}$$

where $\mathbf{V}_t = c_{t|t-1} - \mathbf{g}_t$.

By applying the method of generalised least squares (see e.g. Pollock (1999)), we obtain an estimating equation for $\mathbf{g}_t$ in the form of

$$c_t = (p_{t|t-1}^{-1} + (z_t')^2 (\mathbf{S}_e^2)^{-1})^{-1} (p_{t|t-1}^{-1} c_{t|t-1} + z_t' (\mathbf{S}_e^2)^{-1} y_t) = p_t (p_{t|t-1}^{-1} c_{t|t-1} + z_t' (\mathbf{S}_e^2)^{-1} y_t), \quad (\text{D.7})$$

where

$$p_t = (p_{t|t-1}^{-1} + (z_t')^2 (\mathbf{S}_e^2)^{-1})^{-1} \quad (\text{D.8})$$

is the variance of the estimator.

To give equation (D.7) a form, which is amenable to a recursive procedure, we consider the identity

$$\begin{aligned} p_{t|t-1}^{-1} c_{t|t-1} + z_t' (\mathbf{S}_e^2)^{-1} y_t &= (p_t^{-1} - (z_t')^2 (\mathbf{S}_e^2)^{-1}) c_{t|t-1} + z_t' (\mathbf{S}_e^2)^{-1} y_t \\ &= p_t^{-1} c_{t|t-1} + z_t' (\mathbf{S}_e^2)^{-1} (y_t - z_t' c_{t|t-1}) \end{aligned}$$

Using this on the RHS of equation (D.7), and noting that from (D.3) $c_{t|t-1} = c_{t-1}$, gives

$$\begin{aligned} c_t &= c_{t-1} + p_t z_t' (\mathbf{S}_e^2)^{-1} (y_t - z_t' c_{t-1}) \\ &= c_{t-1} + \mathbf{k}_t (y_t - z_t' c_{t-1}) = c_{t-1} + \mathbf{k}_t (y_t - w_{t-1} c_{t-1}) = (1 - \mathbf{k}_t w_{t-1}) c_{t-1} + \mathbf{k}_t y_t \end{aligned} \quad (\text{D.9})$$

where $\mathbf{k}_t = p_t z_t' (\mathbf{S}_e^2)^{-1} = p_{t|t-1} z_t' f_{t|t-1}^{-1} = p_{t-1} z_t' f_{t|t-1}^{-1} = p_{t-1} w_{t-1} f_{t|t-1}^{-1}$ is commonly described as the Kalman gain. Equation (D.9) is equation (4.4) in the main text.

Using (D.8), we can show that

$$\begin{aligned}
\mathbf{k}_t &= p_t z_t' (\mathbf{S}_e^2)^{-1} \\
&= \left(p_{t|t-1}^{-1} + (z_t')^2 (\mathbf{S}_e^2)^{-1} \right)^{-1} z_t' (\mathbf{S}_e^2)^{-1} \\
&= p_{t|t-1} z_t' \left((z_t')^2 p_{t|t-1} + \mathbf{S}_e^2 \right)^{-1} \\
&= \frac{1}{z_t'} \left[\frac{p_{t|t-1} (z_t')^2}{p_{t|t-1} (z_t')^2 + \mathbf{S}_e^2} \right]
\end{aligned} \tag{D.10}$$

where $p_{t|t-1} (z_t')^2$ is the portion of the prediction error variance due to uncertainty in $c_{t|t-1}$ and $\mathbf{S}_e^2$ is the portion of the prediction error variance due to the random shock $\mathbf{e}_t$. We can easily see that

$$\left| \frac{\partial \mathbf{k}_t}{\partial (p_{t|t-1} (z_t')^2)} \right| > 0,$$

suggesting that as uncertainty with $c_{t|t-1}$ increases, relatively more weight is given to new information in the prediction error, $y_t - z_t' c_{t-1}$. This is quite intuitive, since an increase in uncertainty in $c_{t|t-1}$ may be interpreted as a deterioration of the information content of $c_{t|t-1}$, relative to that of $y_t - z_t' c_{t-1}$.

Equation (D.8) can be rewritten as

$$p_t = p_{t|t-1} - p_{t|t-1} z_t' \left((z_t')^2 p_{t|t-1} + \mathbf{S}_e^2 \right)^{-1} z_t p_{t|t-1} \tag{D.11}$$

Combining equation (D.11) with (D.10), and noting that from (D.5) $p_{t|t-1} = p_{t-1}$, the so-called Ricatti equation – which provides a means for generating the variance of the state prediction – can be written in recursive form as

$$p_t = p_{t-1} - \mathbf{k}_t z_t' p_{t-1} = p_{t-1} - \mathbf{k}_t w_{t-1} p_{t-1} \tag{D.12}$$

which is equation (4.5) in the main text.