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Publication date:
1994

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Citation for published version (APA):

Herings, P. J. J., & Talman, A. J. J. (1994). *Intersection theorems with a continuum of intersection points*. (CentER Discussion Paper; Vol. 1994-79). Unknown Publisher.

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Intersection Theorems with a Continuum of Intersection Points *

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*This research is part of the VF-program “Competition and Cooperation”.

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Abstract

In all existing intersection theorems conditions are given under which a certain subset of a collection of sets has a non-empty intersection. In this paper conditions are formulated under which the intersection is a continuum of points satisfying some interesting topological properties. In this sense the intersection theorems considered in this paper belong to a new class. The intersection theorems are formulated on the unit cube and it is shown that both the vector of zeroes and the vector of ones lie in the same component of the intersection. This is interesting for some specific applications. The theorems give a generalization of the well-known lemmas of Knaster, Kuratowski, and Mazurkiewicz, of Sperner, of Shapley, and of Ichiishi. Moreover the results can be used to sharpen the usual formulation of the Sperner Lemma on the cube.

1 Introduction

In intersection theorems conditions are given under which a certain subset of a collection of sets has a non-empty intersection. Well-known intersection theorems on the unit simplex are given in Knaster, Kuratowski, and Mazurkiewicz (1929) (KKM Lemma), Sperner (1928) and Scarf (1967) (Sperner Lemma), Shapley (1973) (KKMS Lemma), Gale (1984) (Gale Lemma), and Ichiishi (1988) (Ichiishi Lemma). Intersection theorems can be used to prove the existence of solutions to mathematical programming problems, economic equilibrium existence problems, and solutions to game theoretic problems. The KKM Lemma and the Sperner Lemma can be used to prove Brouwer's fixed point theorem, and also to show the existence of an equilibrium in an exchange economy with or without production. Both the KKMS Lemma and the Ichiishi Lemma are very useful when showing the non-emptiness of the core of a cooperative game, see Shapley (1973), Ichiishi (1988), and Shapley and Vohra (1991). In Gale (1984) an intersection theorem is used to show the existence of an equilibrium in an economy with indivisible commodities. In order to prove the existence of a Nash equilibrium in a non-cooperative game it is useful to formulate intersection theorems on the cube or even more general the simplotope, see for example van der Laan and Talman (1993).

It is possible to generalize the above mentioned intersection theorems and to formulate intersection theorems on a polytope. In for example Ichiishi and Idzik (1991) an intersection theorem on a polytope is derived which generalizes both the KKM Lemma and the Gale Lemma. In van der Laan, Talman, and Yang (1994) a general theorem on the polytope is stated. Most of the results stated above can be derived from this theorem. Moreover, this theorem makes it possible to formulate analogs of the KKM, Sperner, KKMS, and Ichiishi Lemma on the polytope.

In all the intersection theorems stated above conditions are given under which a certain subset of a collection of sets has a non-empty intersection. The sets in the collection form a closed covering of a simplex, a simplotope, or a polytope. In this paper intersection theorems are formulated with a continuum of intersection points. Hence these intersection theorems belong to a new class. Let I_n denote the set of integers $\{1, \dots, n\}$ for some natural number n . Let $Q^n = \{q \in \mathbb{R}^n \mid 0 \leq q_j \leq 1, \forall j \in I_n\}$ denote the n -dimensional unit cube. Conditions are given on a collection of subsets covering the cube such that certain subsets of this collection have an intersection consisting of a continuum of points. Moreover, the intersection has some interesting topological properties. It will be shown that it has a component, i.e., a maximally connected subset, containing both the vertex being the vector of zeroes and the vertex being the vector of ones. Finally, the intersection theorems formulated in this paper generalize the KKM, Sperner, KKMS, and Ichiishi Lemma on the unit simplex and also lead to a strengthening of the usual formulation of

the Sperner Lemma on the cube. There is a close relationship between the intersection theorems of this paper and the equilibrium existence problem in economies with price rigidities as introduced in Drèze (1975). The intersection theorems of this paper can be used to show the existence of a continuum of equilibria in these economies.

In Section 2 some mathematical preliminaries are given. A correspondence satisfying the conditions of a total excess demand correspondence in an economy with price rigidities is introduced and using Browder's fixed point theorem some interesting properties of this correspondence are derived. These results are used in Section 3 to formulate several intersection theorems on the cube, among which the analogs of the KKM Lemma and the Sperner Lemma. Conditions are given under which a certain subset of a collection of subsets covering the cube has a non-empty intersection, containing, among other points, the vector of zeroes and the vector of ones. Moreover, the vector of zeroes and the vector of ones lie in the same component of the intersection. Using one of the intersection theorems of Section 3 it is possible to strengthen the usual formulation of the Sperner Lemma on the cube. In Section 3 also intersection theorems with a continuum of intersection points generalizing the KKMS Lemma and the Ichiishi Lemma are given. The proofs of all intersection theorems of Sections 3 are derived from the correspondence introduced in Section 2. Therefore this correspondence unifies these intersection theorems. In Section 4 attention is focused on some well-known intersection theorems on the unit simplex, where the existence of a non-empty intersection (in general not a continuum of points) of a certain subset of a collection of sets covering the simplex is guaranteed. Both the Sperner Lemma and the KKM Lemma, which are in some sense each others dual, will be derived from an intersection theorem of Section 3. It is also possible to derive the KKMS Lemma and its dual, the Ichiishi Lemma, from another intersection theorem given in Section 3.

2 Some Preliminaries

In the following 0^n will denote an n -dimensional vector containing only zeroes and 1^n an n -dimensional vector of ones. The closure of a subset S of some topological space will be denoted by $\text{cl}(S)$. The convex hull of a subset S of some Euclidean space will be denoted by $\text{co}(S)$. For the remainder of the paper it will be useful to consider a correspondence $\zeta : Q^n \rightarrow \mathbb{R}^n$ satisfying the following assumption.

Assumption A The correspondence $\zeta : Q^n \rightarrow \mathbb{R}^n$ satisfies:

1. ζ is a correspondence with a closed graph satisfying $\zeta(q)$ is non-empty and convex for every $q \in Q^n$ and $\bigcup_{q \in Q^n} \zeta(q)$ is bounded;

2. $\forall q \in Q^n$, $\exists z \in \zeta(q)$ such that for every $j \in I_n$, $q_j = 0$ implies $z_j \geq 0$, and $q_j = 1$ implies $z_j \leq 0$;
3. $\forall q \in Q^n$, $\forall z \in \zeta(q)$, $\exists p \in \mathbb{R}_{++}^n$ such that $p \cdot z = 0$.

Assumption A.1 guarantees that the correspondence ζ is upper semi-continuous. Assumption A.2 specifies a boundary condition for ζ and Assumption A.3 is equivalent to the condition that if $z \in \zeta(q)$ for some $q \in Q^n$, then $z_j > 0$ for any $j \in I_n$ implies $z_k < 0$ for some $k \in I_n$, and $z_j < 0$ for any $j \in I_n$ implies $z_k > 0$ for some $k \in I_n$. If $n = 1$, a correspondence ζ satisfying Assumption A necessarily equals the correspondence associating with every element in Q^1 the set $\{0\}$. A picture of a correspondence $\zeta : Q^2 \rightarrow \mathbb{R}^2$ satisfying Assumption A is drawn in Figure 1. In Figure 1 only the z_1 -axis has been drawn, which

Figure 1: A correspondence $\zeta : Q^2 \rightarrow \mathbb{R}^2$ satisfying Assumption A

is sufficient to determine the zero points of ζ by Assumption A.3. For every $q \in Q^2$ the set $\zeta(q)$ in Figure 1 consists of one element, except when $q_1 = 1$ or $q_2 = 0$. Notice that the set of zero points of ζ is given by $\{q \in Q^2 \mid q_1 = 1 \text{ or } q_2 = 0\}$.

Models of economies with price rigidities as introduced in Drèze (1975) yield excess demand correspondences satisfying Assumption A as is shown in Herings (1992). In Theorem 2.7 interesting properties of the set of zero points of a correspondence ζ satisfying Assumption A are given. Since the intersection theorems developed in the next section are

all proved using Theorem 2.7 this shows that there is a close relationship between the equilibrium existence problem in economies with price rigidities and the class of intersection theorems to be considered in this paper. In fact, it is possible to show the existence of a continuum of equilibria in economies with price rigidities using the intersection theorems of this paper.

The following definitions, which can be found in for instance Armstrong (1983), will be useful later on.

Definition 2.1 (Connectedness)

A topological space X is connected if it is not the union of two non-empty disjoint, closed sets.

A subset of a topological space is connected if it becomes a connected space when given the induced topology. Intuitively, a connected set is a set which is of one piece.

Definition 2.2 (Component)

The component of an element x in a topological space X is the union of all connected subsets of X containing x .

It is not difficult to show that a component is connected and considering Definition 2.1 the component of an element x in a topological space X is the largest connected subset of X containing x . The collection of all components of a topological space X partitions X .

Let a correspondence ζ satisfying Assumption A be given. Then the set Z_ζ of zero points of ζ is defined as follows, $Z_\zeta = \{q \in Q^n \mid 0^n \in \zeta(q)\}$. Theorem 2.7 states that there exists a connected set of zero points $Z_\zeta^c \subset Z_\zeta$ containing the two extreme points 0^n and 1^n of Q^n . In order to show Theorem 2.7 some preliminary work should be done first.

It will be useful in the proof of Theorem 2.7 to extend a correspondence ζ satisfying Assumption A such that it is defined on $\mathbb{R}^n$. For every $q \in Q^n$ choose an element $\hat{z}(q) \in \zeta(q)$ satisfying $\hat{z}_j(q) \geq 0$ if $q_j = 0$ and $\hat{z}_j(q) \leq 0$ if $q_j = 1$. Assumption A on ζ guarantees that $\hat{z}(q)$ can be chosen in this way for every $q \in Q^n$. For a non-empty compact set $S \subset \mathbb{R}^n$ define the correspondence $\Pi_S : \mathbb{R}^n \rightarrow S$ as the orthogonal projection on S , so $\Pi_S(x) = \{s' \in S \mid s' \in \arg \min_{s \in S} \|x - s\|_2\}$, $\forall x \in \mathbb{R}^n$. It is not difficult to show that Π_S is a continuous function if S is convex, see for example Mas-Colell (1985). Let $\hat{\zeta} : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be the correspondence with graph in $\mathbb{R}^n \times \mathbb{R}^n$ given by the set $\text{cl}(\{(x, \hat{z}(\Pi_{Q^n}(x))) \mid x \in \mathbb{R}^n\})$. Notice that component $j \in I_n$ of the projection function Π_{Q^n} is given by $(\Pi_{Q^n}(x))_j = \max\{0, \min\{x_j, 1\}\}$, $\forall x \in \mathbb{R}^n$.

Lemma 2.3

Let a correspondence $\zeta : Q^n \rightarrow \mathbb{R}^n$ satisfying Assumption A be given. Then the correspondence $\hat{\zeta}$ is non-empty valued and upper semi-continuous. For every $x \in \mathbb{R}^n$, $z \in \hat{\zeta}(x)$ implies $z \in \zeta(\Pi_{Q^n}(x))$. Moreover, for every $x \in \mathbb{R}^n$, $z \in \hat{\zeta}(x)$ and $x_j < 0$ implies $z_j \geq 0$, and $z \in \hat{\zeta}(x)$ and $x_j > 1$ implies $z_j \leq 0$.

Proof

Clearly, $\hat{\zeta}$ is a non-empty valued correspondence. It follows from the boundedness of the set $\cup_{q \in Q^n} \zeta(q)$ that the correspondence $\hat{\zeta}$ is upper semi-continuous. The second statement in Lemma 2.3 follows immediately using the definition of $\hat{\zeta}$ and the upper semi-continuity of ζ . Let $z \in \hat{\zeta}(x)$ for some $x \in \mathbb{R}^n$ with $x_j < 0$ for some $j \in I_n$. Then there exists a sequence $(x^r, z^r)_{r \in \mathbb{N}}$ such that $z^r = \hat{z}(\Pi_{Q^n}(x^r))$ and $\lim_{r \rightarrow \infty} (x^r, z^r) = (x, z)$. Since $x_j < 0$ there exists $r' \in \mathbb{N}$ such that $r \geq r'$ implies $x_j^r < 0$, hence $(\Pi_{Q^n}(x^r))_j = 0$, and $z_j^r \geq 0$. Consequently $z_j \geq 0$. It can be shown in a similar way that $z \in \hat{\zeta}(x)$ and $x_j > 1$ imply $z_j \leq 0$. Q.E.D.

Finally, define the correspondence $\bar{\zeta} : \mathbb{R}^n \rightarrow \mathbb{R}^n$ by $\bar{\zeta}(x) = \zeta(x)$, $\forall x \in Q^n$, and $\bar{\zeta}(x) = \text{co}(\hat{\zeta}(x))$, $\forall x \in \mathbb{R}^n \setminus Q^n$. The following lemma, shown in Todd (1976, p. 56, Theorem 1.4), is useful to derive some properties of $\bar{\zeta}$.

Lemma 2.4

Let $\Phi^1 : S \rightarrow T$ and $\Phi^2 : S \rightarrow T$ be upper semi-continuous correspondences, where $S \subset \mathbb{R}^k$ and $T \subset \mathbb{R}^m$.

1. Let $\Phi : S \rightarrow T$ be defined by $\Phi(s) = \Phi^1(s) \cup \Phi^2(s)$, $\forall s \in S$. Then Φ is upper semi-continuous.
2. Let S be closed and let $\tilde{S} \subset \mathbb{R}^k$ be such that $S \subset \tilde{S}$. Define $\Phi : \tilde{S} \rightarrow T$ by $\Phi(s) = \Phi^1(s)$, $\forall s \in S$, and $\Phi(s) = \emptyset$, $\forall s \in \tilde{S} \setminus S$. Then Φ is upper semi-continuous.
3. Define $\Phi : S \rightarrow T$ by $\Phi(s) = \text{co}(\Phi^1(s))$, $\forall s \in S$. Then Φ is upper semi-continuous.

Lemma 2.5

Let be given a correspondence $\zeta : Q^n \rightarrow \mathbb{R}^n$ satisfying Assumption A. Then $\bar{\zeta}$ is an upper semi-continuous correspondence, and $\bar{\zeta}(q)$ is non-empty and convex for every $q \in Q^n$. For every $x \in \mathbb{R}^n$, $z \in \bar{\zeta}(x)$ implies $z \in \zeta(\Pi_{Q^n}(x))$. Moreover, for every $x \in \mathbb{R}^n$, $z \in \bar{\zeta}(x)$ and $x_j < 0$ implies $z_j \geq 0$, and $z \in \bar{\zeta}(x)$ and $x_j > 1$ implies $z_j \leq 0$.

Proof

The correspondence $\bar{\zeta}$ is upper semi-continuous since the extension to $\mathbb{R}^n$ of the upper semi-continuous correspondence ζ defined on the closed set Q^n , obtained by assigning the empty set to every point $x \in \mathbb{R}^n \setminus Q^n$ is upper semi-continuous by Lemma 2.4.2, the union of this

extension of ζ with the correspondence $\hat{\zeta}$ is upper semi-continuous by Lemma 2.4.1, and a convexified upper semi-continuous correspondence is upper semi-continuous by Lemma 2.4.3. Clearly, $\bar{\zeta}(q)$ is non-empty and convex for every $q \in Q^n$. The other statements in Lemma 2.5 follow immediately from Lemma 2.3. Q.E.D.

The following lemma is a special case of Theorem 3 in Mas-Colell (1974) in the sense that in the theorem convex-valued correspondences are considered, while Mas-Colell treats the more general case of contractible valued correspondences. It is a generalization of Theorem 2 in Browder (1960, p. 186) where the case with continuous functions is considered.

Lemma 2.6

Let $S \subset \mathbb{R}^n$ be a non-empty, compact, convex set, let the correspondence $\Psi : S \times [0, 1] \rightarrow S$ be upper semi-continuous, and let $\Psi(s, t)$ be non-empty and convex, $\forall (s, t) \in S \times [0, 1]$. Then the set $F_\Psi = \{(s, t) \in S \times [0, 1] \mid s \in \Psi(s, t)\}$ contains a component F_Ψ^c such that $(S \times \{0\}) \cap F_\Psi^c \neq \emptyset$ and $(S \times \{1\}) \cap F_\Psi^c \neq \emptyset$.

Theorem 2.7 will be proved as an application of Lemma 2.6.

Theorem 2.7

Let a correspondence $\zeta : Q^n \rightarrow \mathbb{R}^n$ satisfying Assumption A be given. Then the set Z_ζ contains a component Z_ζ^c containing 0^n and 1^n .

Proof

Let Z be a compact, convex set containing $\cup_{q \in Q^n} \zeta(q)$. Define the set $R = \{r \in \mathbb{R}^n \mid \sum_{j=1}^n r_j = 0, r_j \geq -1, \forall j \in I_n\}$. Clearly, the set R is non-empty, compact, and convex. Define the correspondence $\alpha : Z \rightarrow R$ by

$$\alpha(z) = \{\bar{r} \in R \mid \bar{r} \cdot z \geq r \cdot z, \forall r \in R\}, \forall z \in Z.$$

Using the maximum theorem, see for example Hildenbrand (1974, p. 30), it follows immediately that the correspondence α is upper semi-continuous. Define the correspondence $\beta : R \times [1 - n, 2] \rightarrow Z$ by

$$\beta(r, t) = \bar{\zeta}(r + t1^n), \forall (r, t) \in R \times [1 - n, 2].$$

Then β is upper semi-continuous because of the upper semi-continuity of $\bar{\zeta}$ and the continuity of the function assigning $r + t1^n$ to $(r, t) \in R \times [1 - n, 2]$. Define the correspondence $\Psi : Z \times R \times [1 - n, 2] \rightarrow Z \times R$ by

$$\Psi(z, r, t) = \beta(r, t) \times \alpha(z), \forall (z, r, t) \in Z \times R \times [1 - n, 2].$$

Then, obviously, Ψ is upper semi-continuous and non-empty valued. The set $Z \times R$ is non-empty, compact, and convex. Clearly the set $\beta(r, t)$ is convex for every $(r, t) \in R \times [1 - n, 2]$.

Using the convexity of R and the linearity in r of $r \cdot z$ it follows that the set $\alpha(z)$ is convex for every $z \in Z$. Using a straightforward generalization of Lemma 2.6 it follows that there exists a component F_Ψ^c of the set $F_\Psi = \{(z, r, t) \in Z \times R \times [1 - n, 2] \mid (z, r) \in \Psi(z, r, t)\}$ such that $F_\Psi^c \cap (Z \times R \times \{1 - n\}) \neq \emptyset$ and $F_\Psi^c \cap (Z \times R \times \{2\}) \neq \emptyset$. Clearly, $(z^*, r^*, t^*) \in F_\Psi^c$ implies $(z^*, r^*) \in \Psi(z^*, r^*, t^*) = \bar{\zeta}(r^* + t^* 1^n) \times \alpha(z^*)$.

Suppose $\max_{j \in I_n} z_j^* > 0$. Since $z^* \in \zeta(\Pi_{Q^n}(r^* + t^* 1^n))$ there exists by Assumption A.3 some $p^* \in \mathbb{R}_{++}^n$ such that $p^* \cdot z^* = 0$ and therefore there exists a $k \in I_n$ with $z_k^* < 0$. It is easily verified that $r \in \alpha(z)$ for any $z \in Z$ with $z_j > z_{j'}$, $j, j' \in I_n$, implies $r_{j'} = -1$. Consequently $r_k^* = -1$. If $t^* < 1$, then $r_k^* + t^* < 0$ and since $z^* \in \bar{\zeta}(r^* + t^* 1^n)$ this implies $z_k^* \geq 0$, a contradiction. Consider the case $t^* \geq 1$. By definition of α there exists $j' \in I_n$ such that $z_{j'}^* = \max_{j \in I_n} z_j^* > 0$ and $r_{j'}^* > 0$. Hence $r_{j'}^* + t^* > 1$ and since $z^* \in \bar{\zeta}(r^* + t^* 1^n)$ this implies $z_{j'}^* \leq 0$, a contradiction. Consequently, we have that $\max_{j \in I_n} z_j^* \leq 0$. Since $p^* \in \mathbb{R}_{++}^n$ and $p^* \cdot z^* = 0$ this implies $z^* = 0^n$.

Consider the continuous function $f : Z \times R \times [1 - n, 2] \rightarrow Q^n$ defined by $f(z, r, t) = \Pi_{Q^n}(r + t 1^n)$, $\forall (z, r, t) \in Z \times R \times [1 - n, 2]$. Due to the fact that the image of a connected set by a continuous function is connected it holds that $f(F_\Psi^c) \subset Q^n$ is connected. If $q^* \in f(F_\Psi^c)$, then $q^* = \Pi_{Q^n}(r^* + t^* 1^n)$ for some $(z^*, r^*, t^*) \in F_\Psi^c$. Hence $0^n = z^* \in \bar{\zeta}(r^* + t^* 1^n) \subset \zeta(\Pi_{Q^n}(r^* + t^* 1^n)) = \zeta(q^*)$. Consequently, $f(F_\Psi^c) \subset Z_\zeta$. Next, consider the points $(0^n, r^1, 1 - n) \in F_\Psi^c$ and $(0^n, r^2, 2) \in F_\Psi^c$. By definition, $f(0^n, r^1, 1 - n) = \Pi_{Q^n}(r^1 + (1 - n) 1^n)$. Since $r^1 \in R$ it holds for every $j \in I_n$ that $r_j^1 \leq n - 1$, and consequently $\Pi_{Q^n}(r^1 + (1 - n) 1^n) = 0^n$. Since $r^2 \in R$ it holds for every $j \in I_n$ that $r_j^2 \geq -1$, and consequently $\Pi_{Q^n}(r^2 + 2 \cdot 1^n) = 1^n$. Hence $0^n, 1^n \in f(F_\Psi^c)$. Therefore, the set Z_ζ contains a component Z_ζ^c containing both 0^n and 1^n . Q.E.D.

Theorem 2.7 will turn out to be a very useful tool for proving a number of intersection theorems in the next section. Since Theorem 2.7 is used in the proof of all these intersection theorems, Theorem 2.7 can be seen as a unifying theorem.

3 Intersection Theorems with a Continuum of Intersection Points

In the following, for $q \in Q^n$, $I^0(q) = \{j \in I_n \mid q_j = 0\}$, $I^1(q) = \{j \in I_n \mid q_j = 1\}$, $s^0(q)$ denotes the number of elements in the set $I^0(q)$, $s^1(q)$ denotes the number of elements in the set $I^1(q)$, and $s(q) = s^0(q) + s^1(q)$. The j -th unit vector in $\mathbb{R}^n$ will be denoted by e^j . In Theorem 3.1 it will be assumed that if an index $j \in I_n$ is taken, then $j + 1 = 1$ if $j = n$, and $j - 1 = n$ if $j = 1$.

Theorem 3.1

Let $C^1, \dots, C^n$ be closed subsets of Q^n satisfying $\cup_{j=1}^n C^j = Q^n$. Moreover, if for $q \in Q^n$, $q_j = 0$ or $q_{j+1} = 1$, then $q \in C^j$. Then there exists a connected set S in Q^n such that $0^n, 1^n \in S$ and $S \subset \cap_{j=1}^n C^j$.

Proof

Define for every $q \in Q^n$ the set $J(q) = \{j \in I_n \mid q \in C^j\}$. Let $t(q)$ denote the number of elements in the set $J(q)$. Define the correspondence $\zeta : Q^n \rightarrow \mathbb{R}^n$ by

$$\zeta(q) = \text{co} \left(\left\{ e^j - \frac{1}{n} 1^n \mid j \in J(q) \right\} \right).$$

First we verify that ζ satisfies Assumption A. From Lemma 2.4 it follows immediately, using the closedness of the sets C^j , $\forall j \in I_n$, that ζ is upper semi-continuous. Clearly, $\zeta(q)$ is non-empty and convex for every $q \in Q^n$ and $\cup_{q \in Q^n} \zeta(q)$ is bounded. Hence Assumption A.1 is satisfied by ζ .

If, for some $j \in I_n$, $q_j = 0$, then $q \in C^j$ and hence $e^j - \frac{1}{n} 1^n \in \zeta(q)$. If, for some $j \in I_n$, $q_j = 1$, then $q \in C^{j-1}$ and hence $e^{j-1} - \frac{1}{n} 1^n \in \zeta(q)$. Three cases have to be considered.

1. If $q = 1^n$, then consider $z \in \zeta(q)$ with $z = \sum_{j \in I_n} \frac{1}{n} (e^j - \frac{1}{n} 1^n) = 0^n$. So it follows that $z_j \leq 0$ for every $j \in I_n$.
2. If $\forall j \in I_n$, $0 \leq q_j < 1$, then define $z = \sum_{j \in J(q)} \frac{1}{t(q)} (e^j - \frac{1}{n} 1^n)$. It follows immediately that $z \in \zeta(q)$ and $q_j = 0$ implies $z_j = \frac{1}{t(q)} - \frac{1}{n} \geq 0$.
3. If $\exists j_1 \in I_n$ with $q_{j_1} = 1$ and $\exists j_2 \in I_n$ with $q_{j_2} < 1$, then choose some $j' \in I_n$ satisfying $q_{j'} = 1$ and $q_{j'-1} \neq 1$. Define $z = \sum_{j \in I^0(q)} \frac{1}{n} e^j + (1 - \frac{s^0(q)}{n}) e^{j'-1} - \frac{1}{n} 1^n$. Then it is easily verified that $z \in \zeta(q)$. Moreover, $q_j = 0$ implies $z_j \geq \frac{1}{n} - \frac{1}{n} = 0$ and $q_j = 1$ implies $z_j = -\frac{1}{n} < 0$.

The Cases 1, 2, and 3 show that ζ satisfies Assumption A.2.

Since $\forall q \in Q^n$, $\forall z \in \zeta(q)$, $1^n \cdot z = 0$, also Assumption A.3 is satisfied by ζ .

If $0^n \in \zeta(q^*)$ for some $q^* \in Q^n$, or equivalently $q^* \in Z_\zeta$, then obviously $q^* \in C^j$ for every $j \in I_n$, so $q^* \in \cap_{j \in I_n} C^j$. By Theorem 2.7 there is a connected set of points $Z_\zeta^c \subset Z_\zeta$ satisfying $0^n \in Z_\zeta^c$ and $1^n \in Z_\zeta^c$. Consequently, S can be taken equal to Z_ζ^c . Q.E.D.

In Figure 2, Theorem 3.1 is illustrated for the case $n = 2$. The set $C^1 \cap C^2$ consists of four components, and one of them contains both the points $(0, 0)$ and $(1, 1)$. It should be mentioned that Figure 2 illustrates a rather nice case in the sense that the sets C^1 and C^2 have a fairly easy structure. Besides the boundary condition and the requirement that these two sets cover Q^2 , the only requirement made is that the sets C^1 and C^2 are closed. Hence in general much more complicated situations might arise. The above remark is true for all illustrations in the sequel. In Section 4 it will be shown that Theorem 3.1 immediately leads to the well-known Sperner Lemma, see Sperner (1928), Fan (1968), and Scarf (1967, 1973).

Figure 2: Illustration of Theorem 3.1, case $n = 2$.

It should be noticed that it is possible to replace the boundary condition $q_j = 0$ or $q_{j+1} = 1$ implies $q \in C^j$ by the more general condition that there exist permutations $\pi^1 = (\pi_1^1, \dots, \pi_n^1)$ and $\pi^2 = (\pi_1^2, \dots, \pi_n^2)$ of I_n such that for every $j \in I_n$, $\pi_j^1 \neq \pi_j^2$, and $q_{\pi_j^1} = 0$ or $q_{\pi_j^2} = 1$ implies $q \in C^j$. Theorem 3.1 corresponds to the choice $\pi^1 = (1, \dots, n)$ and $\pi^2 = (2, \dots, n, 1)$. Figure 3 gives an easy counterexample for the case in which the condition $\pi_j^1 \neq \pi_j^2$, $\forall j \in I_n$, is not satisfied. Figure 3 corresponds to the case $\pi^1 = (2, 1)$ and $\pi^2 = (2, 1)$. The set $C^1 \cap C^2$ consists of two components, one containing the point 0^n and the other one the point 1^n . The following theorem generalizes both Theorem 3.1 and the more general specification with the two permutations π^1 and π^2 .

Theorem 3.2

Let $C^1, \dots, C^n$ be closed subsets of Q^n satisfying $\cup_{j=1}^n C^j = Q^n$. Moreover, for any $q \in Q^n$, $q_j = 0$ implies $q \in C^j$, and $q_j = 1$ but $q \neq 1^n$ implies $q \in C^k$ for some $k \in I_n \setminus I^1(q)$. Then there exists a connected set S in Q^n such that $0^n, 1^n \in S$ and $S \subset \cap_{j=1}^n C^j$.

Proof

Define for every $q \in Q^n$ the set $J(q) = \{j \in I_n \mid q \in C^j\}$ and define the correspondence $\zeta : Q^n \rightarrow \mathbb{R}^n$ by

$$\zeta(q) = \text{co} \left(\left\{ e^j - \frac{1}{n} 1^n \mid j \in J(q) \right\} \right).$$

Similar to the proof of Theorem 3.1 it can be shown that ζ is an upper semi-continuous correspondence satisfying $\zeta(q)$ is non-empty and convex for every $q \in Q^n$ and $\cup_{q \in Q^n} \zeta(q)$

Figure 3: Counterexample, case $n = 2$.

is bounded. Moreover, $\forall q \in Q^n$, $\forall z \in \zeta(q)$, $1^n \cdot z = 0$. So Assumptions A.1 and A.3 are satisfied by ζ . Assumption A.2 remains to be verified. The case with $q = 1^n$ is considered first. For every $0 < \varepsilon \leq 1$ and for every $j \in I_n$ it holds that $1^n - \varepsilon e^j \in C^j$ using the assumptions of Theorem 3.2. Since C^j is closed for every $j \in I_n$, this implies that $1^n \in \cap_{j=1}^n C^j$ and hence $0^n \in \zeta(1^n)$. Next consider the case with $q \in Q^n \setminus \{1^n\}$ and $I^0(q) \cup I^1(q) \neq \emptyset$. Let $k \in I_n \setminus I^1(q)$ be such that $q \in C^k$. Define

$$z = \sum_{j \in I^0(q)} \frac{1}{s(q)} e^j + \frac{s^1(q)}{s(q)} e^k - \frac{1}{n} 1^n.$$

Obviously, $z \in \zeta(q)$. If $q_j = 0$, then $z_j \geq \frac{1}{s(q)} - \frac{1}{n} \geq 0$. If $q_j = 1$, then $z_j = -\frac{1}{n} < 0$. Hence Assumption A.2 is satisfied by ζ . By Theorem 2.7 there exists a connected set Z_ζ^c containing both 0^n and 1^n . If $q^* \in Z_\zeta^c$, then $0^n \in \zeta(q^*)$ and it is easily seen that this implies $q^* \in \cap_{j=1}^n C^j$. So the set Z_ζ^c satisfies all the requirements imposed on the set S . Q.E.D.

For the case $n = 2$ Theorems 3.1 and 3.2 are equivalent. For the case $n \geq 3$ Theorem 3.2 is clearly more general. Therefore it is also possible to derive the Sperner Lemma directly from Theorem 3.2. By symmetry considerations the following dual theorem follows as a corollary to Theorem 3.2.

Theorem 3.3

Let $C^1, \dots, C^n$ be closed subsets of Q^n satisfying $\cup_{j=1}^n C^j = Q^n$. Moreover, for any $q \in Q^n$,

$q_j = 1$ implies $q \in C^j$, and $q_j = 0$ but $q \neq 0^n$ implies $q \in C^k$ for some $k \in I_n \setminus I^0(q)$. Then there exists a connected set S such that $0^n, 1^n \in S$ and $S \subset \cap_{j=1}^n C^j$.

It will be shown in Section 4 that the well-known KKM Lemma of Knaster, Kuratowski, and Mazurkiewicz (1929) follows almost immediately from Theorem 3.3. Since Theorems 3.2 and 3.3 are completely symmetric it should be clear that the KKM Lemma can also be easily derived from Theorem 3.2. Similarly, the Sperner Lemma can be derived from Theorem 3.3.

So far intersection theorems have been considered where a statement is made about the intersection of all the sets covering Q^n . In for example the KKMS Lemma or the Ichiishi Lemma (see Shapley (1973) and Ichiishi (1988), respectively) a statement is made about the intersection of sets in certain subsets of the collection of sets covering Q^n . Theorem 3.4 is also an intersection theorem in this spirit. Moreover, unlike Theorems 3.1-3.3 it is completely symmetric with respect to the assumptions made on the sets in the cover of Q^n .

Theorem 3.4

Let $C^1, \dots, C^n$ and $D^1, \dots, D^n$ be closed subsets of Q^n satisfying $(\cup_{j=1}^n C^j) \cup (\cup_{j=1}^n D^j) = Q^n$. Moreover, for every $q \in Q^n$, $q_j = 0$ implies $q \in C^j$ and $q_j = 1$ implies $q \in D^j$. Then there exists a connected set S such that $0^n, 1^n \in S$ and $q^* \in S$ implies $q^* \in C^j \cap D^j$ for some $j \in I_n$, or $q^* \in \cap_{j=1}^n C^j$, or $q^* \in \cap_{j=1}^n D^j$.

Proof

Define $J^0(q) = \{j \in I_n \mid q \in C^j\}$ and $J^1(q) = \{j \in I_n \mid q \in D^j\}$. Notice that $I^0(q) \subset J^0(q)$ and $I^1(q) \subset J^1(q)$. Define the correspondence $\zeta : Q^n \rightarrow \mathbb{R}^n$ by

$$\zeta(q) = \text{co} \left(\left\{ e^j - \frac{1}{n} 1^n \mid j \in J^0(q) \right\} \cup \left\{ \frac{1}{n} 1^n - e^j \mid j \in J^1(q) \right\} \right).$$

Then it follows immediately from Lemma 2.4, using the closedness of the sets C^j and D^j , $\forall j \in I_n$, that ζ is upper semi-continuous. Moreover, $\zeta(q)$ is non-empty and convex for every $q \in Q^n$. Clearly $\cup_{q \in Q^n} \zeta(q)$ is bounded. Hence Assumption A.1 is satisfied by ζ . Since $\forall q \in Q^n$, $\forall z \in \zeta(q)$, $1^n \cdot z = 0$, Assumption A.3 is also satisfied by ζ .

Consider an element $q \in Q^n$ satisfying $I^0(q) \cup I^1(q) \neq \emptyset$. Define $z \in \zeta(q)$ by

$$z = \sum_{j \in I^0(q)} \frac{1}{s(q)} (e^j - \frac{1}{n} 1^n) + \sum_{j \in I^1(q)} \frac{1}{s(q)} (\frac{1}{n} 1^n - e^j).$$

If $q_j = 0$, then $z_j = \frac{1}{s(q)} - \frac{s^0(q)}{s(q)n} + \frac{s^1(q)}{s(q)n} \geq \frac{n-s^0(q)}{s(q)n} \geq 0$. Similarly it can be shown that $q_j = 1$ implies $z_j \leq \frac{s^1(q)-n}{s(q)n} \leq 0$. Hence ζ satisfies Assumption A.2.

By Theorem 2.7 there is a connected set of points Z_ζ^c satisfying $0^n, 1^n \in Z_\zeta^c$ and $q^* \in Z_\zeta^c$

implies $0^n \in \zeta(q^*)$. Let 0^n be an element of $\zeta(q^*)$ for some $q^* \in Q^n$. Then there exists, for every $j \in I_n$, $\lambda^j \geq 0$ and $\mu^j \geq 0$ such that

$$\sum_{j=1}^n \lambda^j (e^j - \frac{1}{n} 1^n) + \sum_{j=1}^n \mu^j (\frac{1}{n} 1^n - e^j) = 0^n,$$

where $\lambda^j = 0$ if $q \notin C^j$ and $\mu^j = 0$ if $q \notin D^j$, and $\sum_{j=1}^n \lambda^j + \sum_{j=1}^n \mu^j = 1$. Define $\lambda = \sum_{j=1}^n \lambda^j$ and $\mu = \sum_{j=1}^n \mu^j$. For every $j \in I_n$ it holds that $\lambda^j - \frac{1}{n} \lambda = \mu^j - \frac{1}{n} \mu$. Hence $\lambda^j - \mu^j = \frac{1}{n} (\lambda - \mu)$, being independent of j . Three possibilities can occur.

If $\lambda > \mu$, then $\forall j \in I_n$, $\lambda^j - \mu^j > 0$, hence $\forall j \in I_n$, $\lambda^j > 0$, and consequently $q^* \in \cap_{j=1}^n C^j$.

If $\lambda = \mu$, then $\forall j \in I_n$, $\lambda^j = \mu^j$. Since for some $k \in I_n$, $\lambda^k > 0$ or $\mu^k > 0$, it holds that $q^* \in C^k \cap D^k$ for some $k \in I_n$.

If $\lambda < \mu$, then $q^* \in \cap_{j=1}^n D^j$.

Q.E.D.

Theorem 3.4 is illustrated in Figure 4 for the case $n = 2$. It is easily verified that the set

Figure 4: Illustration of Theorem 3.4, case $n = 2$.

$(C^1 \cap D^1) \cup (C^2 \cap D^2) \cup (C^1 \cap C^2) \cup (D^1 \cap D^2)$ consists of two components, one of them containing the points $(0, 0)$ and $(1, 1)$.

We will show that at least one point in the set S lies in the intersection of C^k and D^k for some index $k \in I_n$. It is even possible to show that $(\cap_{j=1}^n C^j) \cap D^k \neq \emptyset$ for some $k \in I_n$ and $(\cap_{j=1}^n D^j) \cap C^k \neq \emptyset$ for some $k \in I_n$.

Theorem 3.5

Let $C^1, \dots, C^n$ and $D^1, \dots, D^n$ be closed subsets of Q^n satisfying $(\cup_{j=1}^n C^j) \cup (\cup_{j=1}^n D^j) = Q^n$. Moreover, for every $q \in Q^n$, $q_j = 0$ implies $q \in C^j$ and $q_j = 1$ implies $q \in D^j$. Then there

exists a connected set S with the properties given in Theorem 3.4 and points $s^1, s^2 \in S$ such that $s^1 \in \cup_{k=1}^n ((\cap_{j=1}^n C^j) \cap D^k)$ and $s^2 \in \cup_{k=1}^n ((\cap_{j=1}^n D^j) \cap C^k)$.

Proof

By Theorem 3.4 there exists a connected set S such that $0^n, 1^n \in S$, and $q^* \in S$ implies $q^* \in C^j \cap D^j$ for some $j \in I_n$, or $q^* \in \cap_{j=1}^n C^j$, or $q^* \in \cap_{j=1}^n D^j$. Clearly, S can be chosen such that it is a closed set. Define $J^0(q)$ and $J^1(q)$ as in the proof of Theorem 3.4 and let $j^0(q)$ and $j^1(q)$ denote the number of elements in these sets, respectively. Define the correspondence $\Phi^0 : S \rightarrow \mathbb{R}$ by $\Phi^0(q) = \{j^0(q)\}$ if $j^0(q) > 0$ and $\Phi^0(q) = \emptyset$ if $j^0(q) = 0$, $\forall q \in S$. Define the correspondence $\Phi^1 : S \rightarrow \mathbb{R}$ by $\Phi^1(q) = \{-j^1(q)\}$ if $j^1(q) > 0$ and $\Phi^1(q) = \emptyset$ if $j^1(q) = 0$, $\forall q \in S$. Finally, define the correspondence $\Phi : S \rightarrow \mathbb{R}$ by $\Phi(q) = \text{co}(\Phi^0(q) \cup \Phi^1(q))$, $\forall q \in S$. Using Lemma 2.4 and Theorem 3.4 it follows easily that Φ is an upper semi-continuous correspondence and $\Phi(q)$ is non-empty and convex for every $q \in Q^n$. It will be shown that $\Phi(S)$ is a connected subset of $\mathbb{R}$ and hence an interval. Suppose $\Phi(S)$ is not connected, then it can be partitioned in two non-empty disjoint sets T^1 and T^2 , both being closed in $\Phi(S)$. By Proposition 1 of Hildenbrand (1974, p.22) both $\Phi^{-1}(T^1)$ and $\Phi^{-1}(T^2)$ are closed in S . Suppose $\Phi^{-1}(T^1) \cap \Phi^{-1}(T^2) \neq \emptyset$ then there exists an element $q \in S$ and points t^1 and t^2 such that $t^1 \in \Phi(q) \cap T^1$ and $t^2 \in \Phi(q) \cap T^2$. Since Φ is convex-valued it holds that $(1 - \lambda)t^1 + \lambda t^2 \in \Phi(q)$ for every $\lambda \in [0, 1]$. Hence there exists a continuous function $f : [0, 1] \rightarrow \Phi(S)$ such that $f(0) = t^1$ and $f(1) = t^2$. So t^1 and t^2 are an element of the same component of $\Phi(S)$, a contradiction. Therefore, $\Phi^{-1}(T^1) \cap \Phi^{-1}(T^2) = \emptyset$. Clearly $\Phi^{-1}(T^1) \cup \Phi^{-1}(T^2) = S$. Hence S is not connected, a contradiction. Consequently $\Phi(S)$ is connected.

It holds that $0^n \in S$, $n \in \Phi(0^n)$, $1^n \in S$, and $-n \in \Phi(1^n)$. Therefore $\Phi(S) = [-n, n]$. Suppose that for every $k \in I_n$, $(\cap_{j=1}^n C^j) \cap D^k \cap S = \emptyset$. Then $\Phi((\cap_{j=1}^n C^j) \cap S) = \{n\}$ and $\Phi(S \setminus \cap_{j=1}^n C^j) \subset [-n, n - 1]$, so $\Phi(S) \subset [-n, n - 1] \cup \{n\}$, a contradiction. Consequently there is a point $s^1 \in S$ such that $s^1 \in (\cap_{j=1}^n C^j) \cap D^k$ for some $k \in I_n$. Similarly it can be shown that there is a point $s^2 \in S$ such that $s^2 \in (\cap_{j=1}^n D^j) \cap C^k$ for some $k \in I_n$. Q.E.D.

Theorem 3.5 strengthens the usual formulation of the analog of the Sperner Lemma on the cube (see Freund (1986) and van der Laan, Talman, and Yang (1994)), which is given in Corollary 3.6.

Corollary 3.6

Let $C^1, \dots, C^n$ and $D^1, \dots, D^n$ be closed subsets of Q^n satisfying $(\cup_{j=1}^n C^j) \cup (\cup_{j=1}^n D^j) = Q^n$. Moreover, for every $q \in Q^n$, $q_j = 0$ implies $q \in C^j$ and $q_j = 1$ implies $q \in D^j$. Then there is an index $k \in I_n$ such that $C^k \cap D^k \neq \emptyset$.

Next intersection theorems with a continuum of intersection points generalizing the KKMS

Lemma and the Ichiishi Lemma will be considered. In order to do this we first give a definition of a balanced collection of sets. Define for every non-empty subset T of I_n the vector e^T by $e_j^T = \frac{1}{|T|}$ if $j \in T$ and $e_j^T = 0$ if $j \notin T$, where $|T|$ denotes the number of elements in the set T . Define the vector $e^\emptyset$ by $e_j^\emptyset = \frac{1}{n}$, $\forall j \in I_n$. Denote the collection of all subsets of I_n by $\mathcal{T}_n$. Notice that $\emptyset \in \mathcal{T}_n$.

Definition 3.7

Let $\mathcal{B}$ be a non-empty collection of elements of $\mathcal{T}_n$, say $\mathcal{B} = \{T^1, \dots, T^m\}$. The collection $\mathcal{B}$ is balanced if there exist positive numbers $\lambda^1, \dots, \lambda^m$ such that $\sum_{i=1}^m \lambda^i = 1$ and $\sum_{i=1}^m \lambda^i e^{T^i} = \frac{1}{n} \mathbf{1}^n$.

Definition 3.7 is slightly more general than the usual definition of balancedness since the empty set is not excluded as an element of a balanced collection of sets. If only non-empty subsets of I_n are considered, then Definition 3.7 reduces to the usual definition of balancedness. In Section 4 it will be shown that the next theorem generalizes the KKMS Lemma and the Ichiishi Lemma.

Theorem 3.8

Let $\{C^T \mid T \in \mathcal{T}_n\}$ be a collection of closed subsets of Q^n satisfying $\cup_{T \in \mathcal{T}_n} C^T = Q^n$. Moreover, for every $q \in Q^n$ with $\emptyset \neq I^0(q) \neq I_n$, $q \in C^T$ for a set $T \in \mathcal{T}_n$ satisfying $I^0(q) \subset T$, and for every $q \in Q^n$ with $\emptyset \neq I^1(q) \neq I_n$, $q \in C^T$ for a set $T \in \mathcal{T}_n$ satisfying $I^1(q) \subset I_n \setminus T$. Then there exists a connected set S such that $0^n, 1^n \in S$ and for every $q^* \in S$ there is a balanced collection $\{T^1, \dots, T^m\}$ of sets in $\mathcal{T}_n$ such that $q^* \in \cap_{i=1}^m C^{T^i}$.

Proof

For every $q \in Q^n$ define the set $J(q) = \{T \in \mathcal{T}_n \mid q \in C^T\}$. Define the correspondence $\zeta : Q^n \rightarrow \mathbb{R}^n$ by

$$\zeta(q) = \text{co} \left(\left\{ e^T - \frac{1}{n} \mathbf{1}^n \mid T \in J(q) \right\} \right).$$

Using Lemma 2.4 and the closedness of the sets C^T , $\forall T \in \mathcal{T}_n$, it follows that ζ is upper semi-continuous. Clearly $\zeta(q)$ is non-empty and convex for every $q \in Q^n$, and $\cup_{q \in Q^n} \zeta(q)$ is bounded, so Assumption A.1 is satisfied by ζ . Assumption A.3 is trivially satisfied by ζ .

Consider the point $q = 0^n$. Due to the boundary condition it holds for every $j \in I_n$ and for every $\varepsilon \in (0, 1]$ that the point $0^n + \varepsilon e^j$ belongs to $C^{I_n \setminus \{j\}}$ or to C^{I_n} . Hence, since each C^T is closed, $0^n \in \cap_{j \in I_n} C^{I_n \setminus \{j\}}$ or $0^n \in C^{I_n}$. Clearly, both the collection $\{I_n \setminus \{j\} \in \mathcal{T}_n \mid j \in I_n\}$ and the collection $\{I_n\}$ are balanced and therefore $0^n \in \zeta(0^n)$. Similarly, since $1^n - \varepsilon e^j$ belongs to $C^{\{j\}}$ or to $C^\emptyset$ for every $\varepsilon \in (0, 1]$ it holds that $1^n \in \cap_{j \in I_n} C^{\{j\}}$ or $1^n \in C^\emptyset$. Hence $0^n \in \zeta(1^n)$ since both the collection $\{\{j\} \in \mathcal{T}_n \mid j \in I_n\}$ and the collection $\{\emptyset\}$ are balanced.

Consider a point $q \in Q^n \setminus \{0^n, 1^n\}$ with $I^0(q) \cup I^1(q) \neq \emptyset$. Let $T^0 \in \mathcal{T}_n$ be a set satisfying $I^0(q) \subset T^0$ and $q \in C^{T^0}$, and let $T^1 \in \mathcal{T}_n$ be a set satisfying $I^1(q) \subset I_n \setminus T^1$ and $q \in C^{T^1}$. If $T^0 = \emptyset$ or $T^1 = \emptyset$, then clearly $0^n \in \zeta(q)$ and Assumption A.2 is satisfied. Hence consider the case $T^0 \neq \emptyset$ and $T^1 \neq \emptyset$. Define $z = \frac{|T^0|}{n}e^{T^0} + \frac{n-|T^0|}{n}e^{T^1} - \frac{1}{n}1^n$. Clearly, $z \in \zeta(q)$. For $j \in I^0(q)$ it holds that $z_j \geq \frac{|T^0|}{n} \frac{1}{|T^0|} - \frac{1}{n} = 0$ and for $j \in I^1(q)$ that $z_j \leq \frac{|T^0|}{n} \frac{1}{|T^0|} - \frac{1}{n} = 0$. Consequently, Assumption A.2 is satisfied.

By Theorem 2.7 there is a connected set of points Z_ζ^c satisfying $0^n, 1^n \in Z_\zeta^c$ and $q^* \in Z_\zeta^c$ implies $0^n \in \zeta(q^*)$. Notice that $0^n \in \zeta(q^*)$ if and only if there exists a balanced collection $\{T^1, \dots, T^m\}$ of sets in $\mathcal{T}_n$ satisfying that $q^* \in \cap_{i=1}^m C^{T^i}$, i.e., there exists $\lambda^{*1}, \dots, \lambda^{*m} > 0$ with $\sum_{i=1}^m \lambda^{*i} = 1$ such that $q^* \in C^{T^i}$ for every $i \in I_m$ and $0^n = \sum_{i=1}^m \lambda^{*i} (\frac{1}{n}1^n - e^{T^i})$, hence $\sum_{i=1}^m \lambda^{*i} e^{T^i} = \frac{1}{n}1^n$. Q.E.D.

Since the boundary condition in Theorem 3.8 is not specified for $q = 0^n$ and $q = 1^n$, it is possible that $C^\emptyset = \emptyset$ or $C^{I_n} = \emptyset$. It should be noticed that the boundary condition specified in Theorem 3.8 is weaker than the condition that for every $q \in Q^n \setminus \{0^n, 1^n\}$ with $\emptyset \neq I^0(q) \cup I^1(q)$, $q \in C^T$ for a set $T \in \mathcal{T}_n$ satisfying $I^0(q) \subset T$ and $I^1(q) \subset I_n \setminus T$. Theorem 3.8 is illustrated in Figure 5. In the illustration n equals 2. In this low-dimensional

Figure 5: Illustration of Theorem 3.8, case $n = 2$.

case the only difference with Theorem 3.1 or Theorem 3.2 is the possibility of non-empty sets $C^\emptyset$ or $C^{\{1,2\}}$. In the case $n = 2$ the minimal balanced collections of sets are given by $\{C^\emptyset\}$, $\{C^{\{1,2\}}\}$, and $\{C^{\{1\}}, C^{\{2\}}\}$. It is easily verified that in Figure 5 the union over all balanced collections of sets $\mathcal{B}$ of the intersection of the sets in $\mathcal{B}$ consists of three components, with one component containing both the points 0^n and 1^n . In case $n \geq 3$ the

situation may be much more complicated than in Theorems 3.1 and 3.2.

By symmetry considerations Theorem 3.9 follows immediately as a corollary to Theorem 3.8. It will be shown in the next section that using Theorem 3.9 it is easy to derive the KKMS Lemma.

Theorem 3.9

Let $\{C^T \mid T \in \mathcal{T}_n\}$ be a collection of closed subsets of Q^n satisfying $\cup_{T \in \mathcal{T}_n} C^T = Q^n$. Moreover, for every $q \in Q^n$ with $\emptyset \neq I^0(q) \neq I_n$, $q \in C^T$ for a set $T \in \mathcal{T}_n$ satisfying $I^0(q) \subset I_n \setminus T$, and for every $q \in Q^n$ with $\emptyset \neq I^1(q) \neq I_n$, $q \in C^T$ for a set $T \in \mathcal{T}_n$ satisfying $I^1(q) \subset T$. Then there exists a connected set S such that $0^n, 1^n \in S$ and for every $q^ \in S$ there is a balanced collection $\{T^1, \dots, T^m\}$ of sets in $\mathcal{T}_n$ such that $q^* \in \cap_{i=1}^m C^{T^i}$.*

4 Intersection Theorems on the Unit Simplex

In this section a number of well-known intersection theorems on the $(n - 1)$ -dimensional unit simplex, $S^n = \{p \in \mathbb{R}_+^n \mid \sum_{j=1}^n p_j = 1\}$, will be shown to follow as corollaries to the theorems of Section 3. Theorem 3.1 leads to the Sperner Lemma. This Lemma is illustrated for the case $n = 3$ in Figure 6. In Figure 6 there is exactly one intersection

Figure 6: Illustration of Sperner Lemma, case $n = 3$.

point. In the proof of Theorem 4.1 a cover $\{C^1, \dots, C^n\}$ of S^n satisfying the conditions of Theorem 4.1 is extended in a straightforward way to a cover $\{\hat{C}^1, \dots, \hat{C}^n\}$ of Q^n satisfying the conditions of Theorem 3.1. Then it follows that there exists a connected set $\hat{S}$ such that $0^n, 1^n \in \hat{S}$ and $\hat{S} \subset \cap_{j=1}^n \hat{C}^j$. It will be shown that this connected set $\hat{S}$ has a non-empty

intersection with the unit simplex S^n . Finally it is shown that $\cap_{j=1}^n \hat{C}^j \cap S^n = \cap_{j=1}^n C^j$. In Theorem 4.1 it is again assumed that if an index $j \in I_n$ is taken, then $j + 1 = 1$ if $j = n$, and $j - 1 = n$ if $j = 1$.

Theorem 4.1 (Sperner Lemma)

Let $C^1, \dots, C^n$ be closed subsets of S^n satisfying $\cup_{j=1}^n C^j = S^n$. Moreover, $p_j = 0$ for some $j \in I_n$ implies $p \in C^j$. Then $\cap_{j=1}^n C^j \neq \emptyset$.

Proof

The case $n = 1$ is trivial, so consider the case $n \geq 2$. For every $j \in I_n$ define the set

$$\hat{C}^j = \{q \in Q^n \mid \Pi_{S^n}(q) \in C^j\} \cup \{q \in Q^n \mid q_j = 0\} \cup \{q \in Q^n \mid q_{j+1} = 1\}.$$

It will be shown that the sets $\hat{C}^1, \dots, \hat{C}^n$ satisfy the conditions of Theorem 3.1. It is easily verified that, using the closedness of the sets $C^1, \dots, C^n$, the continuity of the function Π_{S^n} , and the property that $\cup_{j=1}^n C^j = S^n$, for every $j \in I_n$ the set $\hat{C}^j$ is closed and $\cup_{j=1}^n \hat{C}^j = Q^n$. If, for $q \in Q^n$, $q_j = 0$ or $q_{j+1} = 1$, then clearly $q \in \hat{C}^j$. So the sets $\hat{C}^1, \dots, \hat{C}^n$ satisfy the conditions of Theorem 3.1. Hence there exists a connected set $\hat{S}$ such that $0^n, 1^n \in \hat{S}$ and $\hat{S} \subset \cap_{j=1}^n \hat{C}^j$. Define $f : \hat{S} \rightarrow \mathbb{R}$ by $f(s) = \sum_{j=1}^n s_j$, $\forall s \in \hat{S}$. Since the image of a connected set under a continuous function is connected, $f(0^n) = 0$ and $f(1^n) = n$, there exists $\hat{s} \in \hat{S}$ such that $f(\hat{s}) = 1$, or equivalently $\hat{s} \in \hat{S} \cap S^n \subset \cap_{j=1}^n (\hat{C}^j \cap S^n)$. Clearly $C^j \subset \hat{C}^j \cap S^n$, $\forall j \in I_n$. Suppose there exists an element $\hat{q} \in (\hat{C}^k \cap S^n) \setminus C^k$ for some $k \in I_n$. Then since $\hat{q} \in S^n$ implies that $\Pi_{S^n}(\hat{q}) = \hat{q}$ it holds that $\hat{q}_k = 0$ or $\hat{q}_{k+1} = 1$. Since $\hat{q}_k = 0$ implies $\hat{q} \in C^k$ it holds that $\hat{q}_{k+1} = 1$ and $\hat{q}_k > 0$, yielding a contradiction since $\hat{q} \in S^n$ with $n \geq 2$. Consequently $\hat{C}^j \cap S^n = C^j$, $\forall j \in I_n$, and $\hat{s} \in \cap_{j=1}^n C^j$. Q.E.D.

Theorem 3.3 leads to the KKM Lemma. This lemma is illustrated in Figure 7 for the case $n = 3$. In the proof of Theorem 4.2 a cover $\{C^1, \dots, C^n\}$ of S^n satisfying the conditions of Theorem 4.2 is extended in more or less the same straightforward way as in the proof of Theorem 4.1 to yield a cover $\{\hat{C}^1, \dots, \hat{C}^n\}$ of Q^n satisfying the conditions of Theorem 3.3. Some notation is introduced first. For $J \in \mathcal{T}_n$ let $Q^n(J)$ denote the set $\{q \in Q^n \mid q_j = 0, \forall j \in J\}$. Define $S^n(J) = \{p \in S^n \mid p_j = 0, \forall j \in J\} = S^n \cap Q^n(J)$. Notice that $Q^n(\emptyset) = Q^n$, $S^n(\emptyset) = S^n$, and $S^n(J) \neq \emptyset$ if and only if J is a proper subset of I_n . Denote the collection of all proper subsets of I_n by $\mathcal{T}'_n$, so $\mathcal{T}'_n = \mathcal{T}_n \setminus \{I_n\}$.

Theorem 4.2 (KKM Lemma)

Let $C^1, \dots, C^n$ be closed subsets of S^n satisfying $\cup_{j=1}^n C^j = S^n$. Moreover, for every $p \in S^n$ with $I^0(p) \neq \emptyset$ there exists some $j \in I_n \setminus I^0(p)$ such that $p \in C^j$. Then $\cap_{j=1}^n C^j \neq \emptyset$.

Figure 7: Illustration of KKM Lemma, case $n = 3$.**Proof**

For every $j \in I_n$ define the set

$$\hat{C}^j = \cup_{J \in \mathcal{T}'_n} \{q \in Q^n(J) \mid \Pi_{S^n(J)}(q) \in C^j\} \cup \{q \in Q^n \mid q_j = 1\}.$$

It will be shown that the sets $\hat{C}^1, \dots, \hat{C}^n$ satisfy the conditions of Theorem 3.3. Using the closedness of the sets $C^1, \dots, C^n$, the continuity of the function $\Pi_{S^n(J)}$ for every proper subset J of I_n , and the fact that $\cup_{j=1}^n C^j = S^n$, it holds that for every $j \in I_n$ the set $\hat{C}^j$ is closed and $\cup_{j=1}^n \hat{C}^j = Q^n$. Clearly, $q \in Q^n$ and $q_j = 1$ implies $q \in \hat{C}^j$. Consider $q \in Q^n \setminus \{0^n\}$ with $I^0(q) \neq \emptyset$. Then $q \in Q^n(I^0(q))$ and $\exists k \in I_n \setminus I^0(\Pi_{S^n(I^0(q))}(q)) \subset I_n \setminus I^0(q)$ such that $\Pi_{S^n(I^0(q))}(q) \in C^k$, so $q \in \hat{C}^k$. Consequently the sets $\hat{C}^1, \dots, \hat{C}^n$ satisfy the conditions of Theorem 3.3 and there exists a connected set $\hat{S}$ such that $0^n, 1^n \in \hat{S}$ and $\hat{S} \subset \cap_{j=1}^n \hat{C}^j$. As in the proof of Theorem 4.1 it follows that there exists $\hat{s} \in \hat{S} \cap S^n \subset \cap_{j=1}^n (\hat{C}^j \cap S^n)$. Clearly $C^j \subset \hat{C}^j \cap S^n$. Suppose there exists an element $\hat{q} \in (\hat{C}^k \cap S^n) \setminus C^k$ for some $k \in I_n$. Since $\Pi_{S^n(J)}(\hat{q}) = \hat{q}$ if $\hat{q} \in S^n \cap Q^n(J)$ it follows that $\hat{q}_k = 1$. If $\hat{q}_j = 0$ for every $j \in I_n \setminus \{k\}$, then it follows by the conditions of Theorem 4.2 that $\hat{q} \in C^k$. Hence $\hat{q}_j > 0$ for some $j \neq k$, giving a contradiction since $\hat{q}_k = 1$ and $\hat{q} \in S^n$. Consequently $\hat{C}^j \cap S^n = C^j$, $\forall j \in I_n$, and $\hat{s} \in \cap_{j=1}^n C^j$. Q.E.D.

In Theorem 4.3 the Ichiishi lemma (see Ichiishi (1988)) is derived from Theorem 3.8. The Ichiishi lemma is illustrated in Figure 8. In Figure 8 there is exactly one point for which there is a balanced collection of sets in the cover having an intersection containing this point. In the illustration this balanced collection is not uniquely determined. Both the

Figure 8: Illustration of Ichiishi Lemma, case $n = 3$.

balanced collections $\{\{1, 2\}, \{3\}\}$ and $\{\{1, 3\}, \{2\}\}$ have a non-empty intersection consisting of the same point. Notice that in Figure 8 we have $C^{\{1\}} = \emptyset$ and $C^{\{1, 2, 3\}} = \emptyset$. Denote the collection of all non-empty subsets of I_n by $\mathcal{T}_n^*$, so $\mathcal{T}_n = \mathcal{T}_n^* \cup \{\emptyset\}$.

Theorem 4.3 (Ichiishi Lemma)

Let $\{C^T \mid T \in \mathcal{T}_n^*\}$ be a collection of closed subsets of S^n satisfying $\cup_{T \in \mathcal{T}_n^*} C^T = S^n$. Moreover, for every $p \in S^n$ with $I^0(p) \neq \emptyset$ there is a set $T \in \mathcal{T}_n^*$ satisfying $p \in C^T$ and $I^0(p) \subset T$. Then there is a balanced collection $\{T^1, \dots, T^m\}$ of sets in $\mathcal{T}_n^*$ such that $\cap_{i=1}^m C^{T^i} \neq \emptyset$.

Proof

The case where $C^{I_n} \neq \emptyset$ is trivial, hence consider the case $C^{I_n} = \emptyset$. Define $\hat{C}^\emptyset = \hat{C}^{I_n} = \emptyset$. Define for every $T \in \mathcal{T}_n \setminus \{\emptyset, I_n\}$

$$\hat{C}^T = \cup_{J \in \mathcal{T}_n'} \{q \in Q^n(J) \mid \Pi_{S^n(J)}(2q) \in C^T\} \cup \{q \in Q^n \mid I_n \setminus T \subset I^1(q)\}.$$

It will be shown that the collection of sets $\{\hat{C}^T \mid T \in \mathcal{T}_n\}$ satisfies the conditions of Theorem 3.8. Clearly, for every $T \in \mathcal{T}_n$, $\hat{C}^T$ is closed, and $\cup_{T \in \mathcal{T}_n} \hat{C}^T = Q^n$. Moreover, if $q \in Q^n \setminus \{0^n\}$ with $I^0(q) \neq \emptyset$, then $q \in Q^n(I^0(q))$ and $\exists T \in \mathcal{T}_n$ such that $I^0(\Pi_{S^n(I^0(q))}(2q)) \subset T$ and $\Pi_{S^n(I^0(q))}(2q) \in C^T$. Hence $q \in \hat{C}^T$ while $I^0(q) \subset I^0(\Pi_{S^n(I^0(q))}(2q)) \subset T$. If $q \in Q^n \setminus \{1^n\}$ and $I^1(q) \neq \emptyset$, then $q \in \hat{C}^{I_n \setminus I^1(q)}$. Consequently the collection $\{\hat{C}^T \mid T \in \mathcal{T}_n\}$ satisfies the conditions of Theorem 3.8 and there exists a connected set $\hat{S}$ with the properties stated in Theorem 3.8. As in the proof of Theorem 4.1 it follows that there exists $\hat{s} \in \hat{S} \cap \{q \in Q^n \mid \sum_{j=1}^n q_j = \frac{1}{2}\}$. Hence there is a balanced collection $\{T^1, \dots, T^m\}$ of sets in $\mathcal{T}_n \setminus \{\emptyset, I_n\}$

such that $\hat{s} \in \cap_{i=1}^m \hat{C}^{T^i}$. Since $\hat{s}_j \neq 1$, $\forall j \in I_n$, it holds that $\forall i \in I_m$, $\exists J \in \mathcal{T}'_n$ such that $\hat{s} \in Q^n(J)$ and $\Pi_{S^n(J)}(2\hat{s}) \in C^{T^i}$. Since $\hat{s} \in Q^n(J)$ and $\sum_{j=1}^n \hat{s}_j = \frac{1}{2}$ implies $\Pi_{S^n(J)}(2\hat{s}) = 2\hat{s}$ it holds that $2\hat{s} \in \cap_{i=1}^m C^{T^i}$. Q.E.D.

In Theorem 4.3 a cover of S^n with sets in $\mathcal{T}_n^* = \mathcal{T}_n \setminus \{\emptyset\}$ is considered, which is the usual formulation. Clearly, the statement of Theorem 4.3 is still true if a cover with sets in $\mathcal{T}_n$ is considered, since in the case $C^\emptyset \neq \emptyset$ Theorem 4.3 is trivially true. It is clear that also Theorem 3.9 can be used to derive the Ichiishi Lemma. Similarly the KKMS Lemma can easily be derived from both Theorem 3.8 and Theorem 3.9. In Theorem 4.4 the derivation using Theorem 3.9 will be shown. Theorem 4.4 is illustrated in Figure 9 for the case $n = 3$. In the illustration the unique intersection point is given by the intersection of the sets in

Figure 9: Illustration of KKMS Lemma, case $n = 3$.

the balanced collection $\{\{1, 2\}, \{1, 3\}, \{2, 3\}\}$.

Theorem 4.4 (KKMS Lemma)

Let $\{C^T \mid T \in \mathcal{T}_n^\}$ be a collection of closed subsets of S^n satisfying $\cup_{T \in \mathcal{T}_n^*} C^T = S^n$. Moreover, for every $p \in S^n$ with $I^0(p) \neq \emptyset$ there is a set $T \in \mathcal{T}_n^*$ satisfying $p \in C^T$ and $I^0(p) \subset I_n \setminus T$. Then there is a balanced collection $\{T^1, \dots, T^m\}$ of sets in $\mathcal{T}_n^*$ such that $\cap_{i=1}^m C^{T^i} \neq \emptyset$.*

The proof goes along the same lines as the proof of Theorem 4.3 by using Theorem 3.9 instead of Theorem 3.8 and defining for every $T \in \mathcal{T}_n \setminus \{\emptyset, I_n\}$

$$\hat{C}^T = \cup_{J \in \mathcal{T}'_n} \{q \in Q^n(J) \mid \Pi_{S^n(J)}(2q) \in C^T\} \cup \{q \in Q^n \mid T \subset I^1(q)\}.$$

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C^1

C^2

c^1

c^2