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ATM NETWORK**

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Cost Allocation in a Bank ATM Network ^{*}

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Abstract

We consider a situation in which a group of banks consider connecting their Automated Teller Machines (ATMs) in a network, so that the banks' customers may use ATMs of any bank in the network. The problem studied is that of allocating the total transaction costs arising in the network, among the participating banks. The situation is modeled as a cooperative game with transferable utility. We propose two allocation rules, and discuss their relation to the core and other well-known solution concepts, as well as to population monotonicity.

1 Introduction

Through Automated Teller Machines (ATMs), financial organizations (hereafter called banks) provide service, e.g. cash withdrawals, to their customers. For various reasons, networks of ATMs have formed, consisting of several banks, where customers of one bank may use ATMs of any bank in the network. In such a system, there is a difference between the costs that are incurred by a bank, and the costs that are actually caused by that bank's customers. The banks will seek a method to compensate for such imbalances in network usage, and according to Gow and Thomas (1998), with reference to the UK, this is done by setting *interchange fees*. Every time a customer of bank i uses an ATM of bank j , bank i has to pay a fee f_{ij} to bank j . Setting interchange fees is equivalent to allocating the total cost arising in such a network, and the fee structure will be the result of a negotiation process involving the participating banks.

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The (transaction) costs arising in such a network depends on how transactions are processed. If the ATMs of the network are not easily accessible for the customers of the member banks, the customers will tend to use alternative means of processing their transactions, e.g. withdrawing cash over the counter. Also, the cost of processing a transaction will be higher if the processing involves linking computer systems of different institutions, than if no such links are necessary. The availability, for a particular customer, of ATMs belonging to the network to which his bank is affiliated, and in particular, of ATMs belonging to his own bank, depends on the physical location of the customer as well as of the various ATMs in the network, since the customer need to be physically present at the site of an ATM in order to be able to use it.

Cooperative game theory has proven to be successful to analyse allocation problems that are related to situation from practice. We mention Littlechild (1977), Nouweland et. al. (1996), Fragnelli et al. (2000) and Sánchez-Soriano et al. (2002). Hence, also our ATM cost allocation problem will be modeled as a cooperative game with transferable utility, and in doing so, we explicitly model the location of customers (transactions) and ATMs. A key question is whether there exist cost allocations that insure against break-up of the network. Given such an allocation, it should not be possible for any groups of banks to lower their costs by leaving the network. This requirement is related to the *core* of the corresponding game. Since finding a core allocation means checking a very large number of core inequalities, we would like to be able to deduce such allocations directly from the problem data, i.e. not explicitly considering the game. By relating such "natural" allocations to other solution concepts, such as the Shapley value (Shapley (1953)), the nucleolus (Schmeidler (1969)), and the τ -value (Tijs (1981)), we can learn something about e.g. the location of the allocation within the core. Another interesting question is the properties of the allocation method in a dynamic context. Assuming that there are benefits resulting from cooperation, i.e. the game has a nonempty core, we would like the allocation methods to be such that it facilitates the enlargement of the network. When a new bank wants to join the network, the existing members should not loose by accepting it as a new member. This is related to the concept of population monotonic allocation schemes (cf. Sprumont (1990)).

Our paper is similar in spirit to Gow and Thomas (1998), but our approach differs from theirs in that they do not consider explicitly the location of ATMs and transactions. Another difference is that we do not consider fixed costs. In fact, our *cost savings game* would not be influenced by the inclusion of fixed costs in the manner of Gow and Thomas (1998)¹.

In section 2, we introduce the *ATM-game*. This cost savings game is defined

¹This would not be the case, however, if the number and location of ATMs were endogenously determined in our model.

by aggregating *single-location games* over the set of locations. In section 3 we show that the single-location games correspond to *information market games*, as defined by Muto et. al. (1989) and Potters and Tijs (1989). This correspondence yields many useful results about these games, and makes us able to study the more general ATM-games in section 4. We introduce two allocation rules, the *equal-split rule* and the *transaction-based rule*. Both rules involve aggregating, over the set of locations, allocations proposed for single-location games in section 3, and they only differ with respect to locations where only one bank have ATMs. The equal-split rule yields a core element that coincides with the τ -value, but is not, in general, a population monotonic allocation scheme. In the special case where at most one bank have ATMs in any location, the equal-split rule also coincides with the Shapley value and the nucleolus, and is population monotonic. The transaction-based rule also yields a core element, and moreover, is always population monotonic. By using an example, we illustrate that it is sometimes *the only* population monotonic allocation scheme. In the special case where at least two banks have ATMs in every location, the core consists of a single point, corresponding to the allocation that results from the transaction-based rule.

2 ATM-games

In this section we will introduce formally our model and define a ATM cost savings game. Moreover, we recall some notions from cooperative game theory.

Let N denote the set of banks (players). We define a *location* to be a city or parts thereof, and let L denote the set of locations. Let n_i^ℓ represent the number of transactions of bank $i \in N$ in location $\ell \in L$. For $S \subseteq N$, let $n^\ell(S) := \sum_{i \in S} n_i^\ell$ denote the number of transactions belonging to S in location ℓ . Let A^ℓ be the set of banks that have ATMs in location ℓ . Further, let $L_1 := \{\ell \in L : |A^\ell| = 1\}$ be the set of locations where only one bank is represented, and let $L_M := \{\ell \in L : |A^\ell| > 1\}$ be the set of locations where multiple banks are represented. We will assume that $L = L_1 \cup L_M$, i.e. that there are ATMs in all locations.

With regard to the behaviour of customers, we assume that, if $S \subseteq N$ have formed a network:

- A1 Transactions in a particular location will be processed by an ATM if one or more members of S have an ATM there.
- A2 When a customer of bank $i \in S$ performs a transaction in a location ℓ , and if bank i has ATMs in location ℓ , the customer will use one of the ATMs of bank i .

The transaction costs are assumed to be the same for all banks. The transaction cost will be α if the customer uses an ATM of his own bank. If he uses an ATM of another bank the transaction cost will be β , where $\beta > \alpha$. The cost of non-ATM transactions is complex, since there exist several alternatives to using ATMs, such as withdrawing money over the counter, writing a check to a third person in exchange for cash, or using a cashback facility. In the four-bank example of Gow and Thomas (1998), the cost of cash withdrawal over the counter is used as an approximation of this cost. We shall assume that the cost of a non-ATM transaction is γ , where $\gamma > \beta$.

Suppose S forms a network. Assumptions A1 and A2 imply, for any location $\ell \in L$, that the total amount of transaction costs in location ℓ is given by

$$c^\ell(S) := \begin{cases} \alpha n^\ell(S \cap A^\ell) + \beta n^\ell(S \setminus A^\ell) & \text{if } S \cap A^\ell \neq \emptyset, \\ \gamma n^\ell(S) & \text{otherwise.} \end{cases}$$

Example 2.1 Consider a location ℓ where the three banks A, B , and C , have customers. The numbers of customers of these banks at ℓ are $n_A^\ell = 100, n_B^\ell = 150$, and $n_C^\ell = 200$. Banks A and C have ATMs at ℓ . The cost of an ATM transaction is $\alpha = 1$ for customers serviced by their own bank, and $\beta = 2$ otherwise. Every non-ATM transaction involves cost $\gamma = 10$. In the previous terminology we have $A^\ell = \{A, C\}$. Then these parameters fix the coalitional cost game c^ℓ . The cost at which the coalition $\{A\}$ is able to service all its customers is $c^\ell(\{A\}) = \alpha n_A^\ell = 100$, since $A \in A^\ell$. Similarly, we calculate $c^\ell(\{C\}) = \alpha n_C^\ell = 200$. Since $B \notin A^\ell$, we have $c^\ell(\{B\}) = \gamma n_B^\ell = 1500$. The cost of serving the customers of A and B together are $c^\ell(\{A, B\}) = \alpha n_A^\ell + \beta n_B^\ell = 100 + 300 = 400$. The customers of A and C are all serviced by the ATM of their own bank, hence $c^\ell(\{A, C\}) = \alpha(n_A^\ell + n_C^\ell) = 300$. In this way we compute the cost associated with each coalition, as shown in Figure 2.1. $\triangleleft$

S	$\{A\}$	$\{B\}$	$\{C\}$	$\{A, B\}$	$\{A, C\}$	$\{B, C\}$	N
$c^\ell(S)$	100	1500	200	400	300	500	600

Figure 2.1: The values for c^ℓ in Example .

In order to relate our game to existing literature, it will be convenient to study the corresponding *cost savings game* v^ℓ . The *single-location ATM-game* v^ℓ is

given by, for any $S \subseteq N$,

$$v^\ell(S) := \sum_{i \in S} c^\ell(\{i\}) - c^\ell(S) = \begin{cases} (\gamma - \beta)n^\ell(S \setminus A^\ell) & \text{if } S \cap A^\ell \neq \emptyset, \\ 0 & \text{otherwise.} \end{cases} \quad (2.1)$$

Since all the solution concepts that we will study are relatively invariant under strategic equivalence², all results for a cost savings game can easily be translated into the setting of a cost game.

Example 2.2 Now we return to Example 2.1. The coalitional cost savings for that example are specified in the table below. Notice that the cost savings, i.e. the values of v^ℓ , arise from transactions of banks that do not have ATMs in location ℓ , i.e. bank B in this case. Notice also the zero values of single player coalitions. Single players save no costs, regardless of whether they have ATMs or not. $\triangleleft$

S	$\{A\}$	$\{B\}$	$\{C\}$	$\{A, B\}$	$\{A, C\}$	$\{B, C\}$	N
$v^\ell(S)$	0	0	0	1200	0	1200	1200

Figure 2.2: The values for v^ℓ in Example 2.2.

The *ATM-game* v is obtained by aggregating over the set of locations, i.e. let, for every $S \subseteq N$,

$$v(S) := \sum_{\ell \in L} v^\ell(S) = (\gamma - \beta) \sum_{\ell \in L: S \cap A^\ell \neq \emptyset} n^\ell(S \setminus A^\ell). \quad (2.2)$$

Before studying some properties and solutions of single-location ATM-games, we recall some notions from cooperative game theory. Let (N, g) be some cost savings game. The *core* is defined as

$$C(g) := \{x \in \mathbb{R}^N : x(N) = g(N), x(S) \geq g(S) \ \forall S \subset N\}.$$

In order to define the *Shapley value*, Shapley (1953), denoted $\Phi(g)$, let Π be the set of all orderings of the player set. Take a player $i \in N$ and an order $\pi \in \Pi$. The i th coordinate of the marginal vector $m^\pi(g)$ is given by

$$m_i^\pi(g) := g(\{\pi(1), \dots, \pi(k-1), \pi(k)\}) - g(\{\pi(1), \dots, \pi(k-1)\}),$$

²A solution concept σ is said to be *relatively invariant under strategic equivalence* iff, whenever v is a game with $\sigma(v) \neq \emptyset$, $a > 0$, and $b \in \mathbb{R}^N$, then $\sigma(a + bv) = a + b\sigma(v)$. If c is a cost game, and v is the corresponding cost savings game, then $x \in \sigma(c) \Leftrightarrow y \in \sigma(v)$, where $y_i := c(\{i\}) - x_i$ for all $i \in N$.

where $i = \pi(k)$. The Shapley value is defined as the average, over the set Π , of the marginal vectors, i.e.

$$\Phi(g) := \frac{1}{|\Pi|} \sum_{\pi \in \Pi} m^\pi(g).$$

If g is a convex game, i.e. if

$$g(S) + g(T) \leq g(S \cup T) + g(S \cap T) \quad \forall S, T \subseteq N,$$

then we know from Shapley (1971) that $C(g) = \text{conv}\{m^\pi(g) : \pi \in \Pi\}$, and hence that $\Phi(g) \in C(g)$.

For $x \in \mathbb{R}^N$ such that $x(N) = g(N)$, and for $S \subseteq N$, $e(S, x) := g(S) - x(S)$ is called the *excess value of S with respect to x* . Let $\theta(x)$ be the 2^n -tuple whose components are the excesses $e(S, x)$, $S \subseteq N$, arranged in nonincreasing order. The *nucleolus*, Schmeidler (1969), is the payoff vector x such that $x(N) = g(N)$, and such that $\theta(x)$ is lexicographically minimal.

For each $i \in N$, let $M_i(g) := g(N) - g(N \setminus \{i\})$ denote the i th coordinate of the *utopia vector* $M(g)$. Also, let $m_i(g) := \max_{S \ni i} \{g(S) - \sum_{j \in S \setminus \{i\}} M_j(g)\}$ denote the i th coordinate of the *minimum rights vector* $m(g)$. If g has a nonempty core, then the τ -value $\tau(g)$, Tijs (1981), is given by the convex combination of $M(g)$ and $m(g)$ that satisfies $\sum_{i \in N} \tau_i(g) = g(N)$.

3 Properties and Solutions of Single-Location Games

In this section we consider single-location games as defined in (2.1). First, we show that these games are information market games. Second, we consider some allocations for these games. Here, we distinguish between the cases that there is only one bank in the location and that there are more than one bank in the location.

A single-location game can be related to the class of *information market games*, see Muto et. al. (1989) and Potters et. al. (1989). An information market game consists of a set of players N , where a subset $I \subset N$ possesses information about a (patented) new technology, necessary for producing a new product. The total market for this new product can be partitioned into sub-markets, and the profit realized by a coalition depends on which sub-markets the coalition has access to. Let M_T denote the set of sub-markets that the coalition T has access to, and let r_T denote the profit that can be realized from these sub-markets. A coalition S can realize the profit r_T if it has at least one member with access to the sub-markets M_T , i.e. $S \cap T \neq \emptyset$, as well as at least one member with knowledge of the patented technology, i.e. $S \cap I \neq \emptyset$. Therefore,

the total profit that can be realized by the members of S is given by

$$v_{I,r}(S) = \begin{cases} \sum_{T: T \cap S \neq \emptyset} r_T & \text{if } S \cap I \neq \emptyset \\ 0 & \text{otherwise,} \end{cases} \quad (3.1)$$

thus defining the information market game $(N, v_{I,r})$. Muto et. al. (1989) show that, if $|I| = 1$, then the nucleolus, the τ -value, and (in the convex case) the Shapley value, coincides. Potters and Tijs (1989) show that, if $|I| \geq 2$, and if $r_T = 0$ for all $T \subseteq N$ such that $|T| > 1$, then the core consists of a single point.

Theorem 3.1 *A single-location ATM-game is an information market games.*

Proof. We obtain $v^\ell = v_{I,r}$ by setting $I := A^\ell$ and

$$r_T := \begin{cases} (\gamma - \beta)n_i^\ell & \text{if } T = \{i\} \subset N \setminus A^\ell, \\ 0 & \text{otherwise.} \end{cases}$$

□

3.1 Locations where Only One Bank has ATMs

In this subsection we discuss the single-location game that has only one bank in the location.

Because of Theorem 3.1, and since $|A^\ell| = 1$, this case is covered by Muto et. al. (1989). Moreover, since $r_T = 0$ for all $T \subseteq N$ such that $|T| \geq 2$, we get the following result.

Proposition 3.2 *If $\ell \in L_1$, then the game v^ℓ is convex.*

We denote the bank having ATMs in location ℓ by i^ℓ . From Muto et. al. (1989) and Shapley (1971), respectively, follows the following two characterizations of the core.

Proposition 3.3 *If $\ell \in L_1$, then*

- (i) $C(v^\ell) = \{x \in \mathbb{R}^N : x(N) = v^\ell(N), 0 \leq x_i \leq (\gamma - \beta)n_i^\ell \forall i \in N \setminus \{i^\ell\}\}$
- (ii) $C(v^\ell) = \text{conv}\{m^\pi(v^\ell) : \pi \in \Pi\}$

Proposition 3.3(ii) is useful here, because the vectors of marginal contributions have a simple structure in our case. For any $\pi \in \Pi$, let $S_\pi^\ell := \{i \in N : \pi^{-1}(i) < \pi^{-1}(i^\ell)\}$, i.e. S_π^ℓ is the set of players that precede i^ℓ in the order π .

Proposition 3.4 *If $\ell \in L_1$, then*

$$m_i^\pi(v^\ell) = \begin{cases} 0 & \text{if } i \in S_\pi^\ell, \\ (\gamma - \beta)n_i^\ell(S_\pi^\ell) & \text{if } i = i^\ell, \\ (\gamma - \beta)n_i^\ell & \text{if } i \in N \setminus S_\pi^\ell. \end{cases} \quad (3.2)$$

Proof. Consider an arbitrary player $i \in N$. If $i \neq i^\ell$, and i joins a coalition S , then an additional cost saving of $(\gamma - \beta)n_i^\ell$ will be realized, but only if i^ℓ is already a member of S . If $i = i^\ell$, then i will provide cost savings for all the players that are already in S . $\square$

Example 3.5 Consider a situation where the banks A, B , and C have customers in the location ℓ . Let $n_A^\ell = 200, n_B^\ell = 50$, and $n_C^\ell = 125$. Also, as in Example 2.1, let $\alpha = 1, \beta = 2$, and $\gamma = 10$. Only bank A has ATMs in the location. The values of the game v^ℓ are shown in Figure 3.1, and the marginal vectors are shown in Figure 3.2. From the picture of the core shown in Figure 3.3, the coincidence of the extreme points of the core with the marginal vectors can indeed be verified. Observe that, although there are six different orderings of the player set, there are only four distinct marginal vectors. Both the marginal vectors for which bank A comes first (last) coincides, as is easily seen from (3.2).

S	$\{A\}$	$\{B\}$	$\{C\}$	$\{A, B\}$	$\{A, C\}$	$\{B, C\}$	N
$v^\ell(S)$	0	0	0	400	1000	0	1400

Figure 3.1: The game v^ℓ in Example 3.5.

$\triangleleft$

π	$m_A^\pi(v^\ell)$	$m_B^\pi(v^\ell)$	$m_C^\pi(v^\ell)$
A, B, C	0	400	1000
A, C, B	0	400	1000
B, A, C	400	0	1000
B, C, A	1400	0	0
C, A, B	1000	400	0
C, B, A	1400	0	0

Figure 3.2: Marginal vectors for Example 3.5.

Figure 3.3: The core (solid lines) for Example 3.5.

In a core element, according to Proposition 3.3(i), a bank i that does not have an ATM itself, i.e. $i \neq i^\ell$, should be rewarded a non-negative amount, but no more than the cost savings involving its own transactions, i.e. $(\gamma - \beta)n_i^\ell$. Next, we consider a special core allocation which is a compromise between the lower and upper bound presented by Proposition 3.3(i). We allow bank i to keep half the cost savings arising from its transactions, and award the remaining half of the cost savings to the bank owning the ATM(s), i.e. i^ℓ . This results in the allocation w^ℓ given by, for every $i \in N$,

$$w_i^\ell := \begin{cases} \frac{(\gamma - \beta)}{2} n^\ell(N \setminus \{i\}) & \text{if } i = i^\ell, \\ \frac{(\gamma - \beta)}{2} n_i^\ell & \text{otherwise.} \end{cases} \quad (3.3)$$

Proposition 3.6 *If $\ell \in L_1$, and w^ℓ is given by (3.3), then $w^\ell = \Phi(v^\ell) =$*

$$\nu(v^\ell) = \tau(v^\ell).$$

Proof. Note that, for a player $i \neq i^\ell$, the number of orderings π such that i comes after i^ℓ , i.e. $i \in N \setminus S_\pi^\ell$, is exactly $\frac{|N|}{2}$. The first equality then follows from (3.2), and from the definition of the Shapley value. The rest of the proof follows from Muto et. al. (1989). $\square$

Example 3.7 In Example 3.5, bank A is providing cost savings to B and C by letting their customers use its ATM(s). For bank B , these cost savings amount to $50 \cdot (10 - 2) = 400$, of which B is allowed to keep half, i.e. 200. Bank C 's transactions give rise to cost savings of $125 \cdot 8 = 1000$, and he is allowed to keep 500. The Shapley value, the nucleolus, and the τ -value all coincide with the point $(700, 200, 500)$. $\triangleleft$

3.2 Locations Where Multiple Banks Have ATMs

In this section we discuss the single-location games that has more than one bank in the location.

In this case we propose an allocation where, when cost savings are realized, the entire cost savings go to the owner of the *transactions involved*. Hence, the owners of the ATMs that provide these cost savings, do not get any part of these savings. This allocation x^ℓ is given by, for every $i \in N$,

$$x_i^\ell := \begin{cases} 0 & \text{if } i \in A^\ell, \\ (\gamma - \beta)n_i^\ell & \text{otherwise.} \end{cases} \quad (3.4)$$

From Theorem 3.1 and Potters and Tijs (1989) we have the following result.

Proposition 3.8 *If $\ell \in L_M$, then $C(v^\ell) = \{x^\ell\}$, where x^ℓ is given by (3.4). We also have $x^\ell = \nu(v^\ell) = \tau(v^\ell)$.*

Example 3.9 Recall the situation described in Example 2.1 and Example 2.2. Bank B is the only one that does not have ATMs, so the only cost savings are those involving B 's transactions. According to (3.4), B will be allowed to keep the entire cost savings himself, and this yields the unique core allocation $(0, 1200, 0)$. The game v^ℓ is not convex, since

$$v^\ell(\{A, B\}) - v^\ell(\{B\}) = 1200 > v^\ell(\{A, B, C\}) - v^\ell(\{B, C\}) = 0,$$

and the Shapley value, given by $\Phi(v^\ell) = (200, 800, 200)$, is not a core element. $\triangleleft$

4 Two Allocation Rules for ATM-games

In this section we discuss the ATM-games. We propose two allocations and investigate their relation with existing game theoretical solutions.

In this section we turn to the ATM-game v defined by (2.2). In sections 3.1 and 3.2 we proposed allocation rules for single-location ATM-games. In the following we will discuss two allocation rules for situations with multiple locations. These rules aggregate, over the set of locations, the allocation vectors proposed for single locations. We shall relate the resulting solutions to solution concepts such as the core and the τ -value. Also, we will investigate whether these allocations rule correspond to *population monotonic allocation schemes*, as defined by Sprumont (1990). Let $P(N)$ denote the set of nonempty subsets of N .

Definition 4.1 A vector $d = (d_{iS})_{i \in S, S \in P(N)}$ is a population monotonic allocation scheme of the game v if and only if

$$\sum_{i \in S} d_{iS} = v(S) \quad \forall S \in P(N), \quad (4.1)$$

$$d_{iS} \leq y_{iT} \quad \forall i \in S \subseteq T \in P(N). \quad (4.2)$$

Hence, d specifies an allocation for every game corresponding to the population $S \subseteq N$, where the characteristic function is given by the restriction of v to the members of S . (4.1) expresses that the entire cost $v(S)$ should be covered, and (4.2) that no player shall be made worse off as new players enter the games. In the ATM network situation, population monotonicity ensures that members of an existing network will not object to new banks joining the network.

4.1 The Equal-Split Rule

We first propose an allocation rule that splits the cost savings equally between owners of transactions and the owner of the ATMs. Thus, for $\ell \in L_1$ we choose w^ℓ as defined by (3.3), and for $\ell \in L_M$ we choose x^ℓ according to (3.4). This yields the allocation y given by, for every $i \in N$,

$$\begin{aligned} y_i &:= \sum_{\ell \in L_1} w_i^\ell + \sum_{\ell \in L_M} x_i^\ell \\ &= \frac{\gamma - \beta}{2} \sum_{\ell \in L_1: i \notin A^\ell} n_i^\ell + \frac{\gamma - \beta}{2} \sum_{\ell \in L_1: i \in A^\ell} n^\ell(N \setminus \{i\}) \\ &\quad + (\gamma - \beta) \sum_{\ell \in L_M: i \notin A^\ell} n_i^\ell. \end{aligned} \quad (4.3)$$

Since the equal-split rule adds core elements of the games v^ℓ , the following result is obvious.

Theorem 4.2 *The equal-split rule gives a core element of the game v .*

Example 4.3 Consider a situation with two locations, i.e. $L := \{1, 2\}$, and three banks, i.e. $N = \{A, B, C\}$. In location 1 the banks have 100, 150, and 200 transactions, respectively. Here, bank A and C have ATMs. In location 2 the banks have 200, 50, and 125 transactions, respectively, and only bank A has ATMs there. The locations correspond to those described in Example 2.1 and 3.5, respectively. As before, $\alpha = 1$, $\beta = 2$, and $\gamma = 10$. The values of the resulting game v are shown in Figure 4.1, and a picture of the core in Figure 4.2. In location 1, where both bank A and C have ATMs, the equal-split rule prescribes the allocation $(0, 1200, 0)$, and in location 2, where only bank A has ATMs, the allocation $(700, 200, 500)$. Summing the allocation vectors, we get $y := (700, 1400, 500)$. From Figure 4.2 it can be seen that this is a point in the relative interior of the core. It coincides with the τ -value, but not with the Shapley value or the nucleolus, given by the allocation vectors $(900, 1000, 700)$ and $(950, 1150, 500)$, respectively.

S	$\{A\}$	$\{B\}$	$\{C\}$	$\{A, B\}$	$\{A, C\}$	$\{B, C\}$	N
$v(S)$	0	0	0	1600	1000	1200	2600

Figure 4.1: The game v in Example 4.3.

◁

Figure 4.2: The core (solid lines) for Example 4.3.

In the above example, the allocation returned by the transaction-based rule coincided with the τ -value, and this property holds in the general case.

Theorem 4.4 *If y results from the equal-split rule, then $y = \tau(v)$.*

Proof. Recall the definition in (4.3), and note that, if $\ell \in L_1$, then $w^\ell = \frac{1}{2}(M(v^\ell) + m(v^\ell))$ follows from the proof of Theorem 4.1 in Muto et. al. (1989). For $\ell \in L_M$, we will show that $x^\ell = M(v^\ell) = m(v^\ell)$, and the desired result then follows from $y = \frac{1}{2} \sum_{\ell \in L} (M(v^\ell) + m(v^\ell)) = \frac{1}{2}(M(v) + m(v))$ and $y(N) = v(N)$.

Let $\ell \in L_M$. For every $i \in N$ we have

$$M_i(v^\ell) = v^\ell(N) - v^\ell(N \setminus \{i\}) = \begin{cases} (\gamma - \beta)n_i^\ell & \text{if } i \in N \setminus A^\ell, \\ 0 & \text{if } i \in A^\ell. \end{cases}$$

If $i \in N \setminus A^\ell$, then

$$\begin{aligned} m_i(v^\ell) &= \max\left\{ \max_{\substack{S \cap A^\ell \neq \emptyset \\ S \ni i}} \{v^\ell(S) - \sum_{j \in S \setminus \{i\}} M_j(v^\ell)\}, \max_{\substack{S \cap A^\ell = \emptyset \\ S \ni i}} \{v^\ell(S) - \sum_{j \in S \setminus \{i\}} M_j(v^\ell)\} \right\} \\ &= \max\left\{ (\gamma - \beta) \max_{\substack{S \cap A^\ell \neq \emptyset \\ S \ni i}} \left\{ \sum_{j \in S \setminus A^\ell} n_j^\ell - \sum_{j \in S \setminus A^\ell \setminus \{i\}} n_j^\ell \right\}, \max_{\substack{S \cap A^\ell = \emptyset \\ S \ni i}} \left\{ 0 - \sum_{j \in S \setminus \{i\}} n_j^\ell \right\} \right\} \\ &= \max\{(\gamma - \beta)n_i^\ell, 0\} = (\gamma - \beta)n_i^\ell, \end{aligned}$$

and if $i \in A^\ell$, then

$$\begin{aligned} m_i(v^\ell) &= \max_{S \ni i} \{v^\ell(S) - \sum_{j \in S \setminus \{i\}} M_j(v^\ell)\} \\ &= \max_{S \ni i} \left\{ (\gamma - \beta) \sum_{j \in S \setminus A^\ell} n_j^\ell - (\gamma - \beta) \sum_{j \in S \setminus A^\ell \setminus \{i\}} n_j^\ell \right\} = 0. \end{aligned}$$

□

The equal-split does not always satisfy population monotonicity, as the following example illustrates.

Example 4.5 Consider a situation with two locations, i.e. $L := \{1, 2\}$, and three banks, i.e. $N := \{A, B, C\}$. A has ATMs in both locations, B in location 1, and C in neither location. Hence, $A^1 = \{A, B\}$, and $A^2 := \{A\}$. The number of transactions for each bank $i \in N$ and each location $\ell \in L$ is

$$n_i^\ell := \begin{cases} q & \text{if } i = C \text{ and } \ell = 1, \\ p & \text{otherwise,} \end{cases}$$

where $p < q$. Let $\gamma - \beta := 1$. The cost savings realized by the various coalitions $S \subseteq N$, i.e. the values of the game v , are shown in Figure 4.5. Suppose that B and C have formed a network. The cost savings that they realize are q , involving only the transactions of C in location 1, where B is present. According to the equal-split rule, they will both get a payoff of

$$\frac{q}{2}.$$

Suppose that A wants to join the network. It will provide cost savings for both B and C in location 2, where it is the only bank present, and the total cost savings now increases to $2p + q$. The equal-split rule yields the allocation

$$\left(p, \frac{p}{2}, q + \frac{p}{2}\right).$$

Hence, bank B , since $p < q$, will have its payoff reduced, and it is therefore likely that it will oppose A being accepted as a new participant in the network. ◁

S	$\{A\}$	$\{B\}$	$\{C\}$	$\{A, B\}$	$\{A, C\}$	$\{B, C\}$	N
$v(S)$	0	0	0	p	$q + p$	q	$q + 2p$

Figure 4.2: The game v of Example 4.5.

However, there are situations in which even the equal-split rule satisfies population monotonicity, as the next example shows.

Example 4.6 Consider Example 4.5 again, but now with the assumption that $p = q$, i.e. all the banks have the same number of transactions, in all locations

$\ell \in L$. We will check that the allocation scheme $(y_{iS})_{S \in P(N)}$ corresponding to the equal-split rule is population monotonic. If a bank operate on its own, i.e. if $|S| = 1$, no cost savings will be realized, and we have $y_{iS} = 0$ for all $i \in N$. If all the banks participate in the network, the total cost savings will be $3p$, and the equal-split rule yields the allocation

$$y_N := \left(p, \frac{p}{2}, \frac{3p}{2} \right).$$

Since the equal-split rule always assigns positive payoffs to all banks, we only need to check, for each $S \subset N$ such that $|S| = 2$, that $y_{iS} \leq y_{iN}$. The equal-split rule, when applied to the network formed by S , gives bank $i \in S$ the payoff

$$y_{iS} = \begin{cases} \frac{p}{2} & \text{if } S = \{A, B\}, \\ p & \text{if } S = \{A, C\}, \\ \frac{p}{2} & \text{if } S = \{B, C\}, \end{cases}$$

hence population monotonicity is satisfied. $\triangleleft$

Finally, we consider a special case where the equal-split rule does satisfy population monotonicity, and several well-known solution concepts coincides.

Theorem 4.7 *If $L = L_1$, then*

- (i) *the equal-split rule is a population monotonic allocation scheme, and*
- (ii) *if the allocation y results from the equal-split rule, then $y = \Phi(v) = \nu(v) = \tau(v)$.*

Proof. (i) Let, for any $S \subseteq N$, $L_1^S := \{\ell \in L : |A^\ell \cap S| = 1\}$, and $L_M^S := \{\ell \in L : |A^\ell \cap S| \geq 2\}$. The result follows from (4.3), $\gamma > \beta$, $n_i^\ell \geq 0$ for all $i \in N$, and by noting that if $S \subseteq T$, then $L_1^S \subseteq L_1^T$.

(ii) This result follows from the additivity of the Shapley value, and Muto et. al. (1989). $\square$

4.2 The Transaction-Based Rule

Suppose that banks with ATMs are given no reward for the cost savings that they provide for the banks without ATMs. The cost savings are rewarded to the bank owning the *transactions* for which the savings are realized. Thus, for every $\ell \in L$, we choose x^ℓ as defined by (3.4), and then we sum over the set of locations. This yields the allocation vector z given by, for every $i \in N$,

$$z_i := \sum_{\ell \in L} x^\ell = (\gamma - \beta) \sum_{\ell \in L: i \notin A^\ell} n_i^\ell. \quad (4.4)$$

Since (3.4) yields core elements for each of the games v^ℓ , $\ell \in L$, the following result follows easily.

Theorem 4.8 *The transaction-based rule gives a core element of v .*

Example 4.9 We apply (3.4) to location 1 and 2, respectively, and get the allocations vectors $(0, 1200, 0)$ and $(0, 400, 1000)$. By summing these, we get the allocation vector $z := (0, 1600, 1000)$. From Figure 4.2 we see that this corresponds to one of the extreme points of the core³. $\triangleleft$

We saw in section 4.1 that the equal-split rule is not necessarily population monotonic. The transaction-based rule is better in this respect.

Theorem 4.10 *The transaction-based rule is a population monotonic allocation scheme.*

Proof. The result follows from (4.4), $\gamma > \beta$, $n_i^\ell \geq 0$ for all $i \in N$, and by noting that if $S \subseteq T$, then $L_1^S \cup L_M^S \subseteq L_1^T \cup L_M^T$. $\square$

We showed, in Example 4.6, that there are cases where even the equal-split rule is population monotonic. Next we provide an example where the transaction-based rule is the *only* population monotonic allocation scheme.

Example 4.11 Consider a situation with two locations, i.e. $L := \{1, 2\}$, and three banks, i.e. $N := \{A, B, C\}$. A has ATMs in both locations. Bank B has ATMs only in location 1, and C only in location 2. The cost savings for one transaction is $\gamma - \beta := 1$, and every bank $i \in N$ has $n_i^\ell := p$ transactions in location $\ell = 1, 2$. The core of the game corresponding to the grand coalition

S	$\{A\}$	$\{B\}$	$\{C\}$	$\{A, B\}$	$\{A, C\}$	$\{B, C\}$	N
$v(S)$	0	0	0	p	p	$2p$	$2p$

Figure 4.3: The game v of Example 4.11.

³That this is not so in general can be verified by e.g. adding a location 3 to the example, where $n_i^3 = 100$ for all $i \in N$, and $A^3 = \{B\}$.

consists of the allocation $(0, p, p)$, which is returned by the transaction-based rule. In the sub-game corresponding to the coalition $\{A, B\}$, where the value p is to be allocated between bank A and B , the transaction-based rule and the equal-split rule return the allocations $(0, p)$ and $(\frac{p}{2}, \frac{p}{2})$, respectively. However, the former allocation is the only one that satisfies population monotonicity. In the sub-game corresponding to the coalition $\{A, C\}$, where the value p is to be allocated between bank A and C , the two rules return the allocations $(0, p)$ and $(\frac{p}{2}, \frac{p}{2})$, respectively, and again the only allocation that satisfies population monotonicity is the one returned by the transaction-based rule. In the sub-game corresponding to the coalition $\{B, C\}$, where the value $2p$ is to be allocated between bank B and C , both rules return the allocation (p, p) , which is the only allocation satisfying population monotonicity. $\triangleleft$

Finally, we consider the special case where all locations are served by at least two banks. In this case the core consists of a single point, corresponding to the allocation returned by the transaction-based rule.

Theorem 4.12 *If $L = L_M$, and y results from the transaction-based rule, then $C(v) = \{z\}$.*

Proof. Because of Theorem 4.8, we only need to show that $\bar{z} \in C(v)$ implies $\bar{z} = z$. From $\bar{z} \in C(v)$ and $|A^\ell| > 1$ for all $\ell \in L$ follows

$$\begin{aligned}\bar{z}_i &\leq v(N) - v(N \setminus \{i\}) = (\gamma - \beta) \sum_{\ell \in L_M: i \notin A^\ell} n_i^\ell = z_i & \forall i \in N, \\ \bar{z}(N) &= (\gamma - \beta) \sum_{\ell \in L_M} \sum_{i \in N \setminus A^\ell} n_i^\ell = (\gamma - \beta) \sum_{i \in N} \sum_{\ell \in L_M: i \notin A^\ell} n_i^\ell = z(N), \\ \bar{z}_i &\geq v(\{i\}) \geq 0 & \forall i \in N,\end{aligned}$$

hence the desired result. $\square$

Remark 4.13 If $L = L_M$, then the equal-split rule and the transaction-based rule coincides, so if z is given by transaction-based rule, we have $z = \tau(v)$. Also, since Theorem 4.12 implies that the core consists of only one point, this point must be the nucleolus, and therefore we have $z = \nu(v)$.

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