Tilburg University

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Publication date:
2021

Document Version
Early version, also known as pre-print

[Link to publication in Tilburg University Research Portal](#)

Citation for published version (APA):
van Buggenum, H. (2021). *Coexistence of Money and Interest-Bearing Bonds*. (CentER Discussion Paper; Vol. 2021-019). CentER, Center for Economic Research.

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No. 2021-019

**COEXISTENCE OF MONEY AND
INTEREST-BEARING BONDS**

By

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19 July 2021

ISSN 0924-7815
ISSN 2213-9532

Coexistence of Money and Interest-Bearing Bonds*

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July 19, 2021

Abstract

This paper revisits how coexistence of money and bonds can make a society better off. For this purpose, a model is constructed in which payment instruments matter for settling real transactions and savings instruments matter because agents differ in how they discount future utility. Because bonds and money differ in their characteristics as payment and savings instruments, the model is able to explain the coexistence puzzle for an optimally chosen monetary policy. Such a policy trades-off efficiency in financial markets, in which money is traded for bonds, with efficiency in goods markets, in which money is traded for a real good. Financial markets can achieve a better distribution of savings when agents are constrained by their money holdings, but this is bad for efficiency in goods markets. The former effect can dominate the latter so that optimal policy deviates from the Friedman rule.

Keywords: new monetarism, coexistence puzzle, liquidity, financial markets.

JEL Codes: E40, E44, E50, E52, E62, G12.

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*This paper is based on Chapter 4 of my PhD dissertation, entitled *Banks and Financial Markets in Microfounded Models of Money*, at Tilburg University. I would like to thank Lukas Altermatt, Gaetano Antinolfi, Aleksander Berentsen, Jeffrey Campbell, Francesco Carli, Lucas Herrenbrueck, Tai-Wei Hu, Mohammed Ait Lahcen, Cyril Monnet, Florian Sniekers, Burak Uras, Chris Waller, Randall Wright, and Sophie Zhou for valuable discussions and/or comments. I also thank three anonymous referees, whose comments helped to improve this manuscript. Finally, I thank the participants at the Marrakech Workshop on Monetary Economics organized by the Bernoulli Center for Economics, Tilburg University's Macro Study Group and Internal Seminar Series, the KVS New Paper Session at Leiden University, the 2019 Summer Workshop on Money, Banking, Payments and Finance at the Bank of Canada, the 2019 Belgian Macro Workshop at the University of Gent, the 11th Workshop on Theoretical and Experimental Macroeconomics at the Technical University of Berlin, and the 2021 Society of Economic Dynamics Meetings at the University of Minnesota.

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1 Introduction

Why does fiat money coexist with risk-free nominal bonds that dominate money in terms of return? Since this question has been raised by Hicks (1935), monetary theorists have explained return dominance by arguing that bonds cannot be used to settle real transactions. Return dominance may then make a society worse off, since it implies opportunity costs associated with holding payment instruments. Friedman (1969) therefore argues in favor of zero nominal rates, a policy known as the Friedman rule. Given that coexistence has survived for a long time, a puzzle remains: How can a society be better off with interest-bearing bonds that are a savings instrument but not a payment instrument?

By taking into account the fact that people differ in how they discount future utility,¹ this paper provides an explanation for the coexistence of money and bonds as well as the sub-optimality of the Friedman rule. It develops a model with financial markets, goods markets, and agents facing idiosyncratic shocks to their subjective discount factors. After the realization of shocks, agents trade money and bonds in an over-the-counter financial market, and money and goods in a decentralized goods market. Besides frictions generating an essential role for money in payment, a core friction is that discount factor shocks are private information. Perfect insure against these shocks is therefore inviable, which inhibits an efficient distribution of savings.

The current paper demonstrates that in the presence of the frictions outlined above, the Friedman rule is a sub-optimal policy. The intuition for this result relates to the theory of the second-best (Lipsey & Lancaster, 1956): The Friedman rule maximizes efficiency in decentralized goods markets but inhibits financial markets in achieving an efficient distribution of savings. This is because at the margin, money and bonds are perfect substitutes under the Friedman rule. When policy deviates from the Friedman rule, opportunity costs associated with holding money balances arise and hence agents economize on these balances. Liquidity constraints then become binding, so that the marginal unit of money is valued as a payment instrument. This feature allows financial markets to partially make up for the lack of perfect insurance against discount factor shocks. The reason is that impatient agents care relatively much about the liquidity value of money compared to patient agents. Facing binding liquidity constraints, in the financial market the former therefore sell bonds at a discount to the latter. This implies a net-transfer of savings from impatient to patient agents, which enhances welfare.

Deviating from the Friedman rule also results in welfare losses as agents become constrained by their money holdings in the decentralized goods markets. Optimal policy trades-off welfare benefits from trade in financial markets against these welfare losses. As the benefits arising in financial markets depend positively on the measure of realized transactions, a critical threshold for the indirect liquidity of bonds - the ease of trading

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¹See Lawrance (1991), Carroll and Samwick (1997), Hendricks (2007) and Gelman (2021).

bonds for money - arises. Beyond that threshold, the Friedman rule is sub-optimal because of the reasons outlined above, and the optimal policy regime is characterized by the coexistence of money and interest-bearing bonds. Whether other institutions, like insurance à la Diamond and Dybvig (1983), are better ways to organize trade given the frictions studied here, is a question beyond the scope of this paper.

The current paper makes three contributions to the existing literature, which has been influenced primarily by Kocherlakota (2003). First, it develops a theory explaining coexistence by introducing agents experiencing idiosyncratic shocks to their subjective discount factors. This implies a meaningful role for the distribution of savings instruments, while existing explanations focus on the distribution of payment instruments. Second, the current theory demonstrates an essential role for financial markets, since these institutions channel savings from impatient agents to patient agents. Third, by incorporating matching frictions in financial markets, the theory also offers conditions for optimal coexistence dependent on how easy it is to trade money for bonds.

The current paper is also relevant from a policy perspective. Most importantly, it uncovers previously unexplored consequences of expansionary monetary policies which flood the economy with liquidity to achieve very low interest rates: These policies impair an essential role for financial markets in achieving a favorable distribution of savings. At the same time, the model demonstrates how expansive monetary policies stimulate trade in markets with an essential role for money. Hence, the model rationalizes expansive monetary policies during temporary breakdowns in financial market activity, such as financial crises.² From a development angle, the paper also points towards the importance of accessible, well-developed financial markets.

Additionally, due to a bargaining friction, I find that trading volumes within financial market meetings may be inefficiently large. By restricting the supply of bonds, inefficiently large trading volumes can be suppressed and socially efficient outcomes achieved. This points towards the fact that bonds are special in the sense that they achieve better allocations than when agents could trade equity, the supply of which is not controlled by government, or could use credit in financial market transactions. As an alternative to limiting bond supply, government could also introduce a financial transactions tax.

My theoretical results arise in a standard new monetarist model of money and financial markets, generalized to incorporate stochastic discount factor shocks. This approach allows the distribution of savings to matter without giving up the tractable quasi-linear utility structure pioneered by Lagos and Wright (2005). Due to the timing of preference shocks, by setting the real return on money equal to the inverse of the average subjective discount factor across agents, at the Friedman rule all liquidity constraints are slack. A deviation from the Friedman rule however implies that some liquidity constraints bind.

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²In fact, the model can easily be extended with aggregate shocks to characterize optimal policies in the short-run, for instance as in Berentsen and Waller (2011, 2015).

Trade in decentralized goods markets therefore behaves similarly as in standard models, while trade in financial markets gives rise to non-standard effects as it typically leads to a redistribution of net-savings.³

Introducing discount factor shocks can be motivated for a variety of reasons. First, with infinitely lived agents as in the current model, there is an axiomatic foundation for random discount factor shocks (Higashi, Hyogo, & Takeoka, 2009). With finitely lived agents as in Zhu (2008), random discounting could represent a stochastic probability of death as in Blanchard (1985), or shocks to the degree of altruism as in Atkeson and Lucas (1992) and Farhi and Werning (2007). Because optimal policy in the current model maximizes lifetime utility of agents, it nests the aforementioned setups with finitely lived agents when those alive are in charge of policy. Also, agents face similar preferences upon entering a time period and therefore optimal policies are dynamically consistent.

To introduce financial markets characterized by matching frictions, the current paper builds on work by Geromichalos and Herrenbrueck (2016), who in turn build on work by Duffie, Gârleanu, and Pedersen (2005). This approach integrates different definitions of liquidity used in monetary economics - direct liquidity, acceptability of an asset in payment for goods - and finance - indirect liquidity, ease of trading an asset for money - into one framework. Such a unified framework helps to develop insights about how indirect and direct liquidity differ. I do so by studying the welfare effects of changing the liquidity characteristics of bonds and I find that ideally, bonds should provide maximum indirect liquidity but minimum direct liquidity. Finally, as a robustness check, I show that the coexistence puzzle can also be explained with Walrasian financial markets or when financial markets are replaced by competitive banks.

The remainder of this paper develops as follows. Section 2 briefly discusses the related literature. Section 3 explains the model environment and Section 4 characterizes equilibria. Section 5 considers optimal coexistence in a baseline economy and Section 6 studies the welfare effects of indirect and direct liquidity. Section 7 considers matters in an economy with Walrasian financial markets and competitive banks. Section 8 concludes the analysis. Proofs of propositions and lemmas are in Appendix D.

2 Related Literature

Early explanations of the coexistence puzzle, by e.g. Bryant and Wallace (1984) and Villamil (1988), rely on seignorage revenues being extracted more efficiently in a coexistence regime. Kocherlakota (2003) shows that in absence of tax considerations, a society can benefit from having both money and bonds. The mechanism identified by Kocherlakota (2003) is transitory and abstracts from costs of coexistence caused by reduced

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³In fact, abstracting from financial market activity, the economy behaves similarly as one in which agents face shocks to the marginal utility of consumption or production in the decentralized goods market.

trade in goods markets. Kocherlakota's (2003) insight has been reproduced in the models of Andolfatto (2011), Berentsen, Huber, and Marchesiani (2014), and Geromichalos and Herrenbrueck (2016). The last two papers use a different mechanism than identified by Kocherlakota (2003). Andolfatto (2011) demonstrates that the transitory mechanism identified by Kocherlakota (2003) persists in steady state. Nevertheless, Andolfatto (2011), Berentsen et al. (2014), and Geromichalos and Herrenbrueck (2016) use models in which the Friedman rule is optimal and at the Friedman rule, bonds are inessential.

The current paper goes a step further; even when the Friedman rule is implementable by lump-sum taxation, coexistence can arise in an optimal policy regime that deviates from the Friedman rule. Shi (1997) and Nosal (2011) obtain non-optimality of the Friedman rule in models where deviating from the Friedman rule increases trade in goods markets through an extensive margin effect. That means, more matches between buyers and sellers get realized. Shi (2008), who builds on Shi (1997), shows how coexistence of money and bonds can improve welfare when the Friedman rule is not optimal. The mechanism highlighted in the current paper differs from that of Shi (1997, 2008) and Nosal (2011), as it operates through trades within financial market matches while taking as given the measure of matches realized in both goods markets and financial markets.

Boel and Camera (2006) study an economy with permanently patient and impatient agents. In their framework, the real return on holding money is limited by patient agents' subjective discount factor. Therefore, when inflation is as low as possible, impatient agents still face a cost of carrying money and remain to be liquidity constrained in some markets. Boel and Camera (2006) show that coexistence of money and bonds helps to re-distribute money to impatient agents, thereby improving the distribution of money. In the current framework, coexistence helps to improve the distribution of savings, but hurts the distribution of money. Boel and Waller (2019) study the same setup as Boel and Camera (2006) but with idiosyncratic shocks to agents' subjective discount factors. At the Friedman rule, impatient agents are, for a similar reason as in Boel and Camera (2006), constrained by their money holdings. With a different timing of preference shocks, the current model does not share this feature; at the Friedman rule no agent is constrained by his or her money holdings. Moreover, in the models of Boel and Camera (2006) and Boel and Waller (2019), there is no role for financial markets.

Finally, frictional goods markets in the current model relate to work of Trejos and Wright (2016). They study monetary exchange with assets that bear heterogeneous valuations, but consider only a single indivisible asset and do not explicitly separate goods markets from financial markets. Moreover, in the model of Trejos and Wright (2016) heterogeneous asset valuations arise because agents care differently about dividends. In the current model, agents care differently about future utility. This allows for heterogeneous valuations of fiat money, which bears zero dividend by definition.

3 The Environment

Time is discrete and continues forever. In each time period $t \geq 0$, three markets convene sequentially: (i) a *centralized market* (CM), (ii) an *over-the-counter financial market* (OTC), and (iii) a *decentralized goods market* (DGM). The economy is populated by a unit measure of infinitely lived households and a government. There are two goods that are both fully perishable and perfectly divisible: *CM goods* and *DGM goods*. CM and DGM goods can be traded only in, respectively, the CM and the DGM.

Government has a monopoly on issuing three storable and tradable nominal objects, labeled as *money*, *notes*, and *bonds*. Money is intrinsically useless and perfectly divisible, and notes and bonds are perfectly divisible claims on money. Without loss, I only consider one period lived notes and bonds that when issued in CM t , mature in CM $t + 1$. Both assets are normalized to pay one unit of money at maturity and cannot be redeemed prematurely. Money, notes, and bonds can always be traded in the CM and the OTC. However, only notes and money can be traded in the DGM. Bonds therefore provide only some indirect liquidity. Moreover, money and notes are not the same because money can always be traded in the DGM, while notes cannot always be traded. Notes thus combine features of bonds and money, and introducing them in the model allows to assess the difference between indirect and direct liquidity of assets other than money.

3.1 Households

Period t flow utility enjoyed by a household is given by the function

$$\mathcal{U}(y, \bar{y}, q, \bar{q}) = U(y) - \bar{y} + u(q) - \bar{q},$$

where y denotes consumption of CM goods, $\bar{y}$ production of CM goods, q consumption of DGM goods, and $\bar{q}$ production of DGM goods. In the DGM there is a single-coincidence of wants as households cannot produce and consume simultaneously; with equal probability a household becomes either a buyer or a seller. Function u is twice continuously differentiable and satisfies $u(0) = 0$, $u' > 0$, $u'' < 0$, $\lim_{q \rightarrow 0} u'(q) = \infty$, and $\lim_{q \rightarrow \infty} u'(q) = 0$. Function U is also twice continuously differentiable, and satisfies $U' > 0$, $U'' < 0$, $\lim_{y \rightarrow 0} U'(y) = \infty$, and $\lim_{y \rightarrow \infty} U'(y) = 0$. Let q^* and y^* be such that $u'(q^*) = 1$ and $U'(y^*) = 1$.

Households discount utility from period $t + 1$ to period t with factor $\beta\delta_t$, where $\beta \in (0, 1)$. After CM t has convened, all households draw a $\delta_t \in \{\delta^I, \delta^P\}$ from a distribution that is the same for all households and all time periods. Let π^i be the probability that $\delta_t = \delta^i$, and normalize $0 < \delta^I < 1 < \delta^P$ and $\pi^I\delta^I + \pi^P\delta^P = 1$. During period t , I refer to households that draw $\delta_t = \delta^I$ as *impatient* and to those that draw $\delta_t = \delta^P$ as *patient*.

Let $\zeta_{\xi_t} = \prod_{s=1}^{s=t-1} \delta^{\xi_t(s)}$ and $\mathcal{P}_{\xi} = \prod_{s=1}^{s=t-1} \pi^{\xi_t(s)}$, where $\xi_t(s)$ is the s -th element of a

vector $\xi_t \in \{I, P\}^t$. Here, $\xi_t \in \{I, P\}^t$ is used to index a household in period t according to its past preference shocks. At the beginning of period $t = 0$, with full commitment and the possibility to transfer goods across households, households or a social planner would choose period t allocations to maximize

$$\begin{aligned} \sum_{\xi_t \in \{I, P\}^t} \mathcal{P}_{\xi_t} \zeta_{\xi_t} [U(y_{\xi_t}) - \bar{y}_{\xi_t} + u(q_{\xi_t})/2 - \bar{q}_{\xi_t}/2], \\ \text{s.t.} \quad \sum_{\xi_t \in \{I, P\}^t} \mathcal{P}_{\xi_t} [y_{\xi_t} - \bar{y}_{\xi_t}] = 0 \quad \text{and} \quad \sum_{\xi_t \in \{I, P\}^t} \mathcal{P}_{\xi_t} [q_{\xi_t} - \bar{q}_{\xi_t}] = 0. \end{aligned}$$

For $t = 0$, all households are the same and so $y = \bar{y} = y^*$ and $q = \bar{q} = q^*$. For $t \geq 1$, households differ and allocations satisfy $\zeta_{\xi_t} U'(y_{\xi_t}) = \zeta_{\xi_t} u'(q_{\xi_t}) = (\delta^I)^t$, $\pi_{\{I, \dots, I\}} \bar{y}_{\{I, \dots, I\}} = \sum_{\xi_t \in \{I, P\}^t} y_{\xi_t}$, and $\pi_{\{I, \dots, I\}} \bar{q}_{\{I, \dots, I\}} = \sum_{\xi_t \in \{I, P\}^t} q_{\xi_t}$; households that were impatient in all preceding periods ($\delta_s = \delta^I$ for $s = 0, \dots, t-1$) become producers in time period t .

Commitment to the allocations above is infeasible in the decentralized economy because agents are anonymous in the DGM and types are unobservable in the CM. Therefore, in the DGM sellers refuse to produce and as of period $t = 1$, all households will claim that they were patient. As argued by Kocherlakota (1998), money can help to alleviate sellers' lack of commitment to production. Because discount factor shocks are revealed already after the CM has convened, money may also help to overcome households' lack of commitment to type-contingent allocations. The current paper shows that introducing bonds and financial markets helps to further overcome this lack of commitment.

3.1.1 DGM for Households

The DGM is a decentralized market, with bilateral trade between buyers and sellers. Here, I make four assumptions. First, buyers meet with a seller only once, but sellers may meet with more than one buyer. This is to nest a setup with Walrasian pricing. Let σ_{ij} denote the measure of buyers with $\delta^i \in \{\delta^I, \delta^P\}$, matched to sellers with $\delta^j \in \{\delta^I, \delta^P\}$. Let $\Sigma = \{\sigma_{II}, \sigma_{IP}, \sigma_{PI}, \sigma_{PP}\}$ summarize the measure of realized matches. Second, to generate a role for money and notes as means of payment to overcome the single-coincidence of wants in the DGM, record-keeping and monitoring are sufficiently bad to render credit arrangements infeasible. Third, because money is often argued to be the most liquid asset in an economy, money can be used in all DGM transactions. Notes can only be used by a fraction $\chi \in (0, 1)$ of randomly selected buyers, where χ proxies the direct liquidity of notes. Bonds can never be used in the DGM; they are a mere savings instrument. Fourth, to avoid complexities related to asymmetric information in bargaining, matched households may observe each others preferences. Because of anonymity, this information about preferences can only be used within a match.

Consider a meeting between a buyer and a seller in DGM t . Let z be the value of

liquid assets held by the buyer, expressed in terms of CM $t + 1$ goods. Here, $z = m$ in a meeting where the buyer can only transfer money and $z = m + d$ when the buyer can also transfer notes, with m and d denoting the value of the buyer's money and note holdings. Let δ^i and δ^j denote the discount factor shock drawn by the buyer and seller, respectively. Suppose that the seller produces q goods for the buyer and that the buyer transfers assets worth p CM $t + 1$ goods to the seller. Let the value function of entering CM $t + 1$ with money holdings worth m CM goods, real note holdings worth d CM goods, and real bond holdings worth b CM goods be denoted with $W_{t+1}(m, d, b)$. Because of quasi-linear utility, conjecture that the function W is linear: $W_{t+1}(m, d, b) = m + d + b + \bar{W}_{t+1}$. Surplus from the DGM transaction then satisfies

$$\mathcal{S} = u(q) - q - \beta(\delta^i - \delta^j)p. \quad (1)$$

Like in Trejos and Wright (2016), not only the amount of traded goods matters for the surplus, but potentially also the amount of traded assets. Here, the reason is that agents care differently about the savings value of assets.

Individual rationality of the transaction requires $u(q) - \beta\delta^i p \geq 0$ and $\beta\delta^j p - q \geq 0$. Feasibility of the transaction requires $p \leq z$. When the constraint $p \leq z$ is slack, all trades on the Pareto frontier satisfy $\delta^j u'(q) = \delta^i$. Therefore, let q_{ij}^* satisfy $\delta^j u'(q_{ij}^*) = \delta^i$ and observe that $q_{IP}^* > q_{II}^* = q_{PP}^* > q_{PI}^*$; impatient agents have a stronger desire to consume compared to patient agents.

Instead of focusing on a specific market structure to pin down Σ and (q, p) , I follow an approach based on Gu and Wright (2016). For realized matches, it means that there exist strictly increasing and twice continuously differentiable pricing protocols $\Upsilon = \{v_{II}, v_{IP}, v_{PI}, v_{PP}\}$. Here, $v_{ij} : \mathbb{R}_+ \mapsto \mathbb{R}_+$ maps the amount of traded goods in a DGM t match between a buyer with δ^i and a seller with δ^j , into the value of traded assets (expressed in CM $t + 1$ goods).

Assumption 1. $u'(q)/v'_{ij}(q)$ is strictly decreasing in q and $\hat{q}_{ij} : u'(\hat{q}_{ij}) = \delta^i v'_{ij}(\hat{q}_{ij})$ satisfies $\hat{q}_{ij} \leq q_{ij}^*$.

Assumption 1 implies that the buyer's marginal benefits of carrying liquid assets into a DGM match are strictly decreasing. In turn, this property can be microfounded with a variety of trading arrangements.⁴ Also, when $v_{II}(q) = v_{IP}(q) = v_{PI}(q) = v_{PP}(q)$ for all q , Υ nests a setting where households cannot observe each others preferences. It follows

⁴This includes proportional bargaining, gradual bargaining, generalized Nash bargaining when bargaining power of buyers is sufficiently large, and Walrasian pricing.

that trade in a DGM meeting satisfies:

$$q = \begin{cases} v_{ij}^{-1}(z) & \text{if } z < v_{ij}(\hat{q}_{ij}) \\ \hat{q}_{ij} & \text{if } z \geq v_{ij}(\hat{q}_{ij}) \end{cases} \quad \text{and} \quad p = \begin{cases} z & \text{if } z < v_{ij}(\hat{q}_{ij}) \\ v_{ij}(\hat{q}_{ij}) & \text{if } z \geq v_{ij}(\hat{q}_{ij}) \end{cases}. \quad (2)$$

In words, the buyer wishes to consume at most $\hat{q}_{ij}$, which he/she can afford when $z \geq v_{ij}(\hat{q}_{ij})$. Otherwise, the buyer is effectively constrained by his/her liquid asset holdings and consumes only $v_{ij}^{-1}(z) \leq \hat{q}_{ij}$. With regards to market structure, I assume that Σ is fixed.

When a household enters the DGM with liquid assets worth z and a discount factor shock δ^i , it earns an expected utility surplus from its liquid asset holdings given by:

$$\mathcal{L}^i(z) = \sum_{j \in \{I, P\}} \frac{\sigma_{ij}}{\pi^i} [u \circ \min\{v_{ij}^{-1}(z), \hat{q}_{ij}\} - \beta \delta^i \min\{z, v_{ij}(\hat{q}_{ij})\}]. \quad (3)$$

Observe that $\mathcal{L}_z^i(z) \geq 0$ and $\mathcal{L}_{zz}^i(z) \leq 0$, both with strict inequality if and only if $z < \hat{z}^i \equiv \max\{v_{iI}(\hat{q}_{iI}), v_{iP}(\hat{q}_{iP})\}$. Intuitively, $\mathcal{L}$ captures the value of assets as payment instruments for households. This value increases at a decreasing rate, until a household holds liquid assets worth more than $\hat{z}^i$ CM $t+1$ goods. Then, additional liquid asset holdings are no longer used to settle transactions in DGM t but are carried over to CM $t+1$.

Assumption 2. *When unconstrained by their holdings of liquid assets, impatient households spend more on DGM goods than patient households: $\hat{z}^I > \hat{z}^P$.*

Assumption 2 is satisfied for a broad variety of trading arrangements (see Appendix C) and also when households' preferences are private information during the DGM. Because a household may become a seller in DGM t , it earns an additional expected utility surplus Δ_t^i independent of its asset holdings. Let $G_t(m, d | \delta^j)$ denote the conditional CDF of real money balances and note holdings carried into DGM t by households with $\delta_t = \delta^j$. It follows that Δ_t^i satisfies

$$\Delta_t^i = \sum_{j \in \{I, P\}} \frac{\sigma_{ji}}{\pi^i} \iint \left\{ \begin{aligned} &\chi [-c(\min\{v_{ji}^{-1}(m' + d'), \hat{q}_{ij}\}) + \beta \delta^i \min\{m' + d', v_{ji}(\hat{q}_{ij})\}] \\ &+ (1 - \chi) [-c(\min\{v_{ji}^{-1}(m'), \hat{q}_{ij}\}) + \beta \delta^i \min\{m', v_{ji}(\hat{q}_{ij})\}] \end{aligned} \right\} dG_t(m', d' | \delta^j).$$

Given that, when a buyer, the household can trade notes with probability χ , the value of entering DGM t with money, notes, and bonds worth m , d , and b CM $t+1$ goods, respectively, is given by:

$$V_t^i(m, d, b) = \chi \mathcal{L}^i(m + d) + (1 - \chi) \mathcal{L}^i(m) + \Delta_t^i + \beta \delta^i [m + d + b + \bar{W}_{t+1}].$$

3.1.2 OTC for a Household

The OTC is a decentralized financial market in which trade is bilateral and households obtain at most one match. The matching technology is such that a patient (impatient) household is matched to an impatient (resp. patient) household with probability η^P (resp. η^I).⁵ The measure of matches realized during the OTC, denoted with ω , satisfies $\omega = \pi^I \eta^I = \pi^P \eta^P \leq \min\{\pi^I, \pi^P\}$. Here, ω , η^I , and η^P are proxies for the indirect liquidity of notes and bonds. In the OTC, just as in the DGM, credit is infeasible.

Consider a match between households i and j that have drawn discount factor shocks $\delta^i \neq \delta^j$. Let m_i , d_i , and b_i denote the real money, note, and bond holdings of household i , expressed in terms of CM $t + 1$ goods. Define m_j , d_j , and b_j in an analogous fashion. Suppose that household i receives money holdings with net real value l_{ij} from household j and notes with real value k_{ij} , and that household j receives bonds with net real value a_{ij} from household i . Note that these quantities can be negative. Private surplus of the transaction is

$$\begin{aligned} \mathcal{F}_{ij} = & (1 - \chi) \{ [\mathcal{L}^i(l_{ij} + m_i) - \mathcal{L}^i(m_i)] + [\mathcal{L}^j(m_j - l_{ij}) - \mathcal{L}^j(m_j)] \} \\ & + \chi \left\{ \begin{aligned} & [\mathcal{L}^i(l_{ij} + k_{ij} + m_i + d_i) - \mathcal{L}^i(m_i + d_i)] \\ & + [\mathcal{L}^j(m_j + d_j - l_{ij} - k_{ij}) - \mathcal{L}^j(m_j + d_j)] \end{aligned} \right\} + \beta(\delta^j - \delta^i)(a_{ij} - l_{ij} - k_{ij}). \end{aligned} \quad (4)$$

subject to feasibility constraints $-m_i \leq l_{ij} \leq m_j$, $-d_i \leq k_{ij} \leq d_j$, and $-b_j \leq a_{ij} \leq b_i$, as well as individual rationality conditions

$$\begin{aligned} 0 & \leq \chi [\mathcal{L}^i(l_{ij} + k_{ij} + m_i + d_i) - \mathcal{L}^i(m_i + d_i)] \\ & \quad + (1 - \chi) [\mathcal{L}^i(l_{ij} + m_i) - \mathcal{L}^i(m_i)] - \beta\delta^i(a_{ij} - l_{ij} - k_{ij}), \\ 0 & \leq \chi [\mathcal{L}^j(m_j + d_j - l_{ij} - k_{ij}) - \mathcal{L}^j(m_j + d_j)] \\ & \quad + (1 - \chi) [\mathcal{L}^j(m_j - l_{ij}) - \mathcal{L}^j(m_j)] + \beta\delta^j(a_{ij} - l_{ij} - k_{ij}). \end{aligned}$$

Contrasting Duffie et al. (2005), Geromichalos and Herrenbrueck (2016), and Trejos and Wright (2016), households attach the same value to different assets as savings instruments, but differ in their desire to save. This novelty shows up in Equation (4) through the term $\beta(\delta^j - \delta^i)(a_{ij} - l_{ij} - k_{ij})$, which captures the welfare effect of a re-distribution of savings that occurs when assets do not trade at par.

For simplicity, I focus on a transaction (l_{ij}, k_{ij}, a_{ij}) generated by proportional bargaining. That means, the patient household appropriates a constant share $\alpha \in (0, 1)$ of the

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⁵Matches in which two patient or two impatient households meet can be ignored, because in equilibrium households enter the OTC with identical asset portfolios. Carrying identical asset portfolios, households also truthfully reveal their preferences within a match.

match surplus. Defining $\alpha^i = (1 - \alpha)\mathbb{I}_{i=I} + \alpha\mathbb{I}_{i=P}$, $\alpha^j = 1 - \alpha^i$, $\kappa_{ij} = l_{ij} + k_{ij}$, and

$$\begin{aligned} \tilde{a}_{ij} = & \kappa_{ij} + \frac{1 - \chi}{\beta} \frac{\alpha^j [\mathcal{L}^i(l_{ij} + m_i) - \mathcal{L}^i(m_i)] - \alpha^i [\mathcal{L}^j(m_j - l_{ij}) - \mathcal{L}^j(m_j)]}{\alpha\delta^I + (1 - \alpha)\delta^P} \\ & + \frac{\chi}{\beta} \frac{\alpha^j [\mathcal{L}^i(\kappa_{ij} + m_i + d_i) - \mathcal{L}^i(m_i + d_i)] - \alpha^i [\mathcal{L}^j(m_j + d_j - \kappa_{ij}) - \mathcal{L}^j(m_j + d_j)]}{\alpha\delta^I + (1 - \alpha)\delta^P}, \end{aligned}$$

I find that match surplus $\mathcal{F}_{ij}(m_i, d_i, b_i; m_j, d_j, b_j)$ becomes

$$\begin{aligned} \mathcal{F}_{ij}(\cdot) = & \max_{l_{ij}, \kappa_{ij}} \left\{ (1 - \chi) \frac{\delta^j [\mathcal{L}^i(l_{ij} + m_i) - \mathcal{L}^i(m_i)] + \delta^i [\mathcal{L}^j(m_j - l_{ij}) - \mathcal{L}^j(m_j)]}{\alpha\delta^I + (1 - \alpha)\delta^P} \right. \\ & \left. + \chi \frac{\delta^j [\mathcal{L}^i(\kappa_{ij} + m_i + d_i) - \mathcal{L}^i(m_i + d_i)] - \delta^i [\mathcal{L}^j(m_j + d_j - \kappa_{ij}) - \mathcal{L}^j(m_j + d_j)]}{\alpha\delta^I + (1 - \alpha)\delta^P} \right\} \quad (5) \\ \text{s.t.} \quad & -b_j \leq \tilde{a}_{ij} \leq b_i, \quad -m_i \leq l_{ij} \leq m_j, \quad \text{and} \quad -d_i \leq \kappa_{ij} - l_{ij} \leq d_j. \end{aligned}$$

I will only characterize OTC transactions on the equilibrium path and I will do so once I turn towards the equilibrium analysis. For now, let $F_t(m, d, b)$ denote the joint CDF of real money, note, and bond holdings across households when they enter OTC t .⁶ For a household that has drawn $\delta_t = \delta^i$, the value function of entering OTC t with real money holdings m , real note holdings d , and real bond holdings b , becomes

$$\begin{aligned} O_t^i(m, d, b) = & \eta^i \alpha^i \iiint \mathcal{F}_{ij}(m, d, b; m', d', b') dF_t(m', d', b') \\ & + (1 - \chi)\mathcal{L}^i(m) + \chi\mathcal{L}^i(m + d) + \Delta_t^i + \beta\delta^i(m + d + b + \bar{W}_{t+1}). \quad (6) \end{aligned}$$

3.1.3 CM for a Household

The CM is a Walrasian market in which CM goods are used as numeraire. Prices of money, newly issued notes, and newly issued bonds are respectively ϕ_t , φ_t , and ψ_t .

Consider a household that enters CM t with money, notes, and bonds worth respectively $\hat{m}_t$, $\hat{d}_t$, and $\hat{b}_t$ CM t goods. Let m_t , d_t , and b_t denote the real value of money, notes, and bonds, expressed in CM $t + 1$ goods, that the household carries out of CM t . Finally, let τ_t be a real lump-sum tax imposed on households. For a household, the value function of entering CM t is then given by:

$$W_t(\hat{m}_t, \hat{d}_t, \hat{b}_t) = \max_{y_t, \bar{y}_t, m_t, d_t, b_t} \{U(y_t) - \bar{y}_t + \pi^I O_t^I(m_t, d_t, b_t) + \pi^P O_t^P(m_t, d_t, b_t)\} \quad (7a)$$

$$\text{s.t.} \quad [\phi_t m_t + \varphi_t d_t + \psi_t b_t] / \phi_{t+1} + y + \tau_t \leq \hat{m}_t + \hat{d}_t + \hat{b}_t + \bar{y}_t, \quad (7b)$$

$$y_t \geq 0, \quad \bar{y}_t \geq 0, \quad m_t \geq 0, \quad d_t \geq 0, \quad \text{and} \quad b_t \geq 0.$$

⁶There is no need to condition on discount factor shocks. The reason is that these shocks are realized just before the OTC convenes and that these shocks are i.i.d. over time and across households.

Budget constraint (7b) will hold with equality, so we can eliminate $\bar{y}$ in objective function (7a). Following the literature, I render the non-negativity constraint on the production of CM goods slack by assuming that y^* is sufficiently large. We then obtain:

$$W_t(\hat{m}_t, \hat{d}_t, \hat{b}_t) = \max_{\{m_t, d_t, b_t\} \in \mathbb{R}_+^3} \left\{ \sum_{i \in \{I, P\}} \pi^i O_t^i(m_t, d_t, b_t) - \frac{\phi_t m_t + \varphi_t d_t + \psi_t b_t}{\phi_{t+1}} \right\} \quad (8)$$

$$+ U(y^*) - y^* + \hat{m}_t + \hat{d}_t + \hat{b}_t - \tau_t,$$

which, in line with earlier conjectures, can be written as $W_t(\hat{m}_t, \hat{d}_t, \hat{b}_t) = \hat{m}_t + \hat{d}_t + \hat{b}_t + \bar{W}_t$.

Appendix D.1 demonstrates that regardless of the distribution of asset holdings F_t in OTC t , the value function $O_t^i(m, d, b)$ is concave in (m, d, b) . Necessary and sufficient conditions for optimality of assets carried out of CM t are therefore

$$m_t : \quad 0 \geq -\phi_t + \phi_{t+1} \sum_{i \in \{I, P\}} \pi^i \frac{\partial O_t^i(m_t, d_t, b_t)}{\partial m_t} \quad \text{with } = \text{ if } m_t > 0, \quad (9)$$

$$d_t : \quad 0 \geq -\varphi_t + \phi_{t+1} \sum_{i \in \{I, P\}} \pi^i \frac{\partial O_t^i(m_t, d_t, b_t)}{\partial d_t} \quad \text{with } = \text{ if } d_t > 0, \quad (10)$$

$$b_t : \quad 0 \geq -\psi_t + \phi_{t+1} \sum_{i \in \{I, P\}} \pi^i \frac{\partial O_t^i(m_t, d_t, b_t)}{\partial b_t} \quad \text{with } = \text{ if } b_t > 0. \quad (11)$$

3.2 Government

Government is only active in the CM and cannot observe households' preferences. It levies lump-sum taxes, prints money, redeems maturing notes and bonds, and issues new notes and bonds. The supply of money, measured at the end of CM t , is denoted with M_t , the face value of newly notes is denoted with D_t , and the face value of newly issued bonds is denoted with B_t . The government's budget constraint implies that

$$\tau_t = \phi_t(M_{t-1} + D_{t-1} + B_{t-1}) - \phi_t M_t - \varphi_t D_t - \psi_t B_t. \quad (12)$$

4 Equilibrium

We are now in a position to characterize equilibria. Given the importance of interest rates, let me first define

$$1 + i_t^f = \phi_t / (\beta \phi_{t+1}), \quad 1 + i_t^d = \phi_t / \varphi_t \quad \text{and} \quad 1 + i_t^b = \phi_t / \psi_t. \quad (13)$$

Here, i_t^f is the Fisher nominal interest rate. This is the expected nominal return earned by fictitious assets that are only traded in the CM.⁷ A zero Fisher rate is known as the Friedman rule and requires gross inflation, given by ϕ_t/ϕ_{t+1} , to equal the average discount factor β . Then, the return earned by holding money between two CMs exactly compensates a household for expected discounting and it is said that there are no opportunity cost associated with holding money. The nominal return earned by holding bonds (notes) between CM t and CM $t + 1$ is captured by i_t^b (resp. i_t^d).

Because the conditions for optimal portfolio choices in the CM (Equations (9), (10), and (11)) are the same across all households, we can focus on a symmetric state of affairs in which all households leave the CM with identical asset portfolios. Let

$$\mathcal{F}_{ij,x}(m, d, b) = \frac{\partial \mathcal{F}_{ij}(m, d, b; m', d', b')}{\partial x} \Big|_{\substack{m=m' \\ d=d' \\ b=b'}} \quad x \in \{m, d, b\}. \quad (14)$$

Necessary and sufficient conditions for optimal asset portfolios carried out of CM t are then given by:

$$\beta i_t^f \geq \sum_{i \in \{I, P\}} \pi^i \left[\chi \mathcal{L}_z^i(m_t + d_t) + (1 - \chi) \mathcal{L}_z^i(m_t) \right] \quad \text{with } = \text{ if } m_t > 0, \quad (15)$$

$$\beta \frac{i_t^f - i_t^d}{1 + i_t^d} \geq \sum_{i \in \{I, P\}} \pi^i \left[\chi \mathcal{L}_z^i(m_t + d_t) + \alpha^i \eta^i \mathcal{F}_{ij,d}(m_t, d_t, b_t) \right] \quad \text{with } = \text{ if } d_t > 0, \quad (16)$$

$$\beta \frac{i_t^f - i_t^b}{1 + i_t^b} \geq \sum_{i \in \{I, P\}} \pi^i \alpha^i \eta^i \mathcal{F}_{ij,b}(m_t, d_t, b_t) \quad \text{with } = \text{ if } b_t > 0. \quad (17)$$

Definition 1. Given sequence $\{M_t, D_t, B_t\}_{t=0}^\infty$, a symmetric equilibrium is a sequence of prices $\{\phi_t, \varphi_t, \psi_t\}$, interest rates $\{i_t^f, i_t^d, i_t^b\}_{t=0}^\infty$, and portfolio choices $\{m_t, d_t, b_t\}_{t=0}^\infty$ such that for all t :

1. Households maximize utility:

- (a) Optimal CM behavior implies that $\{m_t, d_t, b_t\}$ satisfies Equations (15)-(17).
- (b) Optimal OTC behavior implies that for $x \in \{m, d, b\}$ and $i, j \in \{I, P\}$, we have that $\mathcal{F}_{ij,x}(m_t, d_t, b_t)$ is given by Equation (14) with $\mathcal{F}_{ij}(m, d, b; m', d', b')$ given by Equation (5)
- (c) Optimal DGM behavior implies that for $i \in \{I, P\}$, we have that $\mathcal{L}^i(m_t)$ and $\mathcal{L}^i(m_t + d_t)$ are given by Equation (3).

2. Interest rates satisfy Equation (13).

3. Markets clear; $\phi_{t+1}M_t = m_t$, $\phi_{t+1}B_t = b_t$, and $\phi_{t+1}D_t = d_t$.

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⁷Because of quasi-linear utility, such an asset needs to earn an expected gross real return $1/\beta$.

4.1 OTC Transactions in Symmetric Equilibrium

To characterize OTC transactions in symmetric equilibrium, let l and k denote the real value of money and notes transferred from the impatient to the patient household in an OTC match and let a denote the real face value of bonds transferred from the impatient to the patient household in an OTC match. With three assets, characterizing OTC trades in symmetric equilibrium is rather involved. Fortunately, matters can be simplified by following a two step procedure.

The first step; consider a baseline economy with only money and bonds. With proportional bargaining, surplus of an OTC match in the symmetric equilibrium can then be written as:

$$\begin{aligned} \tilde{\mathcal{F}}(m, b) &= \max_{-m \leq l \leq m} \frac{\delta^P [\mathcal{L}^I(m+l) - \mathcal{L}^I(m)] + \delta^I [\mathcal{L}^P(m-l) - \mathcal{L}^P(m)]}{\alpha\delta^I + (1-\alpha)\delta^P} \\ \text{s.t.} \quad &-b \leq l + \frac{1}{\beta} \frac{\alpha [\mathcal{L}^I(m+l) - \mathcal{L}^I(m)] - (1-\alpha) [\mathcal{L}^P(m-l) - \mathcal{L}^P(m)]}{\alpha\delta^I + (1-\alpha)\delta^P} \leq b. \end{aligned}$$

Next, define functions $a_I^{-1}(m, b)$ and $a_P^{-1}(m, b)$ that satisfy

$$\begin{aligned} b &= a_I^{-1} + \frac{1}{\beta} \frac{\alpha [\mathcal{L}^I(m + a_I^{-1}) - \mathcal{L}^I(m)] - (1-\alpha) [\mathcal{L}^P(m - a_I^{-1}) - \mathcal{L}^P(m)]}{\alpha\delta^I + (1-\alpha)\delta^P}, \\ b &= a_P^{-1} + \frac{1}{\beta} \frac{(1-\alpha) [\mathcal{L}^P(m + a_P^{-1}) - \mathcal{L}^P(m)] - \alpha [\mathcal{L}^I(m - a_P^{-1}) - \mathcal{L}^I(m)]}{\alpha\delta^I + (1-\alpha)\delta^P}. \end{aligned}$$

These functions describe the maximum amount of money that a household, given its bond holdings, can buy from its OTC trading partner. Let $\hat{l}(m)$ denote the transfer of money in case households are effectively unconstrained by their bond holdings and let $\hat{a}(m)$ be the associated transfer of bonds, both expressed in terms of CM $t+1$ goods. These quantities satisfy

$$\delta_P \mathcal{L}_z^I(\hat{l}+m) \leq \delta_I \mathcal{L}_z^P(m-\hat{l}) \text{ if } \hat{l} < m, \quad \text{and} \quad \delta_P \mathcal{L}_z^I(\hat{l}+m) \geq \delta_I \mathcal{L}_z^P(m-\hat{l}) \text{ if } \hat{l} > -m. \quad (19)$$

$$\beta(\hat{a} - \hat{l})[\alpha\delta^I + (1-\alpha)\delta^P] = \alpha[\mathcal{L}^I(\hat{l}+m) - \mathcal{L}^I(m)] - (1-\alpha)[\mathcal{L}^P(m-\hat{l}) - \mathcal{L}^P(m)]. \quad (20)$$

When $2m < \hat{z}^I + \hat{z}^P$, $\hat{l}(m)$ is pinned down uniquely by Equation (19) and when $2m \geq \hat{z}^I + \hat{z}^P$, we can assume without loss that

$$\hat{l}(m) = \begin{cases} \hat{z}^I - m & \text{if } m < \hat{z}^I \\ 0 & \text{if } m \geq \hat{z}^I \end{cases}.$$

When households carry identical portfolios, the solution to Equation (18) then involves:

$$l(m, b) = \max\{-a_P^{-1}(m, b), \min\{a_I^{-1}(m, b), \hat{l}(m)\}\} \\ \text{and } a(m, b) = \max\{-b, \min\{b, \hat{a}(m)\}\}. \quad (21)$$

For the current paper's main contribution, it suffices to understand matters when $2m \geq \hat{z}^I + \hat{z}^P$ and households are unconstrained by their bond holdings. Then, together an impatient and a patient household hold enough money to be unconstrained in the DGM. When $m < \hat{z}^I$ there are gains from OTC trade because without it, the impatient household will be constrained by its money holdings in the DGM. Money will thus be transferred from the patient to the impatient household, until both are unconstrained in the DGM. When $m \geq \hat{z}^I$, there are no gains from trade and OTC activity vanishes. To conclude the first step, formally define surplus from OTC trade in a baseline economy:

$$\tilde{\mathcal{F}}(m, b) = \frac{\delta^P [\mathcal{L}^I(l(m, b) + m) - \mathcal{L}^I(m)] + \delta^I [\mathcal{L}^P(-l(m, b) + m) - \mathcal{L}^P(m)]}{\alpha\delta^I + (1 - \alpha)\delta^P}. \quad (22)$$

The second step implies that in an economy with money, notes, and bonds, there exist $g_1(m, d, b)$ and $g_2(m, d, b)$ (Appendix D.3 provides details) such that

$$\mathcal{F}(m, d, b) = (1 - \chi)\tilde{\mathcal{F}}(m, g_1(m, d, b)) + \chi\tilde{\mathcal{F}}(m + d, g_2(m, d, b)), \quad (23)$$

where $\mathcal{F}(m, d, b) = \mathcal{F}_{ij}(m, d, b; m', d', b')|_{m'=m, d'=d, b'=b}$. In words, the OTC match surplus is a weighted average of the surplus in two different baseline OTC meetings. In the first meeting, households carry money with real value m and bonds with real value g_1 . In the second meeting, households carry money with real value $m + d$ and bonds with real value g_2 . Observe that either g_1 or g_2 can be negative, which is infeasible in a baseline economy. However, Equations (21) and (22) allow for negative values of b . With a negative value for b , these equations imply that the pattern of OTC trade reverses.

4.2 Welfare

Utilitarian welfare as of period $t = 0$, denoted with $\mathcal{W}_0$, equals expected lifetime utility of a household, so I use this as the most appropriate measure for welfare. Define $\mathcal{L}^i(z)/\theta^i(z)$ as the social surplus from a household's liquid asset holdings z in the DGM, with the household in question having drawn $\delta_t = \delta^i$. Using Equations (1) and (2), we find

$$\frac{\mathcal{L}^i(z)}{\theta^i(z)} = \sum_{j \in \{I, P\}} \frac{\sigma_{ij}}{\pi^i} [(u - c) \circ \min\{v_{ij}^{-1}(z), \hat{q}_{ij}\} - \beta(\delta^i - \delta^j) \min\{z, v_{ij}(\hat{q}_{ij})\}]. \quad (24)$$

Note that $\theta^i(z) \in (0, 1]$ and that $\mathcal{L}^i(z)/\theta^i(z)$ is increasing in z for $0 < z < \hat{z}^i$, as Pareto efficiency implies that the social surplus of matches can only improve when buyers can

spend more.⁸ Let $\mathcal{E}(m_t, d_t, b_t)$ capture the period t externalities of OTC trade on sellers in the DGM. This quantity can be decomposed using a two step procedure. First, in a baseline economy with only money and bonds, $\tilde{\mathcal{E}}(m_t, b_t)$ characterizes externalities of trade within an OTC match. Second, in an economy with money, notes, and bonds, externalities from OTC trade can be expressed as a weighted average

$$\mathcal{E}(m_t, d_t, b_t) = (1 - \chi)\tilde{\mathcal{E}}(m_t, g_1(m_t, d_t, b_t)) + \chi\tilde{\mathcal{E}}(m_t + d_t, g_2(m_t, d_t, b_t)), \quad (25)$$

with g_1 and g_2 defined as before, and

$$\begin{aligned} \tilde{\mathcal{E}}(m, b) = & [1 - \theta^I(m + l)]\mathcal{L}^I(m + l)/\theta^I(m + l) - [1 - \theta^I(m)]\mathcal{L}^I(m)/\theta^I(m) \\ & + [1 - \theta^P(m - l)]\mathcal{L}^P(m - l)/\theta^P(m - l) - [1 - \theta^P(m)]\mathcal{L}^P(m)/\theta^P(m). \end{aligned} \quad (26)$$

Lemma 1. *Utilitarian welfare in a symmetric equilibrium satisfies the recursive relationship $\mathcal{W}_t = \mathcal{W}(m_t, d_t, b_t) + \beta\mathcal{W}_{t+1}$, where*

$$\begin{aligned} \mathcal{W}(m, d, b) = & \chi \left[\pi^I \frac{\mathcal{L}^I(m + d)}{\theta^I(m + d)} + \pi^P \frac{\mathcal{L}^P(m + d)}{\theta^P(m + d)} \right] + (1 - \chi) \left[\pi^I \frac{\mathcal{L}^I(m)}{\theta^I(m)} + \pi^P \frac{\mathcal{L}^P(m)}{\theta^P(m)} \right] \\ & + \omega [\mathcal{F}(m, d, b) + \mathcal{E}(m, d, b)] + U(y^*) - y^*. \end{aligned} \quad (27)$$

Optimal policy therefore implements a sequence $\{m_t, d_t, b_t\}_{t=0}^\infty$ to maximize

$$\mathcal{W}_0 = \sum_{t=0}^{\infty} \beta^t \mathcal{W}(m_t, d_t, b_t).$$

Intuitively, flow welfare in the economy can be decomposed into three terms. First, the social benefits from DGM trade if all households would enter the DGM with the asset portfolios they carried out of the CM; $(1 - \chi) [\pi^I \mathcal{L}^I(m)/\theta^I(m) + \pi^P \mathcal{L}^P(m)/\theta^P(m)] + \chi [\pi^I \mathcal{L}^I(m + d)/\theta^I(m + d) + \pi^P \mathcal{L}^P(m + d)/\theta^P(m + d)]$. Second, the social benefits from an OTC match multiplied by the measure of OTC matches; $\omega [\mathcal{F}(m, d, b) + \mathcal{E}(m, d, b)]$. Third, the benefits from trade in CM goods; $U(y^*) - y^*$.

With government active only during the CM, optimal policies are dynamically consistent; when $\{m_s^*, d_s^*, b_s^*\}_{s=t}^\infty$ maximizes $\mathcal{W}_t$, then $\{m_s^*, d_s^*, b_s^*\}_{s=t+1}^\infty$ maximizes $\mathcal{W}_{t+1}$. Without loss, I therefore consider symmetric steady state equilibria (SSE) with stationary policies described by a tuple $\langle \gamma, d, b \rangle$. Here, $\gamma = M_t/M_{t-1}$ is the growth rate of money supply, $d = \phi_{t+1}D_t$ the real face value of notes, and $b = \phi_{t+1}B_t$ the real face value of bonds.

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⁸Alternatively, when households do not observe each others preferences, so that Pareto efficiency may not be attainable, one can argue that $\mathcal{L}^i(z)/\theta^i(z)$ should be increasing as otherwise some sellers become worse off as buyers spend more, which the sellers can prevent by limiting production of DGM goods.

4.3 Properties of Monetary Steady State Equilibria

In SSE, CM prices satisfy the law of motions $\phi_{t+1} = \gamma\phi_t/\gamma$, $\varphi_{t+1} = \varphi_t/\gamma$, and $\psi_{t+1} = \psi_t/\gamma$, as real quantities need to be constant. From the characterization of interest rates, we find that the Fisher interest rate satisfies $\beta i^f = \gamma - \beta$. Given that the real face value of notes and bonds is determined by policy, we can use Equation (15) to describe SSE by means of a stationary value for $m = \phi_{t+1}M_t$:

Lemma 2. *Given government policy $\langle \gamma, d, b \rangle$, SSE is described by an $m > 0$ that solves*

$$\gamma - \beta = \omega[(1 - \alpha)\mathcal{F}_{IP,m}(m, d, b) + \alpha\mathcal{F}_{PI,m}(m, d, b)] \\ + \chi[\pi^I \mathcal{L}_z^I(m + d) + \pi^P \mathcal{L}_z^P(m + d)] + (1 - \chi)[\pi^I \mathcal{L}_z^I(m) + \pi^P \mathcal{L}_z^P(m)]. \quad (28)$$

Given d and b , for each $m > 0$ there is a γ that solves Equation (28). This γ is continuous in m and satisfies $m \geq \hat{z}^I \Leftrightarrow \gamma = \beta$, $m < \hat{z}^I \Leftrightarrow \gamma > \beta$, and $\lim_{\gamma \downarrow \beta} m = \hat{z}^I$. Moreover, in SSE $i^d = i^b = i^f = 0 \Leftrightarrow \gamma = \beta$ and $0 < i^d \leq i^b \leq i^f \Leftrightarrow \gamma > \beta$.

Due to the non-linearity of Equation (28), it is difficult to prove that there is a unique SSE when policy is not at the Friedman rule. Nevertheless, all $m > 0$ can be implemented in SSE as long as γ is chosen appropriately and what matters is the following:

First, when monetary policy implements the Friedman rule, the Fisher rate is zero and there are no opportunity cost associated with holding money. Then, households choose money holdings to ensure that they are unconstrained during the DGM and avoid selling notes and/or bonds at a discount during the OTC; $m \geq \hat{z}^I$ and OTC trade disappears. Once policy deviates from the Friedman rule, households economize on their money holdings so that $m < \hat{z}^I$. In this sense, abstracting from financial market considerations, (deviating from) the Friedman rule has the same implications for DGM trade and equilibrium money balances as in standard models from the monetary-search literature.

Second, because bonds deliver some indirect liquidity services, the nominal rate earned by bonds can fall short of the Fisher rate. From Equation (17) it follows that bonds command a liquidity premium when households are constrained by their bond holdings during the OTC. Similarly, notes deliver some indirect and some direct liquidity services. Equation (16) implies that notes command a liquidity premium when households are constrained by their note holdings during the OTC or DGM. Because bonds are an imperfect substitute for notes and notes are an imperfect substitute for money, we have that away from the Friedman rule $0 < i^d \leq i^b \leq i^f$.

5 Baseline Economy

To understand optimal policy, consider an economy with only money and bonds. Equation (27) demonstrates that in an optimal policy regime, m and b should maximize

$$\tilde{W}(m, b) = \pi^I \mathcal{L}^I(m) / \theta^I(m) + \pi^P \mathcal{L}^P(m) / \theta^P(m) + \omega[\tilde{\mathcal{F}}(m, b) + \tilde{\mathcal{E}}(m, b)]. \quad (29)$$

Lemma 2 implies that all $m > 0$ can be implemented as SSE, as long as γ is chosen appropriately given b . An optimal policy can thus be found by characterizing a pair $(m, b) \geq 0$ that maximizes Equation (29), and then using Equation (28) to find γ .

Proposition 1. *There exists an $\tilde{\eta}^I \in (0, 1)$ such that the Friedman rule is sub-optimal if and only if: (i) $\eta^I > \tilde{\eta}^I$ and (ii) $b > 0$. Bonds are irrelevant for welfare at the Friedman rule and therefore, if and only if $\eta^I > \tilde{\eta}^I$ optimal policy implies coexistence of money and interest-bearing bonds.*

Proposition 1 is the main result of the current paper. It shows that when the indirect liquidity of bonds is sufficiently large, optimal policy deviates from the Friedman rule and coexistence of money and interest-bearing bonds arises endogenously. Intuitively, for a slight deviation from the Friedman rule trade in the OTC picks up. Because impatient households need more money to be unconstrained in the DGM than patient households, in the OTC impatient households then start to sell bonds at a discount to patient households. As a result, the distribution of savings improves. If the indirect liquidity of bonds is sufficiently large, there is enough OTC trade to compensate for reduced DGM trade. The reason is that for a small deviation from the Friedman rule, we have that m satisfies $\hat{z}^P < (\hat{z}^I + \hat{z}^P)/2 < m < \hat{z}^I$. Hence, only a measure $\pi^I(1 - \eta^I)$ of impatient households that do not find a match in the OTC, end up being constrained in the DGM.

To relate optimal policy to the nominal rate earned by bonds, when the Friedman rule is sub-optimal, equilibrium cannot represent a liquidity trap under an optimal policy regime; we need $i^b > 0$. Why? Because $i^b = 0$ would imply that the marginal benefits of carrying money out of the CM equal the marginal benefits of carrying bonds out of the CM. That means, money and bonds are perfect substitutes at the margin. In turn, at the margin bonds then trade at par for money in the OTC. At the margin, OTC trade therefore leaves the distribution of savings unaffected and hence the economy would be better off with more money. With proportional bargaining in the OTC, away from the Friedman rule $i^b = 0$ only if $\alpha = 0$. In words, the impatient agent appropriates the full surplus of an OTC meeting. With $\alpha = 0$, the proof of Proposition 1 demonstrates that we have $\tilde{\eta}^I = 1$ so at best, deviating from the Friedman rule does not hurt welfare.

Because OTC trade vanishes when the Friedman rule is implemented, bonds are irrelevant for real allocations and welfare at the Friedman rule. At the same time, deviating from the Friedman rule is only attractive when OTC trade leads to a redistribution of

savings, which in turn requires a positive supply of bonds that in equilibrium happen to earn a positive nominal interest rate. In an optimal policy regime that deviates from the Friedman rule, coexistence of money and interest-bearing bonds therefore arises endogenously. Below, I discuss two issues regarding insight and I consider two DGM trading mechanisms for which $\tilde{\eta}^I$ can be characterized analytically.

5.1 Discussion

First, existing models of monetary economies with financial markets, such those of Berentsen, Camera, and Waller (2007), Li and Li (2013), Berentsen et al. (2014), and Geromichalos and Herrenbrueck (2016), also find that financial market activity contributes to welfare when policy deviates from the Friedman rule. Nevertheless, in these models the Friedman rule is optimal while in my environment I find sub-optimality of the Friedman rule when financial markets are well-developed. The reason for this difference is the additional role for financial markets in the current framework. This becomes evident when using Equations (22) and (26) to write Equation (29) as

$$\begin{aligned}\tilde{W}(m, b) = & \pi^I \left[(1 - \eta^I) \frac{\mathcal{L}^I(m)}{\theta^I(m)} + \eta^I \frac{\mathcal{L}^I(l + m)}{\theta^I(l + m)} \right] \\ & + \pi^P \left[(1 - \eta^P) \frac{\mathcal{L}^P(m)}{\theta^P(m)} + \eta^I \frac{\mathcal{L}^I(m - l)}{\theta^I(m - l)} \right] \\ & + \omega\beta(\delta^P - \delta^I) [a - l] + U(y^*) - y^*,\end{aligned}\tag{30}$$

with l and a given by Equation (21).

In existing models, $\delta^I = \delta^P = 1$ but for reasons other than shocks to discount factors,⁹ typically $\hat{z}^I > \hat{z}^P$ and $\mathcal{L}_z^I(z) \geq \mathcal{L}_z^P(z)$ with strict inequality if and only if $z < \hat{z}^I$. When policy deviates from the Friedman rule in such models, agents economize on their holdings of real money balances and trade in financial markets arises to attain a more efficient distribution of money across heterogeneous agents. However, the Friedman rule remains optimal as benefits from DGM trade $\mathcal{L}^i(z)/\theta^i(z)$ are maximized when $z = \hat{z}^i$; ideally agents should be unconstrained by their money holdings.¹⁰

In the current paper, because $\delta^I < \delta^P$ not only the distribution of money matters, but also the distribution of savings. Trade in financial markets therefore has two effects; a redistribution of money and when bonds and money do not trade at par, a redistribution of savings. It is exactly the latter effect, captured by $\omega\beta(\delta^P - \delta^I)(a - l)$, which implies that the Friedman rule is sub-optimal; to stimulate transfers of savings in financial mar-

⁹For example, idiosyncratic shocks to DGM trading opportunities, to the marginal utility from consuming DGM goods, or to the marginal dis-utility of producing DGM goods.

¹⁰In fact, abstracting from the savings effect of financial market transactions, that is the term $\omega\beta(\delta^P - \delta^I)(a - l)$ in (30), the model with discount factor shocks is isomorphic to a model with idiosyncratic shocks to DGM trading opportunities or to the marginal utility of consuming or producing DGM goods.

kets, policy implements opportunity costs associated with holding money. The proof of Proposition 1 re-confirms this insight, as it implies that $\lim_{\delta^I \uparrow \delta^P} \tilde{\eta}^I = 1$.

Second, at the Friedman rule real money balances are indeterminate as agents are willing to carry any amount of real money balances greater than $\hat{z}^I$ (see Lemma 2). Some may wonder whether this implies that only the limiting allocations with $i^f \rightarrow 0$ (or equivalently $\gamma \rightarrow \beta$) can be improved upon by deviating from the Friedman rule. For instance, only if $i^f = 0$ households are indifferent over the timing of their production of CM goods. At first sight, it therefore seems that a planner can improve welfare by recommending impatient agents to leave the CM with more real money balances than patient agents, rather than recommending the limiting allocations for $i^f \rightarrow 0$. Such recommendations can only improve welfare compared to the limiting allocations if ex-post, patient agents are better off than impatient agents. Because the latter type can perfectly imitate the former, such welfare gains are impossible. In fact, welfare for the limiting allocations with $i^f \rightarrow 0$ equals welfare for all feasible allocations with $i^f = 0$.

5.2 Walrasian DGM

With linear cost of producing DGM goods, impatient sellers choose not to produce in a perfectly competitive DGM; they require a higher price to produce than patient sellers because money earned from production cannot be spend immediately. The pricing protocol and the measure of DGM meetings therefore satisfy:

$$v(q) = q/(\beta\delta^P) \quad \text{and} \quad \Sigma = \{0, \sigma\pi, 0, \sigma(1 - \pi)\},$$

where $\sigma \in (0, 1/2]$ may account for some randomness in the ability of households to finding trading partners. Equation (3) implies that

$$\mathcal{L}^i(z) = \sigma(u \circ \min\{\beta\delta^P z, \hat{q}^i\} - \beta\delta^i \min\{\beta\delta^P z, \hat{q}^i\}), \quad \text{where } u'(\hat{q}^i) = \delta^P/\delta^i.$$

Also, $\hat{z}^I > \hat{z}^P$ and $\mathcal{L}_z^I(z) \geq \mathcal{L}_z^P(z)$, with strict inequality if and only if $z < \hat{z}^I$. With perfect competition, sellers obtain no surplus from DGM trade, so $\theta^I(z) = \theta^P(z) = 1$ and there are no externalities from OTC trade. Equation (29) therefore becomes

$$\tilde{\mathcal{W}}(m, b) = \pi^I \mathcal{L}^I(m) + \pi^P \mathcal{L}^P(m) + \omega \tilde{\mathcal{F}}(m, b) + U(y^*) - y^*.$$

Because there are no externalities from OTC trade, optimal policy is such that households are unconstrained by their bond holdings in the OTC, i.e. $b \geq |\hat{a}(m)|$ since $\tilde{\mathcal{F}}(m, b)$ then attains a maximum for given m . Using Equation (22), we therefore obtain

$$\tilde{\mathcal{W}}(m, b) \Big|_{b \geq |\hat{a}(m)|} = \pi^I \left[\frac{(1 - \eta^I)\delta^P - \alpha(\delta^P - \delta^I)}{\alpha\delta^I + (1 - \alpha)\delta^P} \right] \mathcal{L}^I(m)$$

$$\begin{aligned}
& + \pi^P \left[\frac{(1 - \eta^P)\delta^I + (1 - \alpha)(\delta^P - \delta^I)}{\alpha\delta^I + (1 - \alpha)\delta^P} \right] \mathcal{L}^P(m) \\
& + \omega \frac{\delta^P \mathcal{L}^I(\hat{l}(m) + m) + \delta^I \mathcal{L}^P(m - \hat{l}(m))}{\alpha\delta^I + (1 - \alpha)\delta^P} + U(y^*) - y^*. \quad (31)
\end{aligned}$$

Plotting Equation (31) for various η^I , Figure 1 shows how welfare depends on real money balances and the Fisher rate.

Using Equation (19), it can be verified that $m + \hat{l}(m)$ and $m - \hat{l}(m)$ are increasing in m . Because $\mathcal{L}^i(z)$ is strictly increasing in z until $z = \hat{z}^i$, it follows that not the Friedman rule is sub-optimal only if $\alpha(\delta^P - \delta^I) > (1 - \eta^I)\delta^P$, as otherwise welfare is increasing in m . Moreover, deviating slightly from the Friedman rule implies that $\hat{z}^P < m < \hat{z}^I$, $m + \hat{l}(m) \geq \hat{z}^I$, and $m - \hat{l}(m) \geq \hat{z}^P$. Welfare then improves if $\alpha(\delta^P - \delta^I) > (1 - \eta^I)\delta^P$. Hence, it must be that

$$\tilde{\eta}^I = [\alpha\delta^I + (1 - \alpha)\delta^P] / \delta^P. \quad (32)$$

Equation (32) implies that $\tilde{\eta}^I$ is decreasing in the dispersion of discount factor shocks. Intuitively, if agents become more heterogeneous in how they value savings, OTC trade becomes more important for welfare. As a result, less OTC matches are needed to make up for welfare losses in the DGM. Also, $\tilde{\eta}^I$ is decreasing in α . Intuitively, if the bargaining power of patient agents in the OTC increases, bonds trade at a larger discount and OTC trade leads to a better distribution of savings.

Recall that optimal policy is such that households are unconstrained by their bond holdings. Nevertheless, with a passive supply of bonds,¹¹ it can be shown that the critical threshold to rationalize coexistence remains unchanged. The reason is that small deviations from the Friedman rule ($m = \hat{z}^I - \varepsilon$) leave households unconstrained by their bonds holdings since unconstrained OTC trade is then arbitrarily small (see Equations (19) and (20)). At the same time, small deviations from the Friedman rule improve welfare whenever $\eta^I > \tilde{\eta}^I$. Finally, closed form solutions for optimal policy (characterized in Appendix B) exist when $u(q)$ exhibits a constant relative rate of risk aversion.

5.3 DGM with Random Matching and Proportional Bargainig

Consider random matching in the DGM and that in all DGM matches, buyers appropriate a constant share $\theta \in (0, 1]$ of the match surplus. It follows that

$$v_{ij}(q) = \frac{(1 - \theta)u(q) + \theta q}{\delta^i(1 - \theta) + \delta^j\theta} \quad \text{and} \quad \Sigma = \{\sigma\pi^2, \sigma\pi(1 - \pi), \sigma\pi(1 - \pi), \sigma\pi^2\}, \quad \text{with } \sigma \in (0, 1/2].$$

In Appendix C.1, I verify that both $\hat{z}^I > \hat{z}^P$ and $\mathcal{L}_z^I(z) \geq \mathcal{L}_z^P(z)$ with strict inequality if and only if $z \in (0, \hat{z}^I)$.

¹¹Meaning that b is fixed at some positive level.

(a) Welfare and real money balances.

(b) Welfare and the Fisher nominal rate.

Figure 1: Welfare in the baseline economy for various η^I , when supply of bonds is chosen optimally given the supply of real money balances. Both welfare and real money balances are normalized to one at the Friedman rule. See Appendix A for the parametrization.

Because buyers appropriate a constant share θ of DGM match surplus, $\theta^I(z) = \theta^P(z) = \theta$. With $\theta < 1$, OTC trade causes externalities on sellers. Exploiting properties of proportional bargaining, social surplus from an OTC meeting, $\tilde{\mathcal{F}}^s(m, b) = \tilde{\mathcal{F}}(m, b) + \tilde{\mathcal{E}}(m, b)$, can be written as

$$\begin{aligned} \tilde{\mathcal{F}}^s(m, b) = & \frac{\alpha(1-\theta)\delta^I + [1-\alpha(1-\theta)]\delta^P}{\alpha\delta^I + (1-\alpha)\delta^P} \frac{\mathcal{L}^I(l+m) - \mathcal{L}^I(m)}{\theta} + \\ & + \frac{[\alpha + \theta(1-\alpha)]\delta^I + (1-\alpha)(1-\theta)\delta^P}{\alpha\delta^I + (1-\alpha)\delta^P} \frac{\mathcal{L}^P(m-l) - \mathcal{L}^P(m)}{\theta} \end{aligned}$$

with l determined by Equation (21). The socially optimal transfer of money in an OTC meeting, $\hat{l}^s \geq 0$, satisfies

$$\{\alpha(1-\theta)\delta^I + [1-\alpha(1-\theta)]\delta^P\}\mathcal{L}^I(\hat{l}^s + m) \geq \{[\alpha + \theta(1-\alpha)]\delta^I + (1-\alpha)(1-\theta)\delta^P\}\mathcal{L}^P(m - \hat{l}^s),$$

with equality if $\hat{l}^s < m$. When households are unconstrained by their bond holdings, they however trade $\hat{l} \geq 0$ given by Equation (19). It follows that for intermediate values of m , OTC trade volume can be inefficiently large:

$$\hat{l}^s(m) \leq \hat{l}(m) \quad \text{with strict inequality iff } \theta < 1 \text{ and } \underline{m} < m < [\hat{z}^I + \hat{z}^P]/2,$$

where $\underline{m}$ satisfies

$$\{\alpha(1-\theta)\delta^I + [1-\alpha(1-\theta)]\delta^P\}\mathcal{L}_z^I(2\underline{m}) = \{[\alpha + \theta(1-\alpha)]\delta^I + (1-\alpha)(1-\theta)\delta^P\}\mathcal{L}_z^P(0).$$

The reason for this result is due to the two effects of OTC trade: (i) trade leads to different distribution of money in the DGM and (ii) trade leads to a different distribution of savings. The second effect implies that it is both socially and privately efficient to transfer more money from the patient to the impatient household than what is needed to achieve the best distribution of money in the DGM. Because money is only useful for buyers, households do not take into account welfare losses on sellers that arise in the DGM. As a result, private incentives to transfer money from the patient to the impatient household can be too large. Policy can correct this wrong incentive by choosing b such that $\hat{l}^s(m) = a_I^{-1}(m, b)$. That means, households cannot trade too much money during the OTC because they hold insufficiently many bonds.

The result above points towards the fact that bonds are special compared to equity or credit. If households would be able to trade equity for money in the OTC, where equity could for example be asset with an exogenously specified dividend in the spirit of Lucas (1978), then only for specific parameter values socially optimal transactions in OTC matches can be achieved. Similarly, if households were able to use credit in the OTC, then similar transactions would arise as when households are unconstrained by

their bond holdings. Households would then trade too much when the equilibrium value for money balances is in the interval described above.¹² When only bonds are tradable for money in the OTC, the problem of socially inefficient trades does not arise as long as government has the monopoly on issuing bonds.

In an economy with competitive financial markets, Berentsen et al. (2014) also find that trade volume can be too high, but because of other reasons than studied here. Geromichalos and Herrenbrueck (2016) argue in favor of OTC trade because, according to them, this resolves the externalities identified by Berentsen et al. (2014). I find that when savings matter, trade volume can still be inefficiently large within OTC matches. Restricting the supply of bonds is a way to resolve this inefficiency. When government is active in the OTC, another remedy is a financial transactions tax.

Taking into account the externalities from OTC trade, Equation (29) becomes:

$$\begin{aligned}\tilde{\mathcal{W}}(m, b) = & U(y^*) - y^* + \pi^I \left[1 - \eta^I \frac{\alpha(1-\theta)\delta^I + [1 - \alpha(1-\theta)]\delta^P}{\alpha\delta^I + (1-\alpha)\delta^P} \right] \frac{\mathcal{L}^I(m)}{\theta} \\ & + \pi^P \left[1 - \eta^P \frac{[\alpha + (1-\alpha)\theta]\delta^I + (1-\alpha)(1-\theta)\delta^P}{\alpha\delta^I + (1-\alpha)\delta^P} \right] \frac{\mathcal{L}^P(m)}{\theta} \\ & + \frac{\omega}{\alpha\delta^I + (1-\alpha)\delta^P} \left[\frac{\{\alpha(1-\theta)\delta^I + [1 - \alpha(1-\theta)]\delta^P\}\mathcal{L}^I(l+m)}{+ \{[\alpha + \theta(1-\alpha)]\delta^I + (1-\alpha)(1-\theta)\delta^P\}\mathcal{L}^P(m-l)} \right].\end{aligned}$$

With $l = \hat{l}^s(m)$ when b is set appropriately, with $m + \hat{l}^s(m)$ and $m - \hat{l}^s(m)$ increasing in m , and with $\mathcal{L}^i(z)$ increasing in z until $\hat{z}^i$, it follows that

$$\tilde{\eta}^I = \{\alpha\delta^I + (1-\alpha)\delta^P\} / \{\alpha(1-\theta)\delta^I + [1 - \alpha(1-\theta)]\delta^P\}.$$

Here, $\tilde{\eta}^I$ is increasing in the DGM bargaining power of buyers. With greater bargaining power for buyers, the private benefits of carrying money into the DGM increase because with given money holdings, more goods can be obtained. Money and bonds become less substitutable and bonds therefore trade at a larger discount during the OTC, which improves the distribution of savings. As with a Walrasian DGM and because of similar reasons, $\tilde{\eta}^I$ is decreasing in the dispersion of discount factor shocks and decreasing in α . Even though bonds matter, when the supply of bonds is passive, the critical threshold to rationalize deviations from the Friedman rule remains unchanged. The intuition is again the same as with a Walrasian DGM.

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¹²For many numerical examples, an optimal policy sets $m \in (\underline{m}, [\hat{z}^I + \hat{z}^P]/2)$ so that indeed households are effectively constrained by their bond holdings in the OTC.

6 An Economy with Bonds and Notes

This section introduces notes into the economy. The purpose is to provide sufficient conditions to let three properties hold. First, an economy with money and bonds cannot benefit from the introduction of notes. Second, an economy with only notes and money can exhibit coexistence of money and interest-bearing notes in an optimal policy regime. Third, in an economy with only money and notes, optimized welfare is increasing in the indirect liquidity of notes but decreasing in the direct liquidity of notes. A first candidate for such a sufficient condition is given by:

Condition 1. *At the margin, liquid assets are more valuable for impatient than for patient agents: $\mathcal{L}_z^I(z) \geq \mathcal{L}_z^P(z)$, $\forall z \geq 0$.*

Condition 1 is satisfied for a variety of DGM trading arrangements.¹³ When types are private information during the DGM, so that there is a single pricing protocol $v(q)$ across all DGM meetings, Condition 1 is satisfied trivially because for impatient households, the opportunity cost of spending money on DGM goods are lower than for patient households. As a second candidate, consider:

Condition 2. *Surplus of sellers in a DGM match is increasing in consumption by buyers: $\mathcal{L}^i(z)[1 - \theta^i(z)]/\theta^i(z)$ is increasing in z for $z \in [0, \hat{z}^i]$ and $i \in \{I, P\}$.*

Condition 2 implies that in DGM meetings, sellers cannot be worse off when buyers consume more. Intuitively, this is a weak condition because sellers can restrict their production.

6.1 Introducing Notes in an Economy with Bonds and Money

Consider an economy with only money and bonds. With Condition 1 or 2 satisfied, this economy cannot benefit from the introduction of notes in an optimal policy environment.

Proposition 2. *When Condition 1 or 2 is satisfied, in an economy with money and bonds, introducing notes does not improve welfare in an optimal policy environment.*

Using Equations (23) and (25) in Equation (27) implies that welfare in the three-asset economy can be written as a weighted average of welfare in two baseline economies

$$\mathcal{W}(m, d, b) = (1 - \chi)\tilde{\mathcal{W}}(m, g_1(m, d, b)) + (1 - \chi)\tilde{\mathcal{W}}(m, g_2(m, d, b)).$$

Therefore, introducing notes in an optimal policy environment can only increase welfare when there exists a pair (m', b') such that $\tilde{\mathcal{W}}(m', b') > \max_{\{m, b\} \in \mathbb{R}_+^2} \tilde{\mathcal{W}}(m, b)$. Clearly, this requires $b' < 0$ so that in a baseline OTC meeting, the pattern of trade reverses.

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¹³This includes Walrasian pricing, proportional bargaining, and gradual bargaining over liquid assets.

When Condition 1 holds, the standard OTC trade pattern always involves a transfer of money from the patient to the impatient household. A reversal of trade then implies that the patient sell bonds to the impatient at a discount. From Equation (30), it however follows that OTC trade then negatively contributes to welfare; a contradiction.

When Condition 2 holds, a reversal in the OTC trade pattern also implies that OTC trade will contribute negatively to welfare. The reason is that if it would contribute positively to welfare, this can only be attributed to the externalities from OTC trade. Specifically, when compared to the Friedman rule, externalities from OTC trade must then contribute positively to welfare. But since no household is constrained by his/her money holdings at the Friedman rule, it must imply that in some DGM meetings lower money holdings by buyers make sellers better off.

6.2 Optimal Coexistence with Only Notes and Money

In an economy with only notes and money, existence of a threshold $\tilde{\eta}_\chi^I \in (0, 1)$ to rationalize deviations from the Friedman rule and to explain optimal coexistence is preserved:

Proposition 3. *In an economy with only notes and money, for all $\chi \in (0, 1)$ there exists an $\tilde{\eta}_\chi^I \in (0, 1)$ such that the Friedman rule is sub-optimal if and only if: (i) $\eta^I > \tilde{\eta}_\chi^I$ and (ii) $d > 0$. Because notes are irrelevant for welfare at the Friedman rule, if and only if $\eta^I > \tilde{\eta}_\chi^I$ optimal policy implies coexistence of money and interest-bearing notes.*

Intuitively, when policy is sufficiently close to the Friedman rule, notes provide no direct liquidity services at the margin. Specifically, when policy is close to the Friedman rule, the real value of money is only slightly lower than what is needed to let agents be unconstrained by their money holdings in DGM meetings. With a positive supply of notes, agents are therefore unconstrained by their asset holdings in DGM meetings in which notes can be transacted. Therefore, for small deviations from the Friedman rule an economy with notes behaves the same as an economy with bonds.

Because the introduction of notes in an economy with only bonds does not improve welfare when either Condition 1 or 2 is satisfied, it follows that when policies are determined optimally, an economy with only notes cannot do better than an economy with only bonds. Therefore, in an economy with only notes, the critical threshold $\tilde{\eta}_\chi^I$ can only be larger than in a similar economy with only bonds. Recall that in an economy with only bonds and money, a DGM with Walrasian pricing or random matching and proportional bargaining implies that $\tilde{\eta}^I$ is determined by small deviations from the Friedman rule. Because an economy with only notes then behaves the same as an economy with only bonds, it follows that $\tilde{\eta}_\chi^I$ is independent of χ for an economy with a Walrasian DGM or a DGM with random matching and proportional bargaining.

6.3 Direct Liquidity versus Indirect Liquidity of Notes

How do indirect and direct liquidity of assets other than fiat money matter for welfare? To address this question, consider indirect and direct liquidity of notes in an economy with only notes and money.

6.3.1 Indirect Liquidity of Notes

What happens when OTC search frictions, which characterize indirect liquidity of notes, change? For the sake of this purpose, recall that $\omega = \pi^I \eta^I = \pi^P \eta^P$. Hence, consider what happens to optimized welfare for a change in ω , keeping π^I and π^P unchanged.

Proposition 4. *When policy is chosen optimally in an economy with only notes and money, welfare is strictly increasing in ω if $\eta^I \geq \tilde{\eta}_\chi^I$ and if $\eta^I < \tilde{\eta}_\chi^I$, welfare is independent of small changes in ω .*

Intuitively, when $\eta^I < \tilde{\eta}_\chi^I$ a small increase in the indirect liquidity of notes implies that the Friedman rule remains optimal, so that welfare remains unchanged. When $\eta^I \geq \tilde{\eta}_\chi^I$, a small increase in the indirect liquidity of notes implies that the Friedman rule remains or becomes sub-optimal. In an optimal policy regime, OTC trade then contributes positively to welfare. Since a small increase in the indirect liquidity of notes also increases the measure of OTC matches, welfare improves even when the real value of money and notes remains unchanged.

I do not formally consider how optimal policy is affected by changes in the indirect liquidity of notes. First, a regularity assumption is needed: there should be unique values for $m = \phi_{t+1} M_t$ and $d = \phi_{t+1} D_t$ that maximize welfare. Second, an increase in indirect liquidity has an ambiguous effect on γ .

On the one hand, when in an optimal coexistence regime the indirect liquidity of notes increases, the optimal value of real money balances decreases. Basically, the marginal welfare benefits of stimulating OTC trade through reducing real money balances increase relative to the marginal welfare cost of reduced DGM trade. In turn, the reduction in real money balances puts upward pressure on the Fisher nominal interest rate, inflation, and γ , as the marginal benefits of carrying money into the DGM increase.

On the other hand, increased indirect liquidity puts downward pressure on γ . This is because notes become better in providing insurance against preference shocks: When the probability that an interest-bearing note can be traded in the OTC increases, there is less need for households to hold return dominated money. Fixing real money balances, the Fisher nominal interest rate, inflation, and γ then decrease.

Which of the two opposing effects dominates, depends on the parametrization of the model. This is demonstrated by the closed form solutions in Appendix B. For these solutions, Figure 2a shows how welfare depends on real money balances and the indirect

liquidity of notes. Figures 3a, 4a, and 5a illustrate how optimized welfare, optimized real money balances, and the associated Fisher rate depend on the indirect liquidity of notes.

6.3.2 Direct Liquidity of Notes

Consider a change in the fraction χ of buyers that can use notes as a payment instrument in DGM matches. In other words, what happens when the direct liquidity of notes changes?

Proposition 5. *Suppose that Condition 1 or 2 is satisfied. In an economy with only notes and money, welfare is then globally decreasing in the liquidity of notes. When externalities from OTC trade can be corrected by a financial transactions tax, optimized welfare is linear in the direct liquidity of notes when $\eta^I > \tilde{\eta}^I$ and independent of the direct liquidity of notes when $\eta^I \leq \tilde{\eta}^I$.*

Because the introduction of notes in an economy with bonds cannot improve optimized welfare, we know that welfare in an economy with only notes and money is bounded by optimized welfare in an economy with only money and bonds. Moreover, when optimal policy deviates from the Friedman rule in an economy with both notes and bonds, bonds are essential; $b > 0$ when policy is chosen optimally. The reason is that impatient households in an OTC match need to become better off in terms of DGM liquidity in all DGM matches, i.e. both $l > 0$ and $k + l > 0$. Such transfers however make patient households in DGM matches worse off. These households need to be compensated by receiving bonds at a discount. When optimal policy does not implement the Friedman rule, an economy with only notes is therefore worse off compared to an economy with only bonds.

Because notes become equivalent to bonds when $\chi \rightarrow 0$, the analysis above implies that optimized welfare in an economy with only notes and money, as a function of χ , is maximized when $\chi = 0$. On the other hand, when $\chi \rightarrow 1$ notes become a perfect substitute for money. In an economy with only notes and money, OTC trade then vanishes and by continuity, optimized welfare approaches welfare at the Friedman rule. Because the Friedman rule is always implementable and welfare at the Friedman rule does not depend on χ , it follows that welfare is globally decreasing in the direct liquidity of notes.

The effects of small changes in the direct liquidity of notes are more difficult to assess. The reason is that the supply of notes affects both OTC trade and DGM trade. When externalities from OTC trade can be corrected with a financial transactions tax, or when there are no externalities from OTC trade, notes are no longer needed to ensure socially optimal OTC transactions. The optimal supply of notes then becomes such that households are never constrained by their note holdings. At the margin, notes then

(a) Indirect liquidity.

(b) Direct liquidity.

Figure 2: Welfare, indirect liquidity, and direct liquidity in an economy with only notes and money. Drawn for an optimal supply of notes given the supply of real money balances. See Appendix A for the parametrization.

(a) Optimized welfare and indirect liquidity.

(b) Optimized welfare and direct liquidity.

Figure 3: Optimized welfare, indirect liquidity, and direct liquidity in an economy with only notes and bonds. Welfare is normalized to one at the Friedman rule. See Appendix A for the parametrization.

(a) Optimal real money balances and indirect liquidity.

(b) Optimal real money balances and direct liquidity.

Figure 4: Optimized real money balances, indirect liquidity, and direct liquidity in an economy with only notes and bonds. Real money balances are normalized to one at the Friedman rule. See Appendix A for the parametrization.

(a) Optimal Fisher interest rate and indirect liquidity.

(b) Optimal Fisher interest rate and direct liquidity.

Figure 5: Optimal Fisher interest rate, indirect liquidity, and direct liquidity. See Appendix A for the parametrization.

provide no DGM liquidity and become equivalent to bonds. Reducing the supply of notes then only hurts welfare because DGM trade may reduce and when notes start to provide DGM liquidity services at the margin, notes will trade for money at prices closer to par in the OTC.

When households are unconstrained by their note holdings and with an optimal financial transactions tax, SSE flow welfare (illustrated by Figure 2b) becomes

$$\mathcal{W} = \chi \mathcal{W}^{fr} + (1 - \chi) \tilde{\mathcal{W}}(m, b^*(m)), \quad \mathcal{W}^{fr} \equiv \pi^I \mathcal{L}^I(\hat{z}^I) / \theta^I(\hat{z}^I) + \pi^P \mathcal{L}^P(\hat{z}^P) / \theta^P(\hat{z}^P).$$

Here, $\mathcal{W}^{fr}$ represents flow welfare when policy implements the Friedman rule and $b^*(m)$ is the socially optimal supply of bonds in the baseline economy. It follows that $\tilde{\eta}_\chi^I$ and optimal real money balances are independent of χ . Also, optimized welfare is linearly decreasing in χ , and strictly decreasing when $\eta^I > \tilde{\eta}^I$. From Lemma 2, it follows that in an optimal policy regime, the Fisher rate, inflation, and γ are linearly decreasing in χ . These insights are illustrated by Figures 3b, 4b, and 5b.

6.3.3 Discussion

Propositions 4 and 5 demonstrate that from a welfare perspective, direct and indirect liquidity of assets other than fiat money are different concepts. As in Kocherlakota (2003) but for different reasons, the model economy is best off with money and bonds. Bonds ideally provide as much indirect liquidity as possible, but no direct liquidity. Kocherlakota (2003) only shows that bonds which provide no direct liquidity services are better for welfare than bonds which are perfect substitutes for money; $\chi = 0$ is better than $\chi = 1$. In the current paper, for a different mechanism, I provide conditions such that assets other than fiat money ideally provide no indirect liquidity services. Moreover, these conditions imply that a society cannot benefit from introducing assets which provide some direct liquidity (notes) when assets which provide no direct liquidity (bonds) are already available.

My results regarding the welfare effects of direct and indirect liquidity contrast findings by Geromichalos, Jung, Lee, and Carlos (2019). These authors combine the model of endogenous direct asset liquidity by Lester, Postlewaite, and Wright (2012) with an OTC financial market. Taking inflation and the supply of assets other than fiat money as given, Geromichalos et al. (2019) find that improving direct asset liquidity can positively affect welfare while improving indirect asset liquidity may hurt welfare. Their results arise only when policy deviates from the Friedman rule while in their environment, the Friedman rule is optimal. Though the current paper considers exogenously given parameters for direct and indirect asset liquidity, the opposite findings of Geromichalos et al. (2019) suggest that the welfare effects of changing direct or indirect asset liquidity depend on the policy context. Specifically, it matters whether optimal policy regimes are considered.

7 Walrasian Financial Markets

Walrasian financial markets aggregate preferences differently than OTC markets since households then do not bargain but take prices as given. This turns out to matter for the qualitative welfare effects of deviating from the Friedman rule.

7.1 Modified Environment

I consider a modified environment with only money and bonds in which the OTC market is replaced by a *Walrasian financial market* (WFM) based on Li and Li (2013). In WFM t , bonds trade for money at a relative price $1/(1 + \iota_t)$. Here, ι_t is the nominal rate earned by holding bonds from WFM t until CM $t + 1$. Households can only obtain money in the WFM by selling bonds; credit is infeasible. When households carry sufficiently many bonds, allocations are however equivalent to those when credit is feasible. The WFM can then be interpreted as a bank in spirit of Berentsen et al. (2007), where households can borrow or deposit money between WFM t and CM $t + 1$ at nominal rate ι_t . To maintain a notion of frictions, I assume that an impatient (patient) household can trade with probability η^I (resp. η^P). To allow for a comparison with OTC markets, I impose that $\pi^I \eta^I = \pi^P \eta^P = \omega \leq \min\{\pi^I, \pi^P\}$.

Let l_t^i denote the net amount of money, expressed in terms of CM $t + 1$ goods, acquired during WFM t by a household with $\delta_t = \delta^i$. The value of entering WFM t for a household that carries money worth m_t CM $t + 1$ goods and bonds worth b_t CM $t + 1$ goods, becomes

$$O_t^i(m_t, b_t) = \eta^i \max_{-m_t \leq l_t^i \leq \frac{b_t}{1 + \iota_t}} \{ \mathcal{L}^i(l_t^i + m_t) - \beta \delta^i l_t^i \iota_t \} + (1 - \eta^i) \mathcal{L}^i(m_t) + \beta \delta^i (m_t + b_t + \bar{W}_{t+1}).$$

The amount of money traded by households with access to WFM t then satisfies

$$\mathcal{L}_z^i(l_t^i + m_t) \leq \max\{\beta \delta^i \iota_t, \mathcal{L}_z^i(b_t/(1 + \iota_t) + m_t)\}, \quad \text{with } = \text{ if } -l_t^i > m_t. \quad (33)$$

It can be verified that $O_t^i(m_t, b_t)$ is concave, so we can focus on symmetric equilibria. Equation (7) implies that necessary and sufficient conditions for optimal asset portfolios carried out of CM t are given by:

$$\beta i_t^f \geq \sum_{i \in \{I, P\}} \pi^i \left[\eta^i \max \left\{ \mathcal{L}_z^i \left(\frac{b_t}{1 + \iota_t} + m_t \right), \beta \delta^i \iota_t \right\} + (1 - \eta^i) \mathcal{L}_z^i(m_t) \right] \quad \text{with } = \text{ if } m_t > 0, \quad (34)$$

$$\beta \frac{i_t^f - i_t^b}{1 + i_t^b} \geq \sum_{i \in \{I, P\}} \frac{\pi^i \eta^i \max \left\{ \mathcal{L}_z^i \left(\frac{b_t}{1 + \iota_t} + m_t \right) - \beta \delta^i \iota_t, 0 \right\}}{1 + \iota_t} \quad \text{with } = \text{ if } b_t > 0. \quad (35)$$

Definition 2. Given a policy sequence $\{M_t, B_t\}_{t=0}^\infty$, a symmetric equilibrium is a sequence of prices $\{\phi_t, \psi_t\}_{t=0}^\infty$, interest rates $\{i_t^f, i_t^b, \iota_t\}_{t=0}^\infty$, portfolio choices $\{m_t, b_t\}_{t=0}^\infty$ and financial market trades $\{l_t^I, l_t^P\}_{t=0}^\infty$ such that for all t :

1. Households maximize utility:

- (a) Optimal CM behavior implies that m_t and b_t satisfy Equation (34) and (35).
- (b) Optimal WFM behavior implies that l_t^i satisfies Equation (33) for $i \in \{I, P\}$.
- (c) Optimal DGM behavior implies that for $i \in \{I, P\}$, we have that $\mathcal{L}^i(m_t)$ is given by Equation (3).

2. Interest rates i_t^f and i_t^b satisfy Equation (13).

3. Markets clear; $\phi_{t+1}M_t = m_t$, $\phi_{t+1}B_t = b_t$, and $\pi^I \eta^I l_t^I + \pi^P \eta^P l_t^P = 0$.

Lemma 3. In symmetric equilibria, given m_t and b_t there is a unique ι_t so that WFM t clears; $\pi^I \eta^I l_t^I + \pi^P \eta^P l_t^P = 0$. This ι_t is non-negative and with $\omega = \pi^I \eta^I = \pi^P \eta^P$, it is independent of ω . When $2m_t > \hat{z}^I + \hat{z}^P$, clearing WFM t requires that $\iota_t = 0$; the economy is in a liquidity trap.

Important to mention here is that when household enter WFM t with enough money to let all market participants be unconstrained in the DGM ($2m_t > \hat{z}^I + \hat{z}^P$), equilibrium implies $\iota_t = 0$. That means, money and bonds trade at par and the distribution of savings remains unaffected by financial market activity. The reason is that for some households, the marginal unit of money carried out of the WFM is a mere savings instrument. This requires $\iota_t = 0$ as otherwise, households are better off using bonds as a savings instrument.

To formally characterize welfare, let $\bar{\iota}(m, b) \geq 0$ denote the equilibrium interest rate that clears WFM t when households carry money and bonds worth m and, respectively, b CM $t+1$ goods into WFM t . Let $\bar{l}(m_t, b_t) = l_t^I$ be the solution of Equation (33) when $\iota_t = \bar{\iota}(m_t, b_t)$ and $i = I$. Flow welfare can then be written as

$$\tilde{\mathcal{W}}(m_t, b_t) = \pi^I \mathcal{L}^I(m_t) / \theta^I(m_t) + \pi^P \mathcal{L}^P(m_t) / \theta^P(m_t) + \omega [\tilde{\mathcal{F}}(m_t, b_t) + \tilde{\mathcal{E}}(m_t, b_t)] + U(y^*) - y^*, \quad (36)$$

with $\tilde{\mathcal{E}}(z_t, b_t)$ given by Equation (26) and

$$\begin{aligned} \tilde{\mathcal{F}}(m_t, b_t) = & [\mathcal{L}^I(\bar{l}(m_t, b_t) + m_t) - \mathcal{L}^I(m_t)] + [\mathcal{L}^P(m_t - \bar{l}(m_t, b_t)) - \mathcal{L}^P(m_t)] \\ & + \beta(\delta^P - \delta^I) \bar{\iota}(m_t, b_t) \bar{l}(m_t, b_t). \end{aligned} \quad (37)$$

Focusing on stationary policies $\langle \gamma, b \rangle$ in SSE, I find:

Lemma 4. *Given government policy $\langle \gamma, b \rangle$, SSE is described by an $m > 0$ that solves*

$$\gamma - \beta = \sum_{i \in \{I, P\}} \pi^i \left[\eta^i \max \left\{ \mathcal{L}_z^i \left(\frac{b}{1 + \bar{l}(m, b)} + m \right), \beta \delta^I \bar{l}(m, b) \right\} + (1 - \eta^i) \mathcal{L}_z^i(m) \right]. \quad (38)$$

Let

$$\hat{m} = \begin{cases} \max\{\hat{z}^I, \hat{z}^P\} & \text{if } \eta^I < 1 \\ \max\{\hat{z}^I - b, (\hat{z}^I + \hat{z}^P)/2\} & \text{if } \eta^I = 1 \end{cases}.$$

Given b , there is a unique γ such that some arbitrary $m > 0$ satisfies Equation (38). This γ is continuous in m and $m \geq \hat{m} \Leftrightarrow \gamma = \beta$, $m < \hat{m} \Leftrightarrow \gamma > \beta$, and $\lim_{\gamma \downarrow \beta} m = \hat{m}$.

Contrasting the baseline model, when $m \geq (\hat{z}^I + \hat{z}^P)/2$ we may already be at the Friedman rule. This occurs when impatient agents have perfect access to financial markets. The reason is that with Walrasian pricing, money and bonds trade at par in the WFM when $m \geq (\hat{z}^I + \hat{z}^P)/2$. Then, when unconstrained by their bond holdings, all households are unconstrained by their money holdings in the DGM.

7.2 Optimal Policy and Coexistence

Deviations from the Friedman rule can be optimal in the modified environment:

Proposition 6. *With Walrasian financial markets, depending on trading arrangement in the DGM there may exist a $\tilde{\eta}^I \in (0, 1)$ such that the Friedman rule is sub-optimal if and only if: (i) $\eta^I > \tilde{\eta}^I$ and (ii) $b > 0$. Therefore, if and only if $\eta^I > \tilde{\eta}^I$, optimal policy implies coexistence of money and interest-bearing bonds. With $1 > \eta^I > \tilde{\eta}^I$ small deviations from the Friedman rule reduce welfare, while the Friedman rule remains a sub-optimal policy.*

Proposition 6 implies that Walrasian markets are special because of two reasons. First, the DGM trading arrangement matters for the existence of a critical threshold to rationalize deviations from the Friedman rule: Even when all households manage to trade in the WFM, the Friedman rule can still be the optimal policy. This occurs when in some DGM matches, first-best quantities cannot be attained; there exist $ij \in \{I, P\}^2$ such that $\sigma_{ij}(q_{ij}^* - \hat{q}_{ij}) > 0$. For example, this is the case with Nash bargaining in the DGM. Then, a small deviation from the Friedman rule does not only generate a first-order welfare gain due to increased trade in the WFM, but also a first-order welfare loss because of reduced trade in the DGM. Due to the competitive nature of the WFM, the latter effect can dominate the former when the dispersion in discount factor shocks is sufficiently small. When the DGM is characterized by Walrasian pricing, proportional bargaining, or gradual bargaining, we have $q_{ij}^* = \hat{q}_{ij}$ and deviating from the Friedman rule remains optimal when all agents have access to the WFM.

Second, when almost all impatient agents can trade during the WFM ($\eta^I \rightarrow 0$), small deviations from the Friedman rule negatively affect welfare. The reason is that for small deviations from the Friedman rule ($m = \hat{z}^I - \varepsilon$), the economy fails to escape a liquidity trap; $\bar{l}(m, b) = 0$ when $(\hat{z}^I + \hat{z}^P)/2 < m < \hat{z}^I$ so bonds trade at par for money in the WFM (see Lemma 3). Before the beneficial effects of deviating from the Friedman rule kick in, welfare losses therefore materialize since a measure $\pi^I(1 - \eta^I)$ of impatient households that are unable to trade during the WFM, become constrained by their money holdings in the DGM. With OTC markets, these welfare losses also occur. But, when η^I is sufficiently large, these losses are offset by the beneficial effects of OTC trade because bargaining implies that bonds then do trade at a discount.

For a Walrasian DGM, Figure 6a plots welfare as a function of real money balances for various η^I when the supply of bonds is chosen optimally.¹⁴ Similarly, Figure 6b plots welfare as a function of the Fisher rate. The figures illustrate how small deviations from the Friedman rule reduce welfare when $\eta^I < 1$, while the Friedman rule is a sub-optimal policy when η^I is sufficiently large.

7.3 Welfare in OTC versus WFM

OTC financial markets can achieve better outcomes than Walrasian financial markets. After all, when η^I is sufficiently close to one, welfare can always be improved by deviating from the Friedman rule with OTC markets, while this is not true for Walrasian financial markets. Furthermore, even when welfare can be improved with Walrasian financial markets, OTC financial markets can attain strictly more welfare. Why? Because in an OTC economy, we can replicate trades in Walrasian financial markets by choosing OTC bargaining power appropriately. Moreover, in an OTC economy, by changing the supply of bonds appropriately, we can let bargaining power of patient agents in OTC matches increase without changing the amount of money transferred within OTC matches. Then, more bonds are transferred to the patient agents, so the distribution of savings is further improved, while the distribution of money remains unchanged.

Geromichalos and Herrenbrueck (2016) also argue that OTC financial markets can outperform Walrasian financial markets. In their model, OTC trade avoids the externalities discussed by Berentsen et al. (2014). In the models of Berentsen et al. (2014) and Geromichalos and Herrenbrueck (2016), these effects only arise when policy is not chosen optimally. The distinguishing feature of the current model, is that it shows how OTC financial markets outperform Walrasian markets in an optimal policy environment.

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¹⁴Whenever financial market trade causes no externalities on sellers, the optimal supply of bonds is such that households are unconstrained by their bond holdings during the WFM.

(a) Welfare and real money balances.

(b) Welfare and the Fisher nominal rate.

Figure 6: Welfare with Walrasian financial markets for various η^I when the supply of bonds is chosen optimally given the supply of real money balances. See Appendix A for the parametrization.

8 Conclusion

This paper introduces patient and impatient households in an otherwise standard monetary model with financial markets. In the model, not only the cross-sectional distribution of money but also the cross-sectional distribution of savings matters for welfare. When money and bonds change hands in financial markets, welfare is affected as savings instruments are traded. If and only if the fraction of households that are able to trade in financial markets is sufficiently large, coexistence of money and interest-bearing bonds arises endogenously when policy is set optimally. Ideally, bonds should not be accepted as a payment instrument but should be easy to trade in financial markets. Qualitatively, it matters whether financial markets are characterized by Walrasian pricing or bilateral trade with bargaining. Nevertheless, both trading arrangements can explain the coexistence puzzle.

A Parameters for Figures

Welfare normalized to one at the Friedman rule and real money balances normalized to one at $\hat{z}^I$. For welfare calculations, the term $U(y^*) - y^*$ is ignored. In all figures, I assume Walrasian pricing in the DGM, i.e. $v_{ij}(q) = q/(\beta\delta^P)$ for all $ij \in \{I, P\}^2$ and $\Sigma = \{\sigma_{II}, \sigma_{IP}, \sigma_{PI}, \sigma_{PP}\} = \sigma\{0, \pi^I, 0, \pi^P\}$ with $\sigma \in (0, 0.5]$. I set $u(q) = q^{1-\rho}/(1-\rho)$, and use $\rho = 0.5$, $\sigma = 0.5$, $\beta = 0.99$ and $\pi^I = \pi^P = 0.5$ in all figures. Remaining parameters are as follows.

Figure	FM	Bonds	Notes	δ^I	δ^P	α	ω	χ
1	OTC	Yes	No	0.5	1.5	0.5	$\{0, 1/3, 0.5\}$	0
2a	OTC	No	Yes	0.5	1.5	0.5	$[0, 0.5]$	0
2b	OTC	No	Yes	0.5	1.5	0.5	0.5	$[0, 1]$
3a, 4a, 5a	OTC	Yes	No	0.5	1.5	$\{0.2, 0.35, 0.5\}$	$[0, 0.5]$	0
3b, 4b, 5b	OTC	No	Yes	0.5	1.5	$\{0.2, 0.35, 0.5\}$	0.5	$[0, 1]$
6	WFM	Yes	No	0.6	1.4	NA	0.5	0

Table 1: Specific parameters for figures.

B Closed-Form Solutions with a Walrasian DGM

With a Walrasian DGM, there are closed form solutions for optimal policies in an economy with only bonds or notes when we choose $u(q) = [(q + \varpi)^{1-\rho} - \varpi^{1-\rho}]/(1-\rho)$ with $\rho > 0$, $\varpi = 0$ when $\rho < 1$ and $\varpi \downarrow 0$ when $\rho \geq 1$. This utility function generalizes standard constant relative risk aversion preferences by including ϖ , which forces $u(0) = 0$.

With $v(q) = \beta\delta_I$ and $\Sigma = \{0, \sigma\pi, 0, \sigma(1-\pi)\}$, where $\sigma \in (0, 1/2]$, we obtain $\hat{z}^i = (\beta\delta^P)^{-\rho}\delta^P/\delta^i$ and $\mathcal{L}_z^i(z) = \sigma \max\{(\beta\delta_P)^{1-\rho}z^{-\rho} - \beta\delta^i, 0\}$. It follows from the analysis in Section 5.2 that in an economy with only bonds and money, the optimal supply of bonds is such that households are not constrained by their bond holdings during the OTC. Similarly, from the proof of Proposition 5 it follows that in an economy with only notes and money, the optimal supply of notes is such that households are not constrained by their bond holdings during the OTC and DGM. The amount of money traded in an OTC meeting, l/ϕ_{t+1} , therefore satisfies

$$l(m) = \begin{cases} m \frac{(\delta^P)^{1/\rho} - (\delta^I)^{1/\rho}}{(\delta^P)^{1/\rho} + (\delta^I)^{1/\rho}} & \text{if } 2m < \hat{z}^I + \hat{z}^P \\ (\beta\delta^P)^{-\rho}\delta^P/\delta^I - m & \text{if } 2m \geq \hat{z}^I + \hat{z}^P \end{cases}.$$

Because the optimal supply of notes in an economy with only notes and money is such that households are not constrained by their note holdings during DGM, flow welfare as a function of real money balances carried out of the CM, $m = \phi_{t+1}M_t$, becomes $\mathcal{W} = \chi[\pi^I \mathcal{L}(\hat{z}^I) + \pi^P \mathcal{L}(\hat{z}^P)] + (1-\chi)\widehat{\mathcal{W}}(m) + U(y^*) - y^*$, where:

$$\begin{aligned} \widehat{\mathcal{W}}(m) = & \pi^I \left[\frac{(1-\eta^I)\delta^P - \alpha(\delta^P - \delta^I)}{\alpha\delta^I + (1-\alpha)\delta^P} \right] \mathcal{L}^I(m) \\ & + \pi^P \left[\frac{(1-\eta^P)\delta^I + (1-\alpha)(\delta^P - \delta^I)}{\alpha\delta^I + (1-\alpha)\delta^P} \right] \mathcal{L}^P(m) \\ & + \omega \frac{\delta^P \mathcal{L}^I(l(m) + m) + \delta^I \mathcal{L}^P(m - l(m))}{\alpha\delta^I + (1-\alpha)\delta^P}. \end{aligned}$$

Regardless of considering an economy with only bonds or only notes, real money balances are chosen to maximize $\widehat{\mathcal{W}}(m)$, and the optimal amount of real money balances is independent of χ .

Define $\alpha^P = \alpha$, $\alpha^I = (1-\alpha)$, $l^P(m) = -l^I(m)$, and $l^I(m) = l(m)$. When households are never constrained by bond holdings or note holdings, we obtain for the Fisher rate:

$$\begin{aligned} \frac{i^f [\alpha\delta^I + (1-\alpha)\delta^P]}{\sigma(1-\chi)} = & \sum_{i \in \{I, P\}} \pi^i [\alpha^{j \neq i} \delta^i + \alpha^i (1-\eta^i) \delta^{j \neq i}] \max \left\{ \frac{(\delta_P)^{1-\rho}}{\beta^\rho m^\rho} - \delta^i, 0 \right\} \\ & + \sum_{i \in \{I, P\}} \pi^i \eta^i \alpha^i \delta^{j \neq i} \max \left\{ \frac{(\delta_P)^{1-\rho}}{\beta^\rho (m + l^i(m))^\rho} - \delta^i, 0 \right\}. \end{aligned}$$

When $\eta^I < \tilde{\eta}^I = [\alpha\delta^I + (1 - \alpha)\delta^P]/\delta^P$, we have optimality of the Friedman rule so $m \geq \hat{z}^I \Rightarrow i^f = 0$. When $\eta^I > \tilde{\eta}^I$, the Friedman rule is no longer optimal so $m < \hat{z}^I$ and $i^f > 0$. Define $\Omega = (\delta^P - \delta^I)[\alpha\delta^I + (1 - \alpha)\delta^P] - 2\eta^I\delta^P\{[(\delta^I)^{1/\rho}/2 + (\delta^P)^{1/\rho}/2]^\rho - \delta^I/2 - \delta^P/2\}$. Some algebra implies that the optimal supply of real money balances satisfies

$$\beta\delta^P m^\rho = \begin{cases} \delta^P \left[1 - 2\pi^I \eta^I \frac{\delta^I/2 + \delta^P/2 - [(\delta^I)^{1/\rho}/2 + (\delta^P)^{1/\rho}/2]^\rho}{\alpha\delta^I + (1 - \alpha)\delta^P} \right] & \text{if } \Omega < 0 \\ \frac{\delta^P}{\delta^I} \left[1 - 2\eta^I \frac{\delta^P - [(\delta^I)^{1/\rho}/2 + (\delta^P)^{1/\rho}/2]^\rho}{\alpha\delta^I + (1 - \alpha)\delta^P + \eta^I\delta^P} \right] & \text{if } \Omega \geq 0 \end{cases}.$$

Here optimal policy is such that all households are constrained by their money holding during the DGM when $\Omega < 0$, while when $\Omega > 0$ only patient households that did not get matched during the OTC are unconstrained by their money holdings during the DGM. In the former case we find for the Fisher nominal rate

$$\frac{i^f}{\sigma(1 - \chi)} = \frac{(1 - \pi^I \eta^I)[\alpha\delta^I + (1 - \alpha)\delta^P] + \pi^I \eta^I [(\delta^I)^{1/\rho}/2 + (\delta^P)^{1/\rho}/2]^\rho}{\alpha\delta^I + (1 - \alpha)\delta^P - 2\pi^I \eta^I \{\delta^I/2 + \delta^P/2 - [(\delta^I)^{1/\rho}/2 + (\delta^P)^{1/\rho}/2]^\rho\}} - 1,$$

and in the latter case

$$\begin{aligned} \frac{i^f[\alpha\delta^I + (1 - \alpha)\delta^P]}{\sigma\pi\delta^I(1 - \chi)} &= \frac{\alpha\delta^I + [1 - \alpha(1 - \eta^I)]\delta^P + \eta^I [(\delta^I)^{1/\rho}/2 + (\delta^P)^{1/\rho}/2]^\rho - \delta^P}{(1 + \eta^P)\delta^P - \alpha(\delta^P - \delta^I) + 2\eta^I [(\delta^I)^{1/\rho}/2 + (\delta^P)^{1/\rho}/2]^\rho - \delta^P} \\ &\quad \times [(1 + \eta^P)\delta^P - \alpha(\delta^P - \delta^I)] \\ &\quad - \alpha\delta^I - [1 - \alpha(1 - \eta^I)]\delta^P. \end{aligned}$$

Because optimal policy is such that households are never constrained by their holdings of bonds or notes in an economy with, respectively, only bonds and notes, we also have that $i^b = i^n = i^f$.

The following insights arise from the closed form solution. First, optimal policies exhibit a discontinuity at $\eta^I = \tilde{\eta}^I$. This is because deviations from the Friedman rule, as long as $2m \geq \hat{z}^I + \hat{z}^P$, then do not affect welfare. Second, when $\eta^I > \tilde{\eta}^I$, the optimal value for m is monotonically decreasing in η^I . Equivalently, m is monotonically decreasing in ω when keeping π^I and π^P fixed. However, when $\eta^I < \tilde{\eta}^I$, the associated Fisher rate can be increasing or decreasing in η^I . Third, the optimal value for m does not depend on χ , but the associated Fisher rate is linear in χ .

C DGM Trading Arrangements with Bargaining

Following Gu and Wright (2016), I considered a generic trading mechanism during the DGM which implied existence of pricing protocols $v_{ij} : \mathbb{R}_+ \mapsto \mathbb{R}_+$ for $ij \in \{I, P\}^2$, and subsequently I imposed properties on these protocols to prove propositions. With a common pricing protocol across all DGM meetings, e.g. when considering Walrasian pricing, these properties were satisfied naturally.

Here, I consider what happens if DGM trade is characterized by bargaining, random matching, and households entering a DGM meeting at most once so that $\Sigma = \sigma\{(\pi^I)^2, \pi^I \pi^P, \pi^I \pi^P, (\pi^P)^2\}$, with $\sigma \in (0, 1]$. Note that during the CM, households' past preferences are private information, so that government cannot levy type-contingent taxes. Many bargaining solutions however require that households observe each other's preferences during the DGM. This is not a problem because households are anonymous during the DGM, so that the observability of preferences in DGM matches generates information that can be used only within these matches. There are also bargaining solutions for setups with asymmetric information about agents' preferences, e.g. those provided by Harsanyi and Selten (1972), and Samuelson (1984). Such solutions necessarily imply that the pricing protocols satisfy $v_{II}(q) = v_{IP}(q) = v_{PI}(q) = v_{PP}(q)$ for all q . Then, Assumption 2 and Condition 1 are trivially satisfied.

In what follows, I will investigate properties of the pricing protocols when these are generated by the proportional bargaining solution, the gradual bargaining solution, and the generalized Nash bargaining solution. To show that my results do not depend on linear utility cost of producing DGM goods, I take a more general approach and assume utility cost $c(q)$ of producing DGM goods with $c(0) = 0$, $c' > 0$ and $c'' \geq 0$. When a buyer with $\delta_t = \delta^i$ and seller with $\delta_t = \delta^j$ are matched, all trades on the Pareto frontier then satisfy $\delta^j u'(q) = \delta^i c'(q)$ when the buyer is unconstrained by its liquid asset holdings. Hence, q_{ij}^* now satisfies $\delta^j u'(q_{ij}^*) = \delta^i c'(q_{ij}^*)$ and we still have $q_{IP}^* > q_{II}^* = q_{PP}^* > q_{PI}^*$.

C.1 Random Matching with Proportional Bargaining

Consider proportional bargaining as in Kalai (1977), with buyers receiving a constant share $\theta \in (0, 1]$ of match surplus across all DGM meetings. During DGM t , surplus of a match between a buyer and a seller, where the former has $\delta_t = \delta^i$ and carries liquid assets worth z CM $t + 1$ goods, and the latter has $\delta_t = \delta^j$, is given by $\mathcal{S}_{ij}(x) = \max_{q,p} \{u(q) - c(q) - \beta(\delta^i - \delta^j)p\}$ subject to sharing rule $(1 - \theta)[u(q) - \delta^i \beta p] = \theta[-c(q) + \delta^j \beta p]$ and liquidity constraint $p \leq z$. Substituting out p in the objective function by using the sharing rule, we obtain:

$$\mathcal{S}_{ij}(z) = \max_q \frac{u(q)\delta^j - c(q)\delta^i}{(1 - \theta)\delta^i + \theta\delta^j} \quad \text{s.t. } v_{ij}(q) \leq z, \quad v_{ij}(q) = \frac{1}{\beta} \frac{(1 - \theta)u(q) + \theta c(q)}{(1 - \theta)\delta^i + \theta\delta^j}. \quad (\text{C.1})$$

Note that $v_{ij}(q)$ is increasing in q and, in line with Assumption 1, $u'(q)/v'_{ij}(q)$ is strictly decreasing in q .

A buyer maximizes $u(q) - \beta\delta^i v_{ij}(q)$, s.t. $z \leq v_{ij}(q)$, which is equivalent to solving Program (C.1). Therefore, in line with Assumption 1, $\hat{q}_{ij} = q_{ij}^*$ for all $ij \in \{I, P\}^2$. Because buyers obtain a constant share of match surplus, Condition 2 is also satisfied.

With, random matching and proportional bargaining, we obtain

$$\mathcal{L}^i(z) = \sigma\theta[\pi^I \mathcal{S}_{iI}(z) + \pi^P \mathcal{S}_{iP}(z)]. \quad (\text{C.2})$$

Equation (C.2) has the following properties, implying that Assumption 2 and Condition 1 are satisfied.

Lemma C.1. *With random matching and proportional bargaining, $\hat{z}^I > \hat{z}^P$ and $\mathcal{L}_z^I(z) \geq \mathcal{L}_z^P(z)$, with strict inequality if and only if $0 < z < \hat{z}^I$.*

C.2 Random Matching with Gradual Bargaining

Consider the gradual bargaining solution proposed by O'Neill, Samet, Wiener, and Winter (2004), further studied in the context of monetary-search economies by Rocheteau, Hu, Lebeau, and In (2020). Let $\theta \in (0, 1]$ denote bargaining power of buyers across all DGM meetings. Consider a meeting between a buyer and a seller during DGM t , where the former has $\delta_t = \delta^i$ and carries liquid assets worth x CM $t + 1$ goods, and the latter has $\delta_t = \delta^j$. Suppose the buyer commits to spend at most $\varrho \leq x$ worth of liquid assets (expressed in CM $t + 1$ goods) and that the seller commits to produce at most $\varsigma \geq 0$ units of DGM goods. The zero-normalized bargaining set is then given by:

$$S_{ij}(\varrho, \varsigma) = \left\{ \left(u(q) - \delta^i \beta p, \delta^j \beta p - c(q) \right) \in \mathbb{R}_+^2 : 0 \leq p \leq \varrho, \quad 0 \leq q \leq \varsigma \right\},$$

where $u(q) - \delta^i \beta p \equiv u_b$ is the buyer's surplus and $\delta^j \beta p - c(q) \equiv u_s$ the seller's surplus.

Let the Pareto frontier of $S_{ij}(\varrho, \varsigma)$ be described by the function $\mathcal{H}_{ij}(u_b, u_s, \varrho, \varsigma) = 0$. Straightforward algebra implies:

$$\mathcal{H}_{ij} = \begin{cases} u_b - u \circ c^{-1}(\delta^j \beta \varrho - u_s) + \delta^j \beta \varsigma & \text{if } \delta^j \beta \varrho < c \circ \min\{q_{ij}^*, \varsigma\} + u_s, \\ \delta^j u_b - \delta^j u(\varsigma) + \delta^i c(\varsigma) + \delta^i u_s & \text{if } \varsigma < \min\{q_{ij}^*, c^{-1}(\delta^j \beta \varrho - u_s)\}, \\ \delta^j u_b - \delta^j u(q_{ij}^*) + \delta^i c(q_{ij}^*) + \delta^i u_s & \text{otherwise.} \end{cases}$$

C.2.1 Gradual Bargaining over Liquid Assets

Suppose households bargain gradually over the amount of liquid assets to be traded in a DGM meeting. Thus, let $\varsigma = \infty$. The gradual bargaining solution implies

$$\frac{du_b}{d\varrho} = -\theta \frac{\partial \mathcal{H}_{ij}/\partial \varrho}{\partial \mathcal{H}_{ij}/\partial u_b} \quad \text{and} \quad \frac{du_s}{d\varrho} = -(1-\theta) \frac{\partial \mathcal{H}_{ij}/\partial \varrho}{\partial \mathcal{H}_{ij}/\partial u_s}, \quad \Rightarrow \quad \frac{du_b}{du_s} = \frac{\theta}{1-\theta} \frac{\partial \mathcal{H}/\partial u_s}{\partial \mathcal{H}/\partial u_b}, \quad (\text{C.3})$$

so that Condition 2 is satisfied. If $\delta^j \beta \varrho < c(q_{ij}^*) + u_s$, we obtain that $(1-\theta)c'(q)du_b = \theta u'(q)du_s$. Also observe that $u_b = u(q) - \delta^i \beta \varrho$ and $u_s = \delta^j \beta \varrho - c(q)$. Totally differentiating these two equations, we obtain $du_b = u'(q)dq - \delta^i \beta d\varrho$ and $du_s = \delta^j \beta d\varrho - c'(q)dq$. Substituting these expressions into Equation (C.3) and subsequently rearranging terms, we find $d\varrho/dq = [u'(q)c'(q)] / \{\beta[\theta\delta^j u'(q) + (1-\theta)\delta^i c'(q)]\}$. If $\delta^j \beta \varrho < c(q_{ij}^*) + u_s$, we obtain that $du_b/d\varrho = du_s/d\varrho = 0$. That means the bargaining solution is independent of ϱ . Without loss, we can therefore assume that there exists a pricing protocol

$$v_{ij}(q) = \int_0^q \frac{1}{\beta} \frac{u'(r)c'(r)}{\theta\delta^j u'(r) + (1-\theta)\delta^i c'(r)} dr.$$

A buyer maximizes $u(q) - \beta\delta^i v_{ij}(q)$ subject to liquidity constraint $v_{ij}(q) \leq z$. We find that $\hat{q}_{ij} = q_{ij}^*$, that $v_{ij}(q)$ is increasing in q , and that $u'(q)/v'_{ij}(q)$ is strictly decreasing in q . These properties imply that Assumption 1 is satisfied. Using random matching, we find

$$\mathcal{L}^i(z) = \sigma\theta\beta \sum_{j \in \{I, P\}} \pi^j \int_0^{\min\{v_{ij}^{-1}(z), q_{ij}^*\}} [\delta^j u'(r)/c'(r) - \delta^i] dr.$$

Here, $\mathcal{L}^i(z)$ has the following properties, which imply that Assumption 2 and Condition 1 are satisfied.

Lemma C.2. *With random matching and gradual bargaining over liquid assets, $\hat{z}^I > \hat{z}^P$ and $\mathcal{L}_z^I(z) \geq \mathcal{L}_z^P(z)$, with strict inequality if and only if $z < \hat{z}^I$.*

C.2.2 Gradual Bargaining over DGM Goods

Consider households bargaining gradually over the amount of DGM goods to be traded. Thus, let $\varrho = z$. The gradual bargaining solution implies that as the seller offers more goods for sale:

$$\frac{du_b}{d\varsigma} = -\theta \frac{\partial \mathcal{H}_{ij}/\partial \varsigma}{\partial \mathcal{H}_{ij}/\partial u_b} \quad \text{and} \quad \frac{du_s}{d\varsigma} = -(1-\theta) \frac{\partial \mathcal{H}_{ij}/\partial \varsigma}{\partial \mathcal{H}_{ij}/\partial u_s}, \quad \Rightarrow \quad \frac{du_b}{du_s} = \frac{\theta}{1-\theta} \frac{\partial \mathcal{H}_{ij}/\partial u_s}{\partial \mathcal{H}_{ij}/\partial u_b}, \quad (\text{C.4})$$

so that Condition 2 is satisfied. If $\varsigma < \min\{q_{ij}^*, c^{-1}(\delta^j \beta z - u_s)\}$, we obtain that Equation (C.4) reduces to $(1-\theta)\delta^j du_b = \theta\delta^i du_s$. Also observe that $u_b = u(\varsigma) - \delta^i \beta p$ and $u_s =$

$\delta^j \beta p - c(\varsigma)$. Totally differentiating these two equations, we obtain $du_b = u'(\varsigma)d\varsigma - \delta^i \beta dp$ and $du_s = \delta^j \beta dp - c'(\varsigma)d\varsigma$. Substituting these expressions into $\delta^j(1 - \theta)du_b = \delta^i \theta du_s$ and subsequently rearranging terms, we find $dp/d\varsigma = [\delta^j(1 - \theta)u'(\varsigma) + \delta^i \theta c'(\varsigma)]/[\beta \delta^i \delta^j]$. If $\varsigma \geq \min\{q_{ij}^*, c^{-1}(\delta^j \beta z - u_s)\}$, we obtain $du_b/d\varsigma = du_s/d\varsigma = 0$, so the bargaining solution is independent of ς . Without loss, we can therefore assume there exists a pricing protocol

$$v_{ij}(q) = \frac{1}{\beta} \frac{\delta^j(1 - \theta)u(q) + \delta^i \theta c(q)}{\delta^i \delta^j}.$$

A buyer maximizes $u(q) - \beta \delta^i v_{ij}(q)$ subject to liquidity constraint $v_{ij}(q) \leq z$. In line with Assumption 1, we find that $\hat{q}_{ij} = q_{ij}^*$, $v_{ij}(q)$ is increasing in q , and $u'(q)/v'_{ij}(q)$ is strictly decreasing in q . Using random matching, we obtain

$$\mathcal{L}^i(z) = \sigma \beta \theta \sum_{j \in \{I, P\}} \pi^j \{ [u \circ \min\{v_{ij}^{-1}(z), q_{ij}^*\}] - \delta^i [c \circ \min\{v_{ij}^{-1}(z), q_{ij}^*\}] / \delta^j \}$$

Here, $\mathcal{L}^i(z)$ has the following properties, which implies that Assumption 2 is satisfied but not Condition 1.

Lemma C.3. *With random matching and gradual bargaining over DGM goods, $\hat{z}^I > \hat{z}^P$ and $\lim_{z \rightarrow 0} \mathcal{L}_z^I(z) < \lim_{z \rightarrow 0} \mathcal{L}_z^P(z)$.*

C.3 Random Matching with Nash Bargaining

Consider the generalized version of the bargaining solution provided by Nash (1950), with buyers having bargaining power $\theta \in (0, 1]$ across all DGM meetings. During DGM t , consider a meeting between a buyer and a seller, where the former has $\delta_t = \delta^i$ and carries liquid assets worth z CM $t + 1$ goods, and the latter has $\delta_t = \delta^j$. Suppose the buyer can commit to spend less than $\varrho \leq z$ worth of liquid assets. Terms of trade (q, p) maximize $[u(q) - \beta \delta^i p]^\theta [-c(q) + \beta \delta^j p]^{1-\theta}$ subject to $p \leq \varrho$. Taking the first-order condition w.r.t. q and subsequently using that $p = v_{ij}(q)$, we find the following pricing protocol

$$v_{ij}(q) = \frac{1}{\beta} \frac{(1 - \theta)u(q)c'(q) + \theta u'(q)c(q)}{\delta^i \theta u'(q) + \delta^i (1 - \theta)c'(q)}.$$

A buyer maximizes $u(q) - \beta \delta^i v_{ij}(q)$ subject to liquidity constraint $v_{ij}(q) \leq z$. In line with Assumption 1, we find $\hat{q}_{ij} \leq q_{ij}^*$, with equality if and only if $\theta = 1$. Moreover, $v_{ij}(q)$ is increasing q . As stressed by, amongst other, Lagos and Wright (2005), $u'(q)/v'_{ij}(q)$ may not be strictly decreasing in q on the relevant domain $[0, \hat{q}_{ij}]$. However, since $\lim_{\theta \rightarrow 1} [u'(q)/v'_{ij}(q)] = \beta \delta^j u'(q)/c'(q)$, there exists a $\theta' < 1$ such that, in line with Assumption 1, $u'(q)/v'_{ij}(q)$ is strictly decreasing in q for all $\theta > \theta'$.

With random matching and Nash bargaining, we also find that if θ is sufficiently large,

Assumption 2 and Condition 1 are satisfied:

Lemma C.4. *With random matching and generalized Nash Bargaining during the DGM, when the bargaining power of buyers is sufficiently large we have that $\hat{z}^I > \hat{z}^P$ and $\mathcal{L}_z^I(z) \geq \mathcal{L}_z^P(z)$, with strict inequality if and only if $z < \hat{z}^I$.*

D Proofs and Derivations

D.1 Concavity of the OTC Value Function

Note that we can write $O_t^i(m, d, b) = \iiint O_t^i(m, d, b; m', d', b') dF_t(m', d', b')$, where

$$O_t^i(m, d, b; m', d', b') = \alpha^i \eta^i \mathcal{F}_{ij}(m, d, b; m', d', b') \\ + \chi \mathcal{L}(m + d) + (1 - \chi) \mathcal{L}(m) + \Delta_t^i + \beta \delta^i (m + d + b + \bar{W}_{t+1}).$$

To show that $O_t^i(m, d, b)$ is concave in (m, d, b) regardless of F_t , it therefore suffices to shows that for arbitrary $\{\bar{m}, \bar{d}, \bar{b}\} \in \mathbb{R}_+^3$, $\bar{O}_t^i(m, d, b) = O_t^i(m, d, b; \bar{m}, \bar{d}, \bar{b})$ is concave in (m, d, b) . Specifically, we need $\lambda \bar{O}_t^i(m', d', b') + (1 - \lambda) \bar{O}_t^i(m'', d'', b'') \leq \bar{O}_t^i(m_\lambda, d_\lambda, b_\lambda)$, where $m_\lambda = \lambda m' + (1 - \lambda) m''$, $d_\lambda = \lambda d' + (1 - \lambda) d''$, $b_\lambda = \lambda b' + (1 - \lambda) b''$, and $\lambda \in (0, 1)$. Let $l(m, d, b; \bar{m}, \bar{d}, \bar{b})$ denote the real value of money the household obtains when it manages to find an OTC match, $k(m, d, b; \bar{m}, \bar{d}, \bar{b})$ the real value of notes the household obtains when it manages to find an OTC match, and let $a(m, d, b; \bar{m}, \bar{d}, \bar{b})$ denote the real value of bonds the household gives up when it manages to find an OTC match. These quantities follow from

$$\mathcal{F}_{ij} = \max_{l, k} \left\{ \begin{aligned} & \chi \frac{\delta^j [\mathcal{L}^i(l + k + m + d) - \mathcal{L}^i(m + d)] + \delta^i [\mathcal{L}^j(\bar{m} + \bar{d} - l - k) - \mathcal{L}^j(\bar{m} + \bar{d})]}{\alpha^j \delta^i + \alpha^i \delta^j} \\ & + (1 - \chi) \frac{\delta^j [\mathcal{L}^i(l + m) - \mathcal{L}^i(m)] + \delta^i [\mathcal{L}^j(\bar{m} - l) - \mathcal{L}^j(\bar{m})]}{\alpha^j \delta^i + \alpha^i \delta^j} \end{aligned} \right\} \quad (\text{D.1})$$

s.t.

$$\begin{aligned} -\bar{b} &\leq l + k + \frac{\chi}{\beta} \frac{\alpha^j [\mathcal{L}^i(l + k + m + d) - \mathcal{L}^i(m + d)] - \alpha^i [\mathcal{L}^j(\bar{m} + \bar{d} - l - k) - \mathcal{L}^j(\bar{m} + \bar{d})]}{\alpha^j \delta^i + \alpha^i \delta^j} \\ &+ \frac{1 - \chi}{\beta} \frac{\alpha^j [\mathcal{L}^i(l + m) - \mathcal{L}^i(m)] - \alpha^i [\mathcal{L}^j(\bar{m} - l) - \mathcal{L}^j(\bar{m})]}{\alpha^j \delta^i + \alpha^i \delta^j} \leq b \\ -m &\leq l \leq \bar{m} \quad \text{and} \quad -d \leq k \leq \bar{d}, \end{aligned}$$

where $i \neq j$.

Let $l' = l(m', d', b'; \bar{m}, \bar{d}, \bar{b})$, $l'' = l(m'', d'', b''; \bar{m}, \bar{d}, \bar{b})$, $l_\lambda = l(m_\lambda, d_\lambda, b_\lambda; \bar{m}, \bar{d}, \bar{b})$, $\tilde{l}_\lambda = \lambda l' + (1 - \lambda) l''$, $k' = k(m', d', b'; \bar{m}, \bar{d}, \bar{b})$, $k'' = k(m'', d'', b''; \bar{m}, \bar{d}, \bar{b})$, $k_\lambda = k(m_\lambda, d_\lambda, b_\lambda; \bar{m}, \bar{d}, \bar{b})$, $\tilde{k}_\lambda = \lambda k' + (1 - \lambda) k''$, $a' = a(m', d', b'; \bar{m}, \bar{d}, \bar{b})$, $a'' = a(m'', d'', b''; \bar{m}, \bar{d}, \bar{b})$, $\tilde{a}_\lambda = \lambda a' + (1 - \lambda) a''$, and $a_\lambda = a(m_\lambda, d_\lambda, b_\lambda; \bar{m}, \bar{d}, \bar{b})$. Observe that

$$\bar{O}_t^i(m, d, b) = \alpha^i \eta^i \mathcal{F}_{ij}(m, d, b; \bar{m}, \bar{d}, \bar{b}) + \chi \mathcal{L}^i(m + d) + (1 - \chi) \mathcal{L}^i(m) + \Delta_t^i + \beta \delta^i (m + d + b + \bar{W}_{t+1}).$$

Therefore, the following holds

$$\begin{aligned} & \bar{O}_t^i(\cdot) - \Delta_t^i - \beta \delta^i [m + d + b + \bar{W}_{t+1}] \\ &= \eta^i \left\{ \begin{aligned} & \chi \mathcal{L}^i(l(\cdot) + k(\cdot) + m + d) + (1 - \chi) \mathcal{L}^i(l(\cdot) + m) \\ & - \beta \delta^i [a(\cdot) - l(\cdot) - k(\cdot)] \end{aligned} \right\} \end{aligned}$$

$$\begin{aligned}
& + (1 - \eta^i) [\chi \mathcal{L}^i(m + d) + (1 - \chi) \mathcal{L}^i(m)] \\
& \propto \alpha^i \eta^i \left\{ \begin{aligned} & \chi [\delta^j \mathcal{L}^i(l(\cdot) + k(\cdot) + m + d) + \delta^i \mathcal{L}^j(\bar{m} + \bar{d} - l(\cdot) - k(\cdot))] \\ & + (1 - \chi) [\delta^j \mathcal{L}^i(l(\cdot) + m) + \delta^i \mathcal{L}^j(m - l(\cdot))] \end{aligned} \right\} \\
& + [\alpha^j \delta^i + \alpha^i \delta^j (1 - \eta^i)] [\chi \mathcal{L}^i(m + d) + (1 - \chi) \mathcal{L}^i(m)] \\
& - \eta^i \alpha^i \delta^i [\chi \mathcal{L}^j(\bar{m} + \bar{d}) + (1 - \chi) \mathcal{L}^j(\bar{m})]
\end{aligned}$$

Because $\mathcal{L}^i(z)$ is concave in z , to prove concavity of $\bar{O}_t^i(\cdot)$ in (m, d, b) it suffices to show that either

$$\begin{aligned}
& \chi [\delta^j \mathcal{L}^i(l(\cdot) + k(\cdot) + m + d) + \delta^i \mathcal{L}^j(\bar{m} + \bar{d} - l(\cdot) - k(\cdot))] \\
& + (1 - \chi) [\delta^j \mathcal{L}^i(l(\cdot) + m) + \delta^i \mathcal{L}^j(m - l(\cdot))]
\end{aligned}$$

or

$$\chi \mathcal{L}^i(l(\cdot) + k(\cdot) + m + d) + (1 - \chi) \mathcal{L}^i(l(\cdot) + m) - \beta \delta^i [a(\cdot) - l(\cdot) - k(\cdot)]$$

is concave in (m, d, b) .

When $(\tilde{l}_\lambda, \tilde{k}_\lambda)$ is a feasible solution of Equation (D.1) for $m = m_\lambda$, $d = d_\lambda$ and $b = b_\lambda$, then

$$\begin{aligned}
& \chi [\delta^j \mathcal{L}^i(l_\lambda + k_\lambda + m_\lambda + d_\lambda) + \delta^i \mathcal{L}^j(\bar{m} + \bar{d} - l_\lambda - k_\lambda)] \\
& + (1 - \chi) [\delta^j \mathcal{L}^i(l_\lambda + m_\lambda) + \delta^i \mathcal{L}^j(\bar{m} - l_\lambda)] \\
& \geq \chi [\delta^j \mathcal{L}^i(\tilde{l}_\lambda + \tilde{k}_\lambda + m_\lambda + d_\lambda) + \delta^i \mathcal{L}^j(\bar{m} + \bar{d} - \tilde{l}_\lambda - \tilde{k}_\lambda)] \\
& + (1 - \chi) [\delta^j \mathcal{L}^i(\tilde{l}_\lambda + m_\lambda) + \delta^i \mathcal{L}^j(\bar{m} - \tilde{l}_\lambda)] \\
& \geq \lambda \left\{ \begin{aligned} & \chi [\delta^j \mathcal{L}^i(l' + k' + m' + d') + \delta^i \mathcal{L}^j(\bar{m} + \bar{d} - l' - k')] \\ & + (1 - \chi) [\delta^j \mathcal{L}^i(l' + m') + \delta^i \mathcal{L}^j(\bar{m} - l')] \end{aligned} \right\} \\
& + (1 - \lambda) \left\{ \begin{aligned} & \chi [\delta^j \mathcal{L}^i(l'' + k'' + m'' + d'') + \delta^i \mathcal{L}^j(\bar{m} + \bar{d} - l'' - k'')] \\ & + (1 - \chi) [\delta^j \mathcal{L}^i(l'' + m'') + \delta^i \mathcal{L}^j(\bar{m} - l'')] \end{aligned} \right\}.
\end{aligned}$$

When $(\tilde{l}_\lambda, \tilde{k}_\lambda)$ is an infeasible solution of Equation (D.1) for $m = m_\lambda$, $d = d_\lambda$ and $b = b_\lambda$, then either (i)

$$\begin{aligned}
& \chi \{ \alpha^j [\mathcal{L}^i(\tilde{l}_\lambda + \tilde{k}_\lambda + m_\lambda + d_\lambda) - \mathcal{L}^i(m_\lambda + d_\lambda)] - \alpha^i [\mathcal{L}^j(\bar{m} + \bar{d} - \tilde{l}_\lambda - \tilde{k}_\lambda) - \mathcal{L}^j(\bar{m} + \bar{d})] \} \\
& + (1 - \chi) \{ \alpha^j [\mathcal{L}^i(\tilde{l}_\lambda + m_\lambda) - \mathcal{L}^i(m_\lambda)] - \alpha^i [\mathcal{L}^j(\bar{m} - \tilde{l}_\lambda) - \mathcal{L}^j(\bar{m})] \} \\
& > \beta (b_\lambda - \tilde{l}_\lambda - \tilde{k}_\lambda) [\alpha^j \delta^i + \alpha^i \delta^j] \\
& \geq \lambda \left(\begin{aligned} & \chi \{ \alpha^j [\mathcal{L}^i(l' + k' + m' + d') - \mathcal{L}^i(m' + d')] - \alpha^i [\mathcal{L}^j(\bar{m} + \bar{d} - l' - k') - \mathcal{L}^j(\bar{m} + \bar{d})] \} \\ & + (1 - \chi) \{ \alpha^j [\mathcal{L}^i(l' + m') - \mathcal{L}^i(m')] - \alpha^i [\mathcal{L}^j(\bar{m} - l') - \mathcal{L}^j(\bar{m})] \} \end{aligned} \right)
\end{aligned}$$

$$\begin{aligned}
& + (1 - \lambda) \left(\chi \left\{ \begin{aligned} & \alpha^j [\mathcal{L}^i(l'' + k'' + m'' + d'') - \mathcal{L}^i(m'' + d'')] \\ & - \alpha^i [\mathcal{L}^j(\bar{m} + \bar{d} - l'' - k'') - \mathcal{L}^j(\bar{m} + \bar{d})] \end{aligned} \right\} \right. \\
& \quad \left. + (1 - \chi) \{ \alpha^j [\mathcal{L}^i(l'' + m'') - \mathcal{L}^i(m'')] - \alpha^i [\mathcal{L}^j(\bar{m} - l'') - \mathcal{L}^j(\bar{m})] \} \right) \\
& \geq \chi \left\{ \begin{aligned} & \alpha^j [\lambda \mathcal{L}^i(l' + k' + m' + d') + (1 - \lambda) \mathcal{L}^i(l'' + k'' + m'' + d'') - \mathcal{L}^i(m_\lambda + d_\lambda)] \\ & - \alpha^i [\mathcal{L}^j(\bar{m} + \bar{d} - \tilde{l}_\lambda - \tilde{k}_\lambda) - \mathcal{L}^j(\bar{m} + \bar{d})] \end{aligned} \right\} \\
& \quad + (1 - \chi) \{ \alpha^j [\lambda \mathcal{L}^i(l' + m') + (1 - \lambda) \mathcal{L}^i(l'' + m'') - \mathcal{L}^i(m_\lambda)] - \alpha^i [\mathcal{L}^j(\bar{m} - \tilde{l}_\lambda) - \mathcal{L}^j(\bar{m})] \}
\end{aligned}$$

or (ii)

$$\begin{aligned}
& \chi \{ \alpha^j [\mathcal{L}^i(\tilde{l}_\lambda + \tilde{k}_\lambda + m_\lambda + d_\lambda) - \mathcal{L}^i(m_\lambda + d_\lambda)] - \alpha^i [\mathcal{L}^j(\bar{m} + \bar{d} - \tilde{l}_\lambda - \tilde{k}_\lambda) - \mathcal{L}^j(\bar{m} + \bar{d})] \} \\
& + (1 - \chi) \{ \alpha^j [\mathcal{L}^i(\tilde{l}_\lambda + m_\lambda) - \mathcal{L}^i(m_\lambda)] - \alpha^i [\mathcal{L}^j(\bar{m} - \tilde{l}_\lambda) - \mathcal{L}^j(\bar{m})] \} \\
& < \beta(-\bar{b} - \tilde{l}_\lambda - \tilde{k}_\lambda) [\alpha^j \delta^i + \alpha^i \delta^j].
\end{aligned}$$

When (i) holds, there exists a pair $(l_\lambda^{\mathfrak{h}}, k_\lambda^{\mathfrak{h}})$ such that $l_\lambda^{\mathfrak{h}} \leq \tilde{l}_\lambda$, $l_\lambda^{\mathfrak{h}} + k_\lambda^{\mathfrak{h}} \leq \tilde{l}_\lambda + \tilde{k}_\lambda$, $-m_\lambda \leq l_\lambda^{\mathfrak{h}} \leq \bar{m}$, $-d_\lambda \leq k_\lambda^{\mathfrak{h}} \leq \bar{d}$, and

$$\begin{aligned}
\beta(b_\lambda - l_\lambda^{\mathfrak{h}} - k_\lambda^{\mathfrak{h}}) [\alpha^j \delta^i + \alpha^i \delta^j] & = \chi \left\{ \begin{aligned} & \alpha^j [\mathcal{L}^i(l_\lambda^{\mathfrak{h}} + k_\lambda^{\mathfrak{h}} + m_\lambda + d_\lambda) - \mathcal{L}^i(m_\lambda + d_\lambda)] \\ & - \alpha^i [\mathcal{L}^j(\bar{m} + \bar{d} - l_\lambda^{\mathfrak{h}} - k_\lambda^{\mathfrak{h}}) - \mathcal{L}^j(\bar{m} + \bar{d})] \end{aligned} \right\} \\
& + (1 - \chi) \left\{ \begin{aligned} & \alpha^j [\mathcal{L}^i(l_\lambda^{\mathfrak{h}} + m_\lambda) - \mathcal{L}^i(m_\lambda)] \\ & - \alpha^i [\mathcal{L}^j(\bar{m} - l_\lambda^{\mathfrak{h}}) - \mathcal{L}^j(\bar{m})] \end{aligned} \right\}.
\end{aligned}$$

It follows that $(l_\lambda^{\mathfrak{h}}, k_\lambda^{\mathfrak{h}})$ is a feasible solution, that

$$\begin{aligned}
& \chi \mathcal{L}^j(\bar{m} + \bar{d} - l_\lambda^{\mathfrak{h}} - k_\lambda^{\mathfrak{h}}) + (1 - \chi) \mathcal{L}^j(\bar{m} - l_\lambda^{\mathfrak{h}}) \\
& \geq \chi \mathcal{L}^j(\bar{m} + \bar{d} - \tilde{l}_\lambda - \tilde{k}_\lambda) + (1 - \chi) \mathcal{L}^j(\bar{m} - \tilde{l}_\lambda) \\
& \geq \lambda \{ \chi \mathcal{L}^j(\bar{m} + \bar{d} - l' - k') + (1 - \chi) \mathcal{L}^j(\bar{m} - l') \} \\
& \quad + (1 - \lambda) \{ \chi \mathcal{L}^j(m'' + d'' - l'' - k'') + (1 - \chi) \mathcal{L}^j(\bar{m} - l'') \}
\end{aligned}$$

and

$$\begin{aligned}
& \chi \mathcal{L}^i(l_\lambda^{\mathfrak{h}} + k_\lambda^{\mathfrak{h}} + m_\lambda + d_\lambda) + (1 - \chi) \mathcal{L}^i(l_\lambda^{\mathfrak{h}} + m_\lambda) \\
& \geq \lambda \{ \chi \mathcal{L}^i(l' + k' + m' + d') + (1 - \chi) \mathcal{L}^i(l' + m') \} \\
& \quad + (1 - \lambda) \{ \chi \mathcal{L}^i(l'' + k'' + m'' + d'') + (1 - \chi) \mathcal{L}^i(l'' + m'') \}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
& \chi [\delta^j \mathcal{L}^i(l_\lambda + k_\lambda + m_\lambda + d_\lambda) + \delta^i \mathcal{L}^j(\bar{m} + \bar{d} - l_\lambda - k_\lambda)] \\
& + (1 - \chi) [\delta^j \mathcal{L}^i(l_\lambda + m_\lambda) + \delta^i \mathcal{L}^j(\bar{m} - l_\lambda)]
\end{aligned}$$

$$\begin{aligned}
&\geq \chi[\delta^j \mathcal{L}^i(l_\lambda^h + k_\lambda^h + m_\lambda + d_\lambda) + \delta^i \mathcal{L}^j(\bar{m} + \bar{d} - l_\lambda^h - k_\lambda^h)] \\
&\quad + (1 - \chi)[\delta^j \mathcal{L}^i(l_\lambda^h + m_\lambda) + \delta^i \mathcal{L}^j(\bar{m} - l_\lambda^h)] \\
&\geq \lambda \left\{ \chi[\delta^j \mathcal{L}^i(l' + k' + m' + d') + \delta^i \mathcal{L}^j(\bar{m} + \bar{d} - l' - k')] \right. \\
&\quad \left. + (1 - \chi)[\delta^j \mathcal{L}^i(l' + m') + \delta^i \mathcal{L}^j(\bar{m} - l')] \right\} \\
&\quad + (1 - \lambda) \left\{ \chi[\delta^j \mathcal{L}^i(l'' + k'' + m'' + d'') + \delta^i \mathcal{L}^j(\bar{m} + \bar{d} - l'' - k'')] \right. \\
&\quad \left. + (1 - \chi)[\delta^j \mathcal{L}^i(l'' + m'') + \delta^i \mathcal{L}^j(\bar{m} - l'')] \right\}.
\end{aligned}$$

When (ii) holds, there exists a pair $(l_\lambda^h, k_\lambda^h)$ such that $l_\lambda^h \geq \tilde{l}_\lambda$, $l_\lambda^h + k_\lambda^h \geq \tilde{l}_\lambda + \tilde{k}_\lambda$, $-m_\lambda \leq l_\lambda^h \leq \bar{m}$, $-d_\lambda \leq k_\lambda^h \leq \bar{d}$, and

$$\begin{aligned}
\beta(-\bar{b} - l_\lambda^h - k_\lambda^h)[\alpha^j \delta^i + \alpha^i \delta^j] &= \chi \left\{ \alpha^j [\mathcal{L}^i(l_\lambda^h + k_\lambda^h + m_\lambda + d_\lambda) - \mathcal{L}^i(m_\lambda + d_\lambda)] \right. \\
&\quad \left. - \alpha^i [\mathcal{L}^j(\bar{m} + \bar{d} - l_\lambda^h - k_\lambda^h) - \mathcal{L}^j(\bar{m} + \bar{d})] \right\} \\
&\quad + (1 - \chi) \left\{ \alpha^j [\mathcal{L}^i(l_\lambda^h + m_\lambda) - \mathcal{L}^i(m_\lambda)] \right. \\
&\quad \left. - \alpha^i [\mathcal{L}^j(\bar{m} - l_\lambda^h) - \mathcal{L}^j(\bar{m})] \right\}.
\end{aligned}$$

It follows that $(l_\lambda^h, k_\lambda^h)$ is a feasible solution of Equation (D.1) for $m = m_\lambda$, $d = d_\lambda$ and $b = b_\lambda$, and that

$$\begin{aligned}
&\chi \mathcal{L}^i(l_\lambda^h + k_\lambda^h + m_\lambda + d_\lambda) + (1 - \chi) \mathcal{L}^i(l_\lambda^h + m_\lambda) - \beta \delta^i [-\bar{b} - l_\lambda^h - k_\lambda^h] \\
&\geq \chi \mathcal{L}^i(\tilde{l}_\lambda + \tilde{k}_\lambda + m_\lambda + d_\lambda) + (1 - \chi) \mathcal{L}^i(\tilde{l}_\lambda + m_\lambda) - \beta \delta^i [\tilde{a}_\lambda - \tilde{l}_\lambda - \tilde{k}_\lambda] \\
&\geq \lambda \{ \chi \mathcal{L}^i(l' + k' + m' + d') + (1 - \chi) \mathcal{L}^i(l' + m') - \beta \delta^i [a' - l' - k'] \} \\
&\quad + (1 - \lambda) \{ \chi \mathcal{L}^i(l'' + k'' + m'' + d'') + (1 - \chi) \mathcal{L}^i(l'' + m'') - \beta \delta^i [a'' - l'' - k''] \}.
\end{aligned}$$

Because $(l_\lambda^h, k_\lambda^h)$ feasible solution of Equation (D.1) for $m = m_\lambda$, $d = d_\lambda$ and $b = b_\lambda$, it follows that

$$\begin{aligned}
&\chi \mathcal{L}^i(l_\lambda + k_\lambda + m_\lambda + d_\lambda) + (1 - \chi) \mathcal{L}^i(l_\lambda + m_\lambda) - \beta \delta^i [a_\lambda - l_\lambda - k_\lambda] \\
&\geq \chi \mathcal{L}^i(l_\lambda^h + k_\lambda^h + m_\lambda + d_\lambda) + (1 - \chi) \mathcal{L}^i(l_\lambda^h + m_\lambda) - \beta \delta^i [-\bar{b} - l_\lambda^h - k_\lambda^h] \\
&\geq \chi \mathcal{L}^i(\tilde{l}_\lambda + \tilde{k}_\lambda + m_\lambda + d_\lambda) + (1 - \chi) \mathcal{L}^i(\tilde{l}_\lambda + m_\lambda) - \beta \delta^i [\tilde{a}_\lambda - \tilde{l}_\lambda - \tilde{k}_\lambda] \\
&\geq \lambda \{ \chi \mathcal{L}^i(l' + k' + m' + d') + (1 - \chi) \mathcal{L}^i(l' + m') - \beta \delta^i [a' - l' - k'] \} \\
&\quad + (1 - \lambda) \{ \chi \mathcal{L}^i(l'' + k'' + m'' + d'') + (1 - \chi) \mathcal{L}^i(l'' + m'') - \beta \delta^i [a'' - l'' - k''] \}.
\end{aligned}$$

D.2 Derivatives of OTC Match Surplus

Define $\kappa = l + k$ and consider:

$$\mathcal{F} = \max_{l, \kappa} \left\{ \chi \frac{\delta^P [\mathcal{L}^I(\kappa + m_I + d_I) - \mathcal{L}^I(m_I + d_I)] + \delta^I [\mathcal{L}^P(m_P + d_P - \kappa) - \mathcal{L}^P(m_P + d_P)]}{\alpha \delta^I + (1 - \alpha) \delta^P} \right. \\ \left. + (1 - \chi) \frac{\delta^P [\mathcal{L}^I(l + m_I) - \mathcal{L}^I(m_I)] + \delta^I [\mathcal{L}^P(m_P - l) - \mathcal{L}^P(m_P)]}{\alpha \delta^I + (1 - \alpha) \delta^P} \right\} \\ \text{s.t. } -m_I \leq l \leq m_P, \quad -d_I \leq \kappa - l \leq d_P, \quad \text{and} \quad -b_P \leq a \leq b_P, \quad (\text{D.2})$$

where

$$a - \kappa = \frac{\chi}{\beta} \frac{\alpha [\mathcal{L}^I(\kappa + m_I + d_I) - \mathcal{L}^I(m_I + d_I)] - (1 - \alpha) [\mathcal{L}^P(m_P + d_P - \kappa) - \mathcal{L}^P(m_P + d_P)]}{\alpha \delta^I + (1 - \alpha) \delta^P} \quad (\text{D.3}) \\ + \frac{1 - \chi}{\beta} \frac{\alpha [\mathcal{L}^I(l + m_I) - \mathcal{L}^I(m_I)] - (1 - \alpha) [\mathcal{L}^P(m_P - l) - \mathcal{L}^P(m_P)]}{\alpha \delta^I + (1 - \alpha) \delta^P}.$$

Let μ_I , μ_P , ν_I , ν_P , λ_I , and λ_P denote the Lagrange multipliers associated with the constraints.

When agents enter with the same asset portfolios, i.e. $m_I = m_P = m$, $d_I = d_P = d$, and $b_I = b_P = b$, we obtain the following first-order conditions for l and κ :

$$0 = (1 - \chi) \frac{[\delta^P - \alpha(\lambda_I - \lambda_P)/\beta] \mathcal{L}_z^I(l + m) - [\delta^I + (1 - \alpha)(\lambda_I - \lambda_P)/\beta] \mathcal{L}_z^P(m - l)}{\alpha \delta^I + (1 - \alpha) \delta^P} \quad (\text{D.4})$$

$$- (\mu_P - \mu_I) + (\nu_P - \nu_I) \\ 0 = \chi \frac{[\delta^P - \alpha(\lambda_I - \lambda_P)/\beta] \mathcal{L}_z^I(\kappa + m + d) - [\delta^I + (1 - \alpha)(\lambda_I - \lambda_P)/\beta] \mathcal{L}_z^P(-\kappa + m + d)}{\alpha \delta^I + (1 - \alpha) \delta^P} \quad (\text{D.5}) \\ - (\nu_P - \nu_I) - (\lambda_I - \lambda_P)$$

Define $V_m^i(m, d, b) = \chi \mathcal{L}_z^i(m + d) + (1 - \chi) \mathcal{L}_z^i(m) + \beta \delta^i$. Then, combine (D.4) and (D.5) to obtain:

$$\frac{\lambda_I - \lambda_P}{\beta} = \frac{\delta_P V_m^I(m + l, d + \kappa - l, b) - \delta_I V_m^P(m - l, d - \kappa + l, b) + [\alpha \delta^I + (1 - \alpha) \delta^P] [\mu_I - \mu_P]}{\alpha V_m^I(m + l, d + \kappa - l, b) + (1 - \alpha) V_m^P(m - l, d - \kappa + l, b)}. \quad (\text{D.6})$$

It can then be shown that in a symmetric equilibrium, $\mathcal{F}_{IP,b} = \lambda_I$, $\mathcal{F}_{PI,b} = \lambda_P$,

$$\mathcal{F}_{IP,m} = \frac{\delta^P - \alpha(\lambda_I - \lambda_P)/\beta}{\alpha \delta^I + (1 - \alpha) \delta^P} \left\{ \chi [\mathcal{L}_z^I(\kappa + m + d) - \mathcal{L}_z^I(m + d)] \right. \\ \left. + (1 - \chi) [\mathcal{L}_z^I(l + m) - \mathcal{L}_z^I(m)] \right\} + \mu_I, \\ \mathcal{F}_{PI,m} = \frac{\delta^I + (1 - \alpha)(\lambda_I - \lambda_P)/\beta}{\alpha \delta^I + (1 - \alpha) \delta^P} \left\{ \chi [\mathcal{L}_z^P(m + d - \kappa) - \mathcal{L}_z^P(m + d)] \right. \\ \left. + (1 - \chi) [\mathcal{L}_z^P(m - l) - \mathcal{L}_z^P(m)] \right\} + \mu_P, \\ \mathcal{F}_{IP,d} = \chi \frac{\delta^P - \alpha(\lambda_I - \lambda_P)/\beta}{\alpha \delta^I + (1 - \alpha) \delta^P} [\mathcal{L}_z^I(\kappa + m + d) - \mathcal{L}_z^I(m + d)] + \nu_I, \\ \mathcal{F}_{PI,d} = \chi \frac{\delta^I + (1 - \alpha)(\lambda_I - \lambda_P)/\beta}{\alpha \delta^I + (1 - \alpha) \delta^P} [\mathcal{L}_z^P(m + d - \kappa) - \mathcal{L}_z^P(m + d)] + \nu_P.$$

D.3 Decomposition of Surplus from OTC Trade

Let $l(m, d, b)$ and $\kappa(m, d, b)$ be solutions of (D.2) when $m_I = m_P = m$, $d_I = d_P = d$, and $b_I = b_P = b$. I will show that there exists g_1 and g_2 such that $\mathcal{F}(m, d, b) = (1 - \chi) \tilde{\mathcal{F}}(m, g_1) + \chi \tilde{\mathcal{F}}(m + d, g_2)$ and $\mathcal{E}(m, d, b) = (1 - \chi) \tilde{\mathcal{E}}(m, g_1) + \chi \tilde{\mathcal{E}}(m + d, g_2)$, where

$\mathcal{F}(m, d, b)$ is the RHS of (D.2) for $m_I = m_P = m$, $d_I = d_P = d$, and $b_I = b_P = b$, and $\mathcal{E}(m, d, b)$ are externalities on sellers from an OTC trade.

Here, I use a guess and verify procedure. Suppose that g_1 and g_2 solve

$$\begin{aligned} l(m, g_1) &= \max \left\{ \min\{m - \hat{z}^P, 0\}, \min\{l(m, d, b), \max\{\hat{z}^I - m, 0\}\} \right\} \\ l(m + d, g_2) &= \max \left\{ \min\{m + d - \hat{z}^P, 0\}, \min\{\kappa(m, d, b), \max\{\hat{z}^I - m - d, 0\}\} \right\} \end{aligned}$$

where $l(\cdot)$ is given by Equation (21). Because the solution of (D.2) implies that $m + l(m, b, d) \geq \hat{z}^I \Leftrightarrow m - l(m, b, d) \geq \hat{z}^P$ and $m + d + \kappa(m, b, d) \geq \hat{z}^I \Leftrightarrow m - \kappa(m, b, d) \geq \hat{z}^P$, we obtain for the private surplus of an OTC match.

$$\begin{aligned} \mathcal{F}(\cdot) &= (1 - \chi) \frac{\delta_P[\mathcal{L}^I(l(m, d, b) + m) - \mathcal{L}^I(m)] + \delta^I[\mathcal{L}^P(m - l(m, d, b)) - \mathcal{L}^P(m)]}{\alpha\delta^I + (1 - \alpha)\delta^P} \\ &\quad + \chi \frac{\delta_P[\mathcal{L}^I(\kappa(m, d, b) + m + d) - \mathcal{L}^I(m + d)] + \delta^I[\mathcal{L}^P(m + d - \kappa(m, d, b)) - \mathcal{L}^P(m + d)]}{\alpha\delta^I + (1 - \alpha)\delta^P} \\ &= (1 - \chi) \frac{\delta_P[\mathcal{L}^I(l(m, g_1) + m) - \mathcal{L}^I(m)] + \delta^I[\mathcal{L}^P(m - l(m, g_1)) - \mathcal{L}^P(m)]}{\alpha\delta^I + (1 - \alpha)\delta^P} \\ &\quad + \chi \frac{\delta_P[\mathcal{L}^I(l(m + d, g_2) + m + d) - \mathcal{L}^I(m + d)] + \delta^I[\mathcal{L}^P(m + d - l(m + d, g_2)) - \mathcal{L}^P(m + d)]}{\alpha\delta^I + (1 - \alpha)\delta^P} \\ &= (1 - \chi)\tilde{\mathcal{F}}(m, g_1) + \chi\tilde{\mathcal{F}}(m + d, g_2). \end{aligned}$$

For the externalities of an OTC match, we obtain

$$\begin{aligned} \mathcal{E}(\cdot) &= (1 - \chi) \left\{ \frac{1 - \theta^I(l(m, d, b) + m)}{\theta^I(l(m, d, b) + m)} \mathcal{L}^I(l(m, d, b) + m) - \frac{1 - \theta^I(m)}{\theta^I(m)} \mathcal{L}^I(m) \right. \\ &\quad \left. + \frac{1 - \theta^P(m - l(m, d, b))}{\theta^P(m - l(m, d, b))} \mathcal{L}^P(m - l(m, d, b)) - \frac{1 - \theta^P(m)}{\theta^P(m)} \mathcal{L}^P(m) \right\} \\ &\quad + \chi \left\{ \frac{1 - \theta^I(\kappa(m, d, b) + m + d)}{\theta^I(\kappa(m, d, b) + m + d)} \mathcal{L}^I(\kappa(m, d, b) + m + d) - \frac{1 - \theta^I(m + d)}{\theta^I(m + d)} \mathcal{L}^I(m + d) \right. \\ &\quad \left. + \frac{1 - \theta^P(m + d - \kappa(m, d, b))}{\theta^P(m + d - \kappa(m, d, b))} \mathcal{L}^P(m + d - \kappa(m, d, b)) - \frac{1 - \theta^P(m + d)}{\theta^P(m + d)} \mathcal{L}^P(m + d) \right\} \\ &= (1 - \chi) \left\{ \frac{1 - \theta^I(l(m, g_1) + m)}{\theta^I(l(m, g_1) + m)} \mathcal{L}^I(l(m, g_1) + m) - \frac{1 - \theta^I(m)}{\theta^I(m)} \mathcal{L}^I(m) \right. \\ &\quad \left. + \frac{1 - \theta^P(m - l(m, g_1))}{\theta^P(m - l(m, g_1))} \mathcal{L}^P(m - l(m, g_1)) - \frac{1 - \theta^P(m)}{\theta^P(m)} \mathcal{L}^P(m) \right\} \\ &\quad + \chi \left\{ \frac{1 - \theta^I(l(m + d, g_2) + m + d)}{\theta^I(l(m + d, g_2) + m + d)} \mathcal{L}^I(l(m + d, g_2) + m + d) - \frac{1 - \theta^I(m + d)}{\theta^I(m + d)} \mathcal{L}^I(m + d) \right. \\ &\quad \left. + \frac{1 - \theta^P(m + d - l(m + d, g_2))}{\theta^P(m + d - l(m + d, g_2))} \mathcal{L}^P(m + d - l(m + d, g_2)) - \frac{1 - \theta^P(m + d)}{\theta^P(m + d)} \mathcal{L}^P(m + d) \right\} \\ &= (1 - \chi)\tilde{\mathcal{E}}(m, g_1) + \chi\tilde{\mathcal{E}}(m + d, g_2). \end{aligned}$$

D.4 Proof of Lemma 1

Suppose that a household indexed with $r \in [0, 1]$ enters CM t with money worth $\hat{m}_{r,t}$ CM goods, notes worth $\hat{d}_{r,t}$ CM goods, and bonds worth $\hat{b}_{r,t}$ CM goods. By definition,

$\mathcal{W}_t = \int_0^1 W_t(m_{r,t}, d_{r,t}, b_{r,t}) dr$. In symmetric equilibrium, characterized by a sequence of portfolio choices $\{m_t, d_t, b_t\}_{t=0}^\infty$, we have that

$$\begin{aligned} \phi_t M_{t-1} = m_{t-1} = \int_0^1 \hat{m}_{r,t} dr, \quad \phi_t D_{t-1} = d_{t-1} = \int_0^1 \hat{d}_{r,t} dr, \\ \text{and} \quad \phi_t B_{t-1} = b_{t-1} = \int_0^1 \hat{b}_{r,t} dr. \end{aligned} \quad (\text{D.7})$$

Since $\{m_t, d_t, b_t\}_{t=0}^\infty$ are optimal portfolio choices, they solve Equation (8) and therefore

$$\begin{aligned} W_t(\hat{m}_{r,t}, \hat{d}_{r,t}, \hat{b}_{r,t}) = U(y^*) - y^* + \hat{m}_{r,t} + \hat{d}_{r,t} + \hat{b}_{r,t} - (\phi_t m_t + \varphi_t d_t + \psi_t b_t) / \phi_{t+1} - \tau_t \\ + \pi^I O_t^I(m_t, d_t, b_t) + \pi^P O_t^P(m_t, d_t, b_t). \end{aligned}$$

Using the characterization of O_t^i in Equation (6) in a symmetric equilibrium where F_t is degenerate at (m_t, d_t, b_t) so that $\iiint \mathcal{F}_{ij}(m_t, d_t, b_t; m', d', b') dF_t(m', d', b') = \mathcal{F}(m_t, d_t, b_t)$, we can write:

$$\begin{aligned} W_t(\hat{m}_{r,t}, \hat{d}_{r,t}, \hat{b}_{r,t}) = U(y^*) - y^* + \hat{m}_{r,t} + \hat{d}_{r,t} + \hat{b}_{r,t} - (\phi_t m_t + \varphi_t d_t + \psi_t b_t) / \phi_{t+1} - \tau_t \\ + \pi^I \{ \chi \mathcal{L}^I(m_t + d_t) + (1 - \chi) \mathcal{L}^I(m_t) + \Delta_t^I \} \\ + \pi^P \{ \chi \mathcal{L}^P(m_t + d_t) + (1 - \chi) \mathcal{L}^P(m_t) + \Delta_t^P \} \\ + \omega \mathcal{F}(m_t, d_t, b_t) + \beta W_{t+1}(m_t, d_t, b_t). \end{aligned} \quad (\text{D.8})$$

Since $W_t(\hat{m}_{r,t}, \hat{d}_{r,t}, \hat{b}_{r,t})$ is linear in $\hat{m}_{r,t}$, $\hat{d}_{r,t}$, and $\hat{b}_{r,t}$, we can rewrite Equation (D.8) as:

$$\begin{aligned} W_t(\hat{m}_{r,t}, \hat{d}_{r,t}, \hat{b}_{r,t}) = U(y^*) - y^* + \hat{m}_{r,t} + \hat{d}_{r,t} + \hat{b}_{r,t} - (\phi_t m_t + \varphi_t d_t + \psi_t b_t) / \phi_{t+1} - \tau_t \\ + \pi^I \{ \chi \mathcal{L}^I(m_t + d_t) + (1 - \chi) \mathcal{L}^I(m_t) + \Delta_t^I \} \\ + \pi^P \{ \chi \mathcal{L}^P(m_t + d_t) + (1 - \chi) \mathcal{L}^P(m_t) + \Delta_t^P \} \\ + \omega \mathcal{F}(m_t, d_t, b_t) + \beta W_{t+1}(\hat{m}_{r,t+1}, \hat{d}_{r,t+1}, \hat{b}_{r,t+1}) \\ + \beta \left(m_t - \hat{m}_{r,t+1} + d_t - \hat{d}_{r,t+1} + b_t - \hat{b}_{r,t+1} \right) \end{aligned}$$

Integrating over all households and combining the result with Equations (12) and (D.7) yields:

$$\begin{aligned} \mathcal{W}_t = U(y^*) - y^* + \pi^I \{ \chi \mathcal{L}^I(m_t + d_t) + (1 - \chi) \mathcal{L}^I(m_t) + \Delta_t^I \} \\ + \pi^P \{ \chi \mathcal{L}^P(m_t + d_t) + (1 - \chi) \mathcal{L}^P(m_t) + \Delta_t^P \} + \mathcal{F}(m_t, d_t, b_t) + \beta \mathcal{W}_{t+1} \end{aligned} \quad (\text{D.9})$$

To substitute out Δ_t^I and Δ_t^P , notice that in symmetric equilibrium the conditional CDF $G_t(m', d' | \delta^j)$ has an associated probability distribution given by

$$g_t(m', d' | \delta^j) = \begin{cases} 1 - \eta^j & \text{if } m' = m_t \text{ and } d' = d_t \\ \eta^j & \text{if } m' = m_t + (\mathbb{I}_{j=I} - \mathbb{I}_{j=P}) l(m_t, g_1(m_t, d_t, b_t)) \\ & \text{and } d' = d_t + m_t + (\mathbb{I}_{j=I} - \mathbb{I}_{j=P}) l(m_t + d_t, g_2(m_t, d_t, b_t)) \\ 0 & \text{otherwise} \end{cases}.$$

Using how we have defined $\mathcal{L}^i(z)/\theta^i(z)$ in Equation (24), how we have defined $\mathcal{E}(m, d, b)$ in Equation (25), and how we have defined $\tilde{\mathcal{E}}(m, b)$ in Equation (26), the properties of $g_t(m', d' | \delta^j)$ imply that

$$\begin{aligned} \pi^I \Delta_t^I + \pi^P \Delta_t^P &= \chi \left[\frac{\pi^I \theta^I(m_t + d_t) \mathcal{L}^I(m_t + d_t)}{1 - \theta^I(m_t + d_t)} + \frac{\pi^P \theta^P(m_t + d_t) \mathcal{L}^P(m_t + d_t)}{1 - \theta^P(m_t + d_t)} \right] \\ &+ (1 - \chi) \left[\frac{\pi^I \theta^I(m_t) \mathcal{L}^I(m_t)}{1 - \theta^I(m_t)} + \frac{\pi^P \theta^P(m_t) \mathcal{L}^P(m_t)}{1 - \theta^P(m_t)} \right] + \omega \mathcal{E}(m_t, d_t, b_t). \end{aligned} \quad (\text{D.10})$$

Combining Equations (D.9) and (D.10), we obtain

$$\begin{aligned} \mathcal{W}_t &= U(y^*) - y^* + \pi^I \left[\chi \frac{\mathcal{L}^I(m_t + d_t)}{\theta^I(m_t + d_t)} + (1 - \chi) \frac{\mathcal{L}^I(m_t)}{\theta^I(m_t)} \right] \\ &+ \pi^P \left[\chi \frac{\mathcal{L}^P(m_t + d_t)}{\theta^P(m_t + d_t)} + (1 - \chi) \frac{\mathcal{L}^P(m_t)}{\theta^P(m_t)} \right] \\ &+ \omega [\mathcal{F}(m_t, d_t, b_t) + \mathcal{E}(m_t, d_t, b_t)] + \beta \mathcal{W}_{t+1}. \end{aligned}$$

This proves that welfare in symmetric equilibrium satisfies the recursive relationship $\mathcal{W}_t = \mathcal{W}(m_t, d_t, b_t) + \beta \mathcal{W}_{t+1}$, with $\mathcal{W}(m_t, d_t, b_t)$ given by Equation (27). Q.E.D.

D.5 Proof of Lemma 2

It follows directly from Definition 1 that symmetric equilibria with a positive value for fiat money satisfy

$$\begin{aligned} \beta i_t &= \omega [(1 - \alpha) \mathcal{F}_{IP,m}(m_t, d_t, b_t) + \alpha \mathcal{F}_{PI,m}(m_t, d_t, b_t)] \\ &+ \chi [\pi^I \mathcal{L}_z^I(m_t + d_t) + \pi^P \mathcal{L}_z^P(m_t + d_t)] + (1 - \chi) [\pi^I \mathcal{L}_z^I(m_t) + \pi^P \mathcal{L}_z^P(m_t)]. \end{aligned}$$

In SSE, b_t and d_t are fixed by policy at b and d , and $m_t = m$ is constant over time. Since $\phi_{t+1} M_t = m$ and $M_{t+1} = \gamma M_t$, it follows that $\gamma \phi_{t+1} = \phi_t$. Therefore, $\beta i_t^f = \gamma - \beta$ and in SSE

$$\begin{aligned} \gamma - \beta &= \omega [(1 - \alpha) \mathcal{F}_{IP,m}(m, d, b) + \alpha \mathcal{F}_{PI,m}(m, d, b)] \\ &+ \chi [\pi^I \mathcal{L}_z^I(m + d) + \pi^P \mathcal{L}_z^P(m + d)] + (1 - \chi) [\pi^I \mathcal{L}_z^I(m) + \pi^P \mathcal{L}_z^P(m)]. \end{aligned} \quad (\text{D.11})$$

Any $m > 0$ that solves Equation (D.11) then sufficiently describes a steady state equilibrium. To see why, note that d and b are given by policy, and that interest rates can be found by using (m, b, d) together with $\beta i^f = \gamma - \beta$ in Equations (16) and (17) to yield

$$i^d = \frac{\gamma}{\omega [(1 - \alpha)\mathcal{F}_{IP,d}(m, d, b) + \alpha\mathcal{F}_{PI,d}(m, d, b)] + \chi \sum_{i \in \{I, P\}} \pi^i \mathcal{L}_z^i(m + d) + \beta} - 1,$$

$$i^b = \frac{\gamma}{\omega [(1 - \alpha)\mathcal{F}_{IP,b}(m, d, b) + \alpha\mathcal{F}_{PI,b}(m, d, b)] + \beta} - 1.$$

Given an initial, exogenous supply of money M_0 , the sequence of prices $\{\phi_t, \varphi_t, \psi_t\}_{t=0}^\infty$ is given by

$$\phi_t = m / [\gamma^{t-1} M_0], \quad \varphi_t = \phi_t(1 + i^d), \quad \text{and} \quad \psi_t = \phi_t(1 + i^b).$$

It follows that the obtained sequence of portfolios choices, interest rates, and prices meets Definition 1. Hence, we indeed have an equilibrium.

The next step is to show that $m \geq \max\{\hat{z}^I, \hat{z}^P\} \Leftrightarrow \gamma = \beta$ and $z < \max\{\hat{z}^I, \hat{z}^P\} \Leftrightarrow \gamma > \beta$. Using results from Appendix D.2, we obtain the following for the general equilibrium condition:

$$\begin{aligned} \gamma - \beta &= \pi^I [\chi \mathcal{L}_z^I(m + d) + (1 - \chi) \mathcal{L}_z^I(m)] + \pi^P [\chi \mathcal{L}_z^P(m + d) + (1 - \chi) \mathcal{L}_z^P(m)] + \omega [(1 - \alpha) \mu_I + \alpha \mu_P] \\ &\quad + \omega (1 - \alpha) \frac{\delta^P - \alpha \frac{\lambda_I - \lambda_P}{\beta}}{\alpha \delta^I + (1 - \alpha) \delta^P} \{ \chi [\mathcal{L}_z^I(\kappa + m + d) - \mathcal{L}_z^I(m + d)] + (1 - \chi) [\mathcal{L}_z^I(l + m) - \mathcal{L}_z^I(m)] \} \\ &\quad + \omega \alpha \frac{\delta^I + (1 - \alpha) \frac{\lambda_I - \lambda_P}{\beta}}{\alpha \delta^I + (1 - \alpha) \delta^P} \{ \chi [\mathcal{L}_z^P(m + d - \kappa) - \mathcal{L}_z^P(m + d)] + (1 - \chi) [\mathcal{L}_z^P(m - l) - \mathcal{L}_z^P(m)] \}. \end{aligned}$$

Define $V_m^I = \chi \mathcal{L}_z^I(\kappa + m + d) + (1 - \chi) \mathcal{L}_z^I(l + m) + \beta \delta^I$, $V_m^P = \chi \mathcal{L}_z^P(m + d - \kappa) + (1 - \chi) \mathcal{L}_z^P(m - l) + \beta \delta^P$, and $\kappa \equiv l + k$. Consider $\mu_P = \mu_I = 0$, using Equation (D.6) we obtain:

$$\begin{aligned} \gamma - \beta &= \pi^I \left[1 - \frac{\eta^I (1 - \alpha) V_m^P}{\alpha V_m^I + (1 - \alpha) V_m^P} \right] [\chi \mathcal{L}_z^I(m + d) + (1 - \chi) \mathcal{L}_z^I(m)] \\ &\quad + \pi^P \left[1 - \frac{\eta^P \alpha V_m^I}{\alpha V_m^I + (1 - \alpha) V_m^P} \right] [\chi \mathcal{L}_z^P(m + d) + (1 - \chi) \mathcal{L}_z^P(m)] \\ &\quad + \frac{\omega (1 - \alpha) V_m^P}{\alpha V_m^I + (1 - \alpha) V_m^P} [\chi \mathcal{L}_z^I(\kappa + m + d) + (1 - \chi) \mathcal{L}_z^I(l + m)] \\ &\quad + \frac{\omega \alpha V_m^I}{\alpha V_m^I + (1 - \alpha) V_m^P} [\chi \mathcal{L}_z^P(m + d - \kappa) + (1 - \chi) \mathcal{L}_z^P(m - l)]. \end{aligned}$$

If $m \geq \max\{\hat{z}^I, \hat{z}^P\}$, then $\mathcal{L}_z^I(m) = \mathcal{L}_z^P(m) = 0$ and OTC trade implies that $l = \kappa = 0$. It follows that we have $\gamma = \beta$ if $m \geq \max\{\hat{z}^I, \hat{z}^P\}$. If $m < \max\{\hat{z}^I, \hat{z}^P\}$, then $\mathcal{L}_z^I(m) > 0$ and/or $\mathcal{L}_z^P(m) > 0$. It follows that $\gamma > \beta$ if $m < \max\{\hat{z}^I, \hat{z}^P\}$. Consider $\mu_I > 0$ or $\mu_P > 0$. Then a case in which $\nu_I > 0$ or $\nu_P > 0$, and λ_I or $\lambda_P > 0$, is a degenerate one. Therefore, consider $\nu_I = \nu_P = 0$ or $\lambda_I = \lambda_P = 0$. Define $V_d^I = \chi \mathcal{L}_z^I(\kappa + m + d) + \beta \delta^I$ and

$V_d^P = \chi \mathcal{L}_z^P(m + d - \kappa) + \beta \delta^P$. When $\nu_I = \nu_P = 0$, then Equations (D.4) and (D.5) imply:

$$\begin{aligned} \gamma - \beta = & \pi^I \left[1 - \frac{\eta^I(1-\alpha)V_d^P}{\alpha V_d^I + (1-\alpha)V_d^P} \right] [\chi \mathcal{L}_z^I(m + d) + (1-\chi)\mathcal{L}_z^I(m)] \\ & + \pi^P \left[1 - \frac{\eta^P \alpha V_d^I}{\alpha V_d^I + (1-\alpha)V_d^P} \right] [\chi \mathcal{L}_z^P(m + d) + (1-\chi)\mathcal{L}_z^P(m)] \\ & + \frac{\omega(1-\alpha)V_d^P}{\alpha V_d^I + (1-\alpha)V_d^P} [\chi \mathcal{L}_z^I(\kappa + m + d) + (1-\chi)\mathcal{L}_z^I(l + m)] \\ & + \frac{\omega \alpha V_d^I}{\alpha V_d^I + (1-\alpha)V_d^P} [\chi \mathcal{L}_z^P(m + d - \kappa) + (1-\chi)\mathcal{L}_z^P(m - l)] \\ & + \frac{\omega(1-\chi)}{\alpha V_d^I + (1-\alpha)V_d^P} \max \left\{ 0, (1-\alpha)[V_d^I \mathcal{L}_z^P(m - l) - V_d^P \mathcal{L}_z^I(l + m)], \right. \\ & \left. \alpha[V_d^P \mathcal{L}_z^I(m + l) - V_d^I \mathcal{L}_z^P(m - l)] \right\}. \end{aligned}$$

Again, if $m \geq \max\{\hat{z}^I, \hat{z}^P\}$, then $l = \kappa = \mathcal{L}_z^I(m) = \mathcal{L}_z^P(m) = 0$ and we have $\gamma = \beta$. If $m < \max\{\hat{z}^I, \hat{z}^P\}$, then $\mathcal{L}_z^I(m) > 0$ and/or $\mathcal{L}_z^P(m) > 0$ and so $\gamma > \beta$. When $\lambda_I = \lambda_P = 0$, we can use Equations (D.4) and (D.5) to obtain:

$$\begin{aligned} \gamma - \beta = & \pi^I \left[1 - \frac{\eta^I(1-\alpha)\delta^P}{\alpha \delta_I + (1-\alpha)\delta^P} \right] [\chi \mathcal{L}_z^I(m + d) + (1-\chi)\mathcal{L}_z^I(m)] \\ & + \pi^P \left[1 - \frac{\eta^P \alpha \delta_I}{\alpha \delta_I + (1-\alpha)\delta^P} \right] [\chi \mathcal{L}_z^P(m + d) + (1-\chi)\mathcal{L}_z^P(m)] \\ & + \frac{\omega(1-\alpha)\delta^P}{\alpha \delta_I + (1-\alpha)\delta^P} [\chi \mathcal{L}_z^I(\kappa + m + d) + (1-\chi)\mathcal{L}_z^P(l + m)] \\ & + \frac{\omega \alpha \delta^I}{\alpha \delta_I + (1-\alpha)\delta^P} [\chi \mathcal{L}_z^P(m + d - \kappa) + (1-\chi)\mathcal{L}_z^P(m - l)] \\ & + \frac{\omega(1-\chi) \max \{ (1-\alpha)[\delta^I V_m^P - \delta^P V_m^I], \alpha[\delta^P V_m^I - \delta^I V_m^P], 0 \}}{\alpha \delta_I + (1-\alpha)\delta^P}. \end{aligned}$$

Again, if $m \geq \max\{\hat{z}^I, \hat{z}^P\}$, then $l = \kappa = \mathcal{L}_z^I(m) = \mathcal{L}_z^P(m) = 0$ and we have $\gamma = \beta$. If $m < \max\{\hat{z}^I, \hat{z}^P\}$, then $\mathcal{L}_z^I(m) > 0$ and/or $\mathcal{L}_z^P(m) > 0$ so $\gamma > \beta$. Concluding, $m < \max\{\hat{z}^I, \hat{z}^P\} \Leftrightarrow \gamma > \beta$ and $m \geq \max\{\hat{z}^I, \hat{z}^P\} \Leftrightarrow \gamma = \beta$.

Next, because the objective function and constraints in Program (D.1) are continuous in $l, \kappa, m_I, d_I, b_I, m_P, d_P$, and b_P , it follows that there exist l and κ solving Program (D.1) which are continuous in m_I, d_I, b_I, m_P, d_P , and b_P . When constraints $-m_I \leq l \leq m_P$, $-d_I \leq \kappa - l \leq d_P$, and $-b_P \leq a \leq b_I$, do not bind simultaneously, $\mathcal{F}_{IP,m}, \mathcal{F}_{IP,d}, \mathcal{F}_{IP,b}, \mathcal{F}_{PI,m}, \mathcal{F}_{PI,d}$, and $\mathcal{F}_{PI,b}$ are continuous in m_I, d_I, b_I, m_P, d_P , and b_P . It follows that, generically, γ is continuous in m, d , and b . Moreover, from the equilibrium conditions in Definition 1, it follows that $\lim_{\gamma \downarrow \beta} m = \max\{\hat{z}^I, \hat{z}^P\}$.

Then, consider the nominal rate earned by holding notes two consecutive CMs, i^d .

From Equations (15) and (16), it follows $\text{sgn}(i^d) = \text{sgn}(f)$, where

$$\begin{aligned}
f &= (1 - \chi)[\pi^I \mathcal{L}_z^I(m) + \pi^P \mathcal{L}_z^P(m)] \\
&\quad + \omega\{(1 - \alpha)[\mathcal{F}_{IP,m}(m, d, b) - \mathcal{F}_{IP,d}(m, d, b)] + \alpha[\mathcal{F}_{PI,m}(m, d, b) - \mathcal{F}_{PI,d}(m, d, b)]\} \\
&= (1 - \chi)[\pi^I \mathcal{L}_z^I(m) + \pi^P \mathcal{L}_z^P(m)] + \omega[(1 - \alpha)(\mu_I - \nu_I) + \alpha(\mu_P - \nu_P)] \\
&\quad + \omega(1 - \chi) \frac{(1 - \alpha)\delta^P - \alpha(1 - \alpha)(\lambda_I - \lambda_P)/\beta}{\alpha\delta^I + (1 - \alpha)\delta^P} [\mathcal{L}_z^I(l + m) - \mathcal{L}_z^I(m)] \\
&\quad + \omega(1 - \chi) \frac{\alpha\delta^P + \alpha(1 - \alpha)(\lambda_I - \lambda_P)/\beta}{\alpha\delta^I + (1 - \alpha)\delta^P} [\mathcal{L}_z^P(m - l) - \mathcal{L}_z^P(m)].
\end{aligned}$$

Observe that f is just the RHS of Equation (16) minus the RHS of Equation (15). Consider $\mu_I = \mu_P$, we can substitute out λ_I and λ_P using Equation (D.6) to obtain:

$$\begin{aligned}
f &= (1 - \chi) \left\{ \pi^I \left[1 - \frac{\eta^I(1 - \alpha)V_m^P}{\alpha V_m^I + (1 - \alpha)V_m^P} \right] \mathcal{L}_z^I(m) \right. \\
&\quad \left. + \pi^P \left[1 - \frac{\eta^P \alpha V_m^I}{\alpha V_m^I + (1 - \alpha)V_m^P} \right] \mathcal{L}_z^P(m) \right\} \\
&\quad + \omega(1 - \chi) \frac{(1 - \alpha)V_m^P \mathcal{L}_z^I(l + m) + \alpha V_m^I \mathcal{L}_z^P(m - l)}{\alpha V_m^I + (1 - \alpha)V_m^P} - \omega[(1 - \alpha)\nu_I + \alpha\nu_P].
\end{aligned}$$

We can then substitute out ν_I and ν_P using Equation (D.4) to obtain

$$\begin{aligned}
f &= (1 - \chi) \left\{ \pi^I \left[1 - \frac{\eta^I(1 - \alpha)V_m^P}{\alpha V_m^I + (1 - \alpha)V_m^P} \right] \mathcal{L}_z^I(m) \right. \\
&\quad \left. + \pi^P \left[1 - \frac{\eta^P \alpha V_m^I}{\alpha V_m^I + (1 - \alpha)V_m^P} \right] \mathcal{L}_z^P(m) \right\} \\
&\quad + \omega(1 - \chi) \frac{\min\{V_m^P \mathcal{L}_z^I(l + m), V_m^I \mathcal{L}_z^P(m - l)\}}{\alpha V_m^I + (1 - \alpha)V_m^P}.
\end{aligned}$$

When $\gamma = \beta$, then $m \geq \max\{\hat{z}^I, \hat{z}^P\}$ and $l = \kappa = 0$, so $f = 0$ and $i^d = 0$. When $\gamma > \beta$, then $m < \max\{\hat{z}^I, \hat{z}^P\}$, so $\mathcal{L}_z^I(m) > 0$ and/or $\mathcal{L}_z^P(m) > 0$ and therefore $f > 0$ and $i^d > 0$. Consider $\mu_I > 0$ or $\mu_P > 0$, and restrict attention to generic cases in which either $\nu_I = \nu_P = 0$ or $\lambda_I = \lambda_P = 0$. With $\nu_I = \nu_P = 0$, we can substitute out λ_I and λ_P using Equation (D.5) to obtain

$$\begin{aligned}
f &= (1 - \chi) \left\{ \pi^I \left[1 - \frac{\eta^I(1 - \alpha)V_d^P}{\alpha V_d^I + (1 - \alpha)V_d^P} \right] \mathcal{L}_z^I(m) \right. \\
&\quad \left. + \pi^P \left[1 - \frac{\eta^P \alpha V_d^I}{\alpha V_d^I + (1 - \alpha)V_d^P} \right] \mathcal{L}_z^P(m) \right\} \\
&\quad + \omega(1 - \chi) \frac{(1 - \alpha)V_d^P \mathcal{L}_z^I(l + m) + \alpha V_d^I \mathcal{L}_z^P(m - l)}{\alpha V_d^I + (1 - \alpha)V_d^P} + \omega[(1 - \alpha)\mu_I + \alpha\mu_P].
\end{aligned}$$

Then, substitute out μ_I and μ_P using Equation (D.4) to obtain

$$f = (1 - \chi) \left\{ \pi^I \left[1 - \frac{\eta^I(1 - \alpha)V_d^P}{\alpha V_d^I + (1 - \alpha)V_d^P} \right] \mathcal{L}_z^I(m) \right. \\ \left. + \pi^P \left[1 - \frac{\eta^P \alpha V_d^I}{\alpha V_d^I + (1 - \alpha)V_d^P} \right] \mathcal{L}_z^P(m) \right\} \\ + \omega(1 - \chi) \frac{\max\{V_d^P \mathcal{L}_z^I(l + m), V_d^I \mathcal{L}_z^P(m - l)\}}{\alpha V_d^I + (1 - \alpha)V_d^P}.$$

Again, when $\gamma = \beta$, then $m \geq \max\{\hat{z}^I, \hat{z}^P\}$ and $l = \kappa = 0$, so $f = 0$ and $i^d = 0$. When $\gamma > \beta$, then $m < \max\{\hat{z}^I, \hat{z}^P\}$, so $\mathcal{L}_z^I(m) > 0$ and/or $\mathcal{L}_z^P(m) > 0$ and therefore $f > 0$ and $i^d > 0$. With $\lambda_I = \lambda_P = 0$, we have

$$f = (1 - \chi) \left\{ \pi^I \left[1 - \frac{\eta^I(1 - \alpha)\delta^P}{\alpha\delta^I + (1 - \alpha)\delta^P} \right] \mathcal{L}_z^I(m) + \pi^P \left[1 - \frac{\eta^P \alpha \delta^I}{\alpha\delta^I + (1 - \alpha)\delta^P} \right] \mathcal{L}_z^P(m) \right\} \\ + \omega(1 - \chi) \frac{(1 - \alpha)\delta^P \mathcal{L}_z^I(l + m) + \alpha\delta^I \mathcal{L}_z^P(m - l)}{\alpha\delta^I + (1 - \alpha)\delta^P} \\ + \omega[(1 - \alpha)(\mu_I - \nu_I) + \alpha(\mu_P - \nu_P)].$$

Then, substitute out ν_I and ν_P using Equation (D.5), and μ_I and μ_P using Equation (D.4), to obtain

$$f = (1 - \chi) \left\{ \pi^I \left[1 - \frac{\eta^I(1 - \alpha)\delta^P}{\alpha\delta^I + (1 - \alpha)\delta^P} \right] \mathcal{L}_z^I(m) + \pi^P \left[1 - \frac{\eta^P \alpha \delta^I}{\alpha\delta^I + (1 - \alpha)\delta^P} \right] \mathcal{L}_z^P(m) \right\} \\ + \frac{\omega}{\alpha\delta^I + (1 - \alpha)\delta^P} \max \left\{ \begin{array}{l} \delta^P [\chi \mathcal{L}_z^I(\kappa + m + d) + (1 - \chi) \mathcal{L}_z^I(l + m)], \\ \delta^I [\chi \mathcal{L}_z^P(m + d - \kappa) + (1 - \chi) \mathcal{L}_z^P(m - l)] \end{array} \right\} \\ - \omega\chi \frac{\max\{\delta^P \mathcal{L}_z^I(\kappa + m + d), \delta^I \mathcal{L}_z^P(m + d - \kappa)\}}{\alpha\delta^I + (1 - \alpha)\delta^P},$$

Again, when $\gamma = \beta$, then $m \geq \max\{\hat{z}^I, \hat{z}^P\}$ and $l = \kappa = 0$, so $f = 0$ and $i^d = 0$. When $\gamma > \beta$, then $m < \max\{\hat{z}^I, \hat{z}^P\}$, so $\mathcal{L}_z^I(m) > 0$ and/or $\mathcal{L}_z^P(m) > 0$, and therefore $f > 0$ and $i^d > 0$. It follows that $i^d \geq 0$, with strict inequality if and only if $\gamma > \beta$.

Finally, consider the nominal rate earned by holding bonds between two consecutive CMs. From, Equations (16) and (17) it follows $\text{sgn}(i^b - i^b) = \text{sgn}(g)$, where

$$g = \chi [\pi^I \mathcal{L}_z^I(m + d) + \pi^P \mathcal{L}_z^P(m + d)] + \omega(1 - \alpha) [\mathcal{F}_{IP,d}(m, d, b) - \mathcal{F}_{IP,b}(m, d, b)] \\ + \omega\alpha [\mathcal{F}_{PI,d}(m, d, b) - \mathcal{F}_{PI,b}(m, d, b)] \\ = \chi [\pi^I \mathcal{L}_z^I(m + d) + \pi^P \mathcal{L}_z^P(m + d)] + \omega[(1 - \alpha)(\nu_I - \lambda_I) + \alpha(\nu_P - \lambda_P)] \\ + \omega\chi(1 - \alpha) \frac{\delta^P - \alpha(\lambda_I - \lambda_P)/\beta}{\alpha\delta^I + (1 - \alpha)\delta^P} [\mathcal{L}_z^I(\kappa + m + d) - \mathcal{L}_z^I(m + d)]$$

$$+ \omega \chi \alpha \frac{\delta^I + (1 - \alpha)(\lambda_I - \lambda_P)/\beta}{\alpha \delta^I + (1 - \alpha)\delta^P} [\mathcal{L}_z^P(m + d - \kappa) - \mathcal{L}_z^P(m + d)].$$

Observe that g is just the RHS of Equation (16) minus the RHS of Equation (17). Consider first $\mu_I = \mu_P = 0$. We can substitute out λ_I and λ_P using Equations (D.5) and (D.6) to obtain:

$$g = \chi \left\{ \begin{aligned} &\pi^I \left[1 - \frac{\eta^I(1 - \alpha)V_m^P}{\alpha V_m^I + (1 - \alpha)V_m^P} \right] \mathcal{L}_z^I(m + d) \\ &+ \pi^P \left[1 - \frac{\eta^P \alpha V_m^I}{\alpha V_m^I + (1 - \alpha)V_m^P} \right] \mathcal{L}_z^P(m + d) \end{aligned} \right\} \\ + \omega \min \left\{ \left(\frac{\chi V_m^P \mathcal{L}_z^I(\kappa + m + d)}{\alpha V_m^I + (1 - \alpha)V_m^P} + \nu_I \right), \left(\frac{\chi V_m^I \mathcal{L}_z^P(m + d - \kappa)}{\alpha V_m^I + (1 - \alpha)V_m^P} + \nu_P \right) \right\}.$$

Then, substitute out ν_I and ν_P using Equations (D.4) and (D.6) to find

$$g = \chi \left\{ \begin{aligned} &\pi^I \left[1 - \frac{\eta^I(1 - \alpha)V_m^P}{\alpha V_m^I + (1 - \alpha)V_m^P} \right] \mathcal{L}_z^I(m + d) + \pi^P \left[1 - \frac{\eta^P \alpha V_m^I}{\alpha V_m^I + (1 - \alpha)V_m^P} \right] \mathcal{L}_z^P(m + d) \end{aligned} \right\} \\ + \omega \frac{\min \left\{ \begin{aligned} &\chi V_m^P \mathcal{L}_z^I(\kappa + m + d) + (1 - \chi) \max\{V_m^P \mathcal{L}_z^I(l + m) - V_m^I \mathcal{L}_z^P(m - l), 0\}, \\ &\chi V_m^I \mathcal{L}_z^P(m + d - \kappa) + (1 - \chi) \max\{V_m^I \mathcal{L}_z^P(m - l) - V_m^P \mathcal{L}_z^I(l + m), 0\} \end{aligned} \right\}}{\alpha V_m^I + (1 - \alpha)V_m^P}.$$

When $\gamma = \beta$, then $m \geq \max\{\hat{z}^I, \hat{z}^P\}$ and $l = \kappa = 0$, so $g = 0$ and $i^b = i^d = 0$. When $\gamma > \beta$, then $m < \max\{\hat{z}^I, \hat{z}^P\}$, so $\mathcal{L}_z^I(m) > 0$ and/or $\mathcal{L}_z^P(m) > 0$ and therefore $g \geq 0$ and $i^b \geq i^d$. Consider $\mu_I > 0$ or $\mu_P > 0$, and restrict attention to generic cases in which either $\nu_I = \nu_P = 0$ or $\lambda_I = \lambda_P = 0$. With $\nu_I = \nu_P = 0$, we can substitute out λ_I and λ_P using Equation (D.5) to obtain

$$g = \chi \left\{ \begin{aligned} &\pi^I \left[1 - \frac{\eta^I(1 - \alpha)V_d^P}{\alpha V_d^I + (1 - \alpha)V_d^P} \right] \mathcal{L}_z^I(m + d) \\ &+ \pi^P \left[1 - \frac{\eta^P \alpha V_d^I}{\alpha V_d^I + (1 - \alpha)V_d^P} \right] \mathcal{L}_z^P(m + d) \end{aligned} \right\} \\ + \omega \chi \frac{\min\{V_d^P \mathcal{L}_z^I(\kappa + m + d), V_d^I \mathcal{L}_z^P(m + d - \kappa)\}}{\alpha V_d^I + (1 - \alpha)V_d^P}.$$

Again, when $\gamma = \beta$, then $m \geq \max\{\hat{z}^I, \hat{z}^P\}$ and $l = \kappa = 0$, so $g = 0$ and $i^b = i^d = 0$. When $\gamma > \beta$, then $m < \max\{\hat{z}^I, \hat{z}^P\}$, so $\mathcal{L}_z^I(m) > 0$ and/or $\mathcal{L}_z^P(m) > 0$ and therefore $g \geq 0$ and $i^b \geq i^d$. Finally, consider $\lambda_I = \lambda_P = 0$. Then, we can substitute out ν_I and ν_P

using Equation (D.5) to obtain

$$g = \chi \left\{ \begin{aligned} &\pi^I \left[1 - \frac{\eta^I(1-\alpha)\delta^P}{\alpha\delta^I + (1-\alpha)\delta^P} \right] \mathcal{L}_z^I(m+d) \\ &+ \pi^P \left[1 - \frac{\eta^P\alpha\delta^I}{\alpha\delta^I + (1-\alpha)\delta^P} \right] \mathcal{L}_z^P(m+d) \end{aligned} \right\} \\ + \omega\chi \frac{\max\{\delta^P \mathcal{L}_z^I(\kappa+m+d), \delta^I \mathcal{L}_z^P(m+d-\kappa)\}}{\alpha\delta^I + (1-\alpha)\delta^P}.$$

Again, when $\gamma = \beta$, then $m \geq \max\{\hat{z}^I, \hat{z}^P\}$ and $l = \kappa = 0$, so $g = 0$ and $i^b = i^d = 0$. When $\gamma > \beta$, then $m < \max\{\hat{z}^I, \hat{z}^P\}$, so $\mathcal{L}_z^I(m) > 0$ and/or $\mathcal{L}_z^P(m) > 0$ and therefore $g \geq 0$ and $i^b \geq i^d$. From Equations (16) and (17), it follows directly that $i^d, i^b \leq i^f = (\gamma - \beta)/\beta$. We can conclude $i^d = i^b = i^f = 0 \Leftrightarrow \gamma = \beta$ and $0 < i^d < i^b \leq i^f \Leftrightarrow \gamma > \beta$. Q.E.D.

D.6 Proof of Proposition 1

Suppose first that $m = \phi_{t+1}M_t$ is larger than $(\hat{z}^I + \hat{z}^P)/2$. When the supply of bonds is sufficiently large so that households are unconstrained by their bond holdings in the OTC, meaning $b \geq |\hat{a}(m)|$, we have that

$$l(m, b)|_{2m \geq \hat{z}^I + \hat{z}^P, b \geq |\hat{a}(m)|} = \max\{\hat{z}^I - m, 0\}. \quad (\text{D.12})$$

Combining Equations (18), (27) and (D.12) implies that we can write flow welfare as

$$\tilde{\mathcal{W}}(m, b)|_{\substack{2m \geq \hat{z}^I + \hat{z}^P \\ b \geq |\hat{a}(m)|}} = \pi^I \left[1 - \eta^I \frac{\alpha[1 - \theta^I(m)]\delta^I + [\alpha\theta^I(m) + 1 - \alpha]\delta^P}{\alpha\delta^I + (1-\alpha)\delta^P} \right] \frac{\mathcal{L}^I(m)}{\theta^I(m)} + \mathcal{C},$$

where $\mathcal{C}$ is a term that does not depend on m and b as long as $m \geq (\hat{z}^I + \hat{z}^P)/2$ and $b \geq |\hat{a}(m)|$. Note that welfare is constant in m when $m \geq \hat{z}^I$, and that $m \geq \hat{z}^I \Rightarrow |\hat{a}(z)| = 0$. Observe that

$$\lim_{\eta^I \rightarrow 1} \tilde{\mathcal{W}}(m, b)|_{\substack{2m \geq \hat{z}^I + \hat{z}^P \\ b \geq |\hat{a}(m)|}} = -\pi \frac{\alpha(\delta^I - \delta^P)}{\alpha\delta^I + (1-\alpha)\delta^P} \mathcal{L}^I(m) + \lim_{\eta^I \rightarrow 1} \mathcal{C},$$

Because $\delta^P > \delta^I$, this expression is strictly decreasing in m for $(\hat{z}^I + \hat{z}^P)/2 < m < \hat{z}^I$. Since $\lim_{m \uparrow \hat{z}^I} |\hat{a}(m)| = 0$, there exists an $\varepsilon' > 0$ such that

$$\lim_{\eta^I \rightarrow 1} \tilde{\mathcal{W}}(\hat{z}^I - \varepsilon, b') > \lim_{\eta^I \rightarrow 1} \tilde{\mathcal{W}}(m'', b'') \quad \forall \varepsilon \in (0, \varepsilon'), b' > 0, m'' \geq \hat{m}^I \text{ and } b'' \geq 0.$$

It follows that it is not optimal to implement the Friedman rule when $\eta^I \rightarrow 1$ and $b > 0$.

To show existence of a critical threshold $\tilde{\eta}^I$ to rationalize deviations from the Friedman

rule, suppose either $b = 0$ or $\eta^I = 0$. Because there is no OTC trade, flow welfare satisfies

$$\tilde{\mathcal{W}}(m, b)|_{b=0 \vee \eta^I=0} = \pi^I \mathcal{L}^I(m)/\theta^I(m) + \pi^P \mathcal{L}^P(m)/\theta^P(m) + U(y^*) - y^*,$$

which is strictly increasing in m until $m = \hat{z}^I$ and constant thereafter. Therefore, the Friedman rule is optimal when $\eta^I = 0$ or $b = 0$. Now, suppose that the Friedman rule is not optimal for some $\bar{\eta}^I \in (0, 1)$. Using that $\omega = \pi^I \eta^I$, welfare in that case satisfies

$$\max_{m \geq 0, b \geq 0} \tilde{\mathcal{W}}(m, b) = \pi^I \frac{\mathcal{L}^I(\bar{m})}{\theta^I(\bar{m})} + \pi^P \frac{\mathcal{L}^P(\bar{m})}{\theta^P(\bar{m})} + \pi^I \bar{\eta}^I [\tilde{\mathcal{F}}(\bar{m}, \bar{b}) + \tilde{\mathcal{E}}(\bar{m}, \bar{b})] + U(y^*) - y^*,$$

where $\bar{m} < \hat{z}^I$ and $\bar{b} > 0$. Importantly, $\tilde{\mathcal{F}}(\bar{m}, \bar{b}) + \tilde{\mathcal{E}}(\bar{m}, \bar{b}) > 0$ must hold since

$$\tilde{\mathcal{F}}(m, b)|_{m \geq \hat{z}^I} = \tilde{\mathcal{E}}(m, b)|_{m \geq \hat{z}^I} = 0 \quad \text{and} \quad \left. \frac{\mathcal{L}^i(m)}{\theta^i(m)} \right|_{m < \hat{z}^I} < \left. \frac{\mathcal{L}^i(m)}{\theta^i(m)} \right|_{m \geq \hat{z}^i} \quad \forall i \in \{I, P\}.$$

It follows that the Friedman rule is also not optimal when $\eta_I > \bar{\eta}^I$. This is because more welfare can then be attained by simply fixing the real face value of bonds at $\bar{b}$ and choosing γ such that $m = \bar{m}$.

Concluding, there exists an $\bar{\eta}^I \in (0, 1)$ such that the Friedman rule is not optimal if and only if $\eta_I > \bar{\eta}^I$ and $b > 0$. Because the supply of bonds and OTC activity are irrelevant at the Friedman rule, it follows that an optimal policy $\langle \gamma, b \rangle$ must satisfy $\gamma > \beta$ and $b > 0$ when $\eta_I > \bar{\eta}^I$. When $\eta^I \leq \bar{\eta}^I$, it follows that any $\langle \gamma, b \rangle$ for which $\gamma = \beta$ is an optimal policy. Q.E.D.

D.7 Proof of Proposition 2

Flow welfare in SSE can be expressed as a weighted average of flow welfare in two SSEs of the baseline economy: $\mathcal{W}(m, d, b) = (1 - \chi)\tilde{\mathcal{W}}(m, g_1(m, d, b)) + \chi\tilde{\mathcal{W}}(m + d, g_2(m, d, b))$, where $\tilde{\mathcal{W}}(z, g) = \pi^I \mathcal{L}^I(z)/\theta^I(z) + \pi^P \mathcal{L}^P(z)/\theta^P(z) + \omega[\tilde{\mathcal{F}}(z, g) + \tilde{\mathcal{E}}(z, g)] + U(y^*) - y^*$. Recall that g_1 or g_2 can be negative. Nevertheless, g_1 and g_2 cannot be negative simultaneously.

Suppose first that m , d , and b are such that g_1 and g_2 are both positive. It follows directly that $\mathcal{W}(m', d', b') \leq \max_{\{m, b\} \in \mathbb{R}_+^2} \tilde{\mathcal{W}}(m, b)$ for all $\{m', d', b'\} \in \mathbb{R}_+^3$. That means, if we choose policy optimally, introducing notes can only improve welfare when either g_1 or g_2 is negative. So now, suppose that m , b , and d are chosen such that either g_1 or g_2 is indeed negative. The introduction of notes only improves welfare when there exist $z' > 0$ and $g' < 0$ so that $\tilde{\mathcal{W}}(z', g') > \max_{\{m, b\} \in \mathbb{R}_+^2} \tilde{\mathcal{W}}(m, b)$. When $\hat{l}(z') \geq 0$, with $g' < 0$ money is transferred from the impatient to the patient households, which in turn implies a transfer of savings from the impatient to the patient households. Because deviating from the Friedman rule can only be attractive when OTC trade implies a transfer of savings from the impatient to the patient, $\tilde{\mathcal{W}}(z', g') < \max_{\{m, b\} \in \mathbb{R}_+^2} \tilde{\mathcal{W}}(m, b)$ when $\hat{l}(z') \geq 0$. It

follows that $\tilde{\mathcal{W}}(z', g') > \max_{\{m, b\} \in \mathbb{R}_+^2} \tilde{\mathcal{W}}(m, b)$ only if $\hat{l}(z') < 0$.

When $\hat{l}(z') < 0$, it means that when households in the baseline economy enter OTC t with money worth z' CM $t + 1$ goods, they find it attractive to transfer money from the impatient household to the patient households. This requires that $\delta^P \mathcal{L}_z^I(z') \leq \delta^I \mathcal{L}_z^P(z')$ and because $\delta^P > \delta^I$, it implies that $\mathcal{L}_z^I(z') < \mathcal{L}_z^P(z')$. Clearly, Condition 1 contradicts that $\hat{l}(z') < 0$.

When $\hat{l}(z') < 0$ and $g' < 0$, we have that $\tilde{\mathcal{W}}(z', g') \geq \max_{\{m, b\} \in \mathbb{R}_+^2} \tilde{\mathcal{W}}(m, b)$ requires $\pi^I \mathcal{L}^I(z')/\theta^I(z') + \pi^P \mathcal{L}^P(z')/\theta^P(z') + \omega[\tilde{\mathcal{F}}(z', g') + \tilde{\mathcal{E}}(z', g')] > \pi^I \mathcal{L}^I(\hat{z}^I)/\theta^I(\hat{z}^I) + \pi^P \mathcal{L}^P(\hat{z}^P)/\theta^P(\hat{z}^P)$. Because $g' < 0$ implies that OTC trade patterns are reversed, $\tilde{\mathcal{F}}(z', g') < 0$ when $g' < 0$. It follows that we then need:

$$\begin{aligned} \pi \frac{\mathcal{L}^I(\hat{z}^I)}{\theta^I(\hat{z}^I)} + \pi^P \frac{\mathcal{L}^P(\hat{z}^P)}{\theta^P(\hat{z}^P)} &< \pi^I \left\{ \eta^I \left[\frac{[1 - \theta^I(l' + z')]\mathcal{L}^I(l' + z')}{\theta^I(l' + z')} + \mathcal{L}^I(z') \right] \right. \\ &\quad \left. + (1 - \eta^I) \frac{\mathcal{L}^I(z')}{\theta^I(z')} \right\} \\ &\quad + \pi^P \left\{ \eta^P \left[\frac{[1 - \theta^P(z' - l')]\mathcal{L}^P(z' - l')}{\theta^P(z' - l')} + \mathcal{L}^P(z') \right] \right. \\ &\quad \left. + (1 - \eta^P) \frac{\mathcal{L}^P(z')}{\theta^P(z')} \right\}, \end{aligned}$$

which in turn requires that there exists $i \in \{I, P\}$ and $z < \hat{z}^i$ such that $\mathcal{L}^i(\hat{z}^i)[1 - \theta^i(\hat{z}^i)]/\theta^i(\hat{z}^i) < \mathcal{L}^i(z)[1 - \theta^i(z)]/\theta^i(z)$. That means in some DGM meetings, sellers become worse off when buyers consume more. Clearly, this contradicts Condition 2. Q.E.D.

D.8 Proof of Proposition 3

Set the supply of illiquid bonds to zero: $b = 0$. Then, suppose that $m = \phi_{t+1} M_t$ is larger than $(\hat{z}^I + \hat{z}^P)/2$. Let the supply of notes be such that households are effectively unconstrained by their note holdings in all OTC and DGM meetings; $d \geq \tilde{d}(m)$.

At the margin, notes provide no DGM liquidity and become equivalent to bonds. Also, because $2m > \hat{z}^I + \hat{z}^P$, all households are unconstrained by their money holdings in the DGM, except for impatient households that did not obtain an OTC match when $m < \hat{z}^I$. Specifically, the real value of money balances acquired by impatient agents in OTC matches satisfies.

$$l(m, d, 0)|_{2m \geq \hat{z}^I + \hat{z}^P, d \geq \tilde{d}(m)} = \max\{\hat{z}^I - m, 0\}$$

and the real value of money plus notes acquired by the impatient in OTC matches satisfies

$$\kappa(m, d, 0)|_{2m \geq \hat{z}^I + \hat{z}^P, d \geq \tilde{d}(m)} = -\frac{1 - \chi}{\beta} \frac{(1 - \alpha)[\mathcal{L}^I(\hat{z}^I) - \mathcal{L}^I(\min\{m, \hat{z}^I\})]}{\alpha\delta^I + (1 - \alpha)\delta^P}.$$

It follows that

$$\tilde{d}(m) = \max \left\{ \begin{array}{l} \max\{\hat{z}^I - m, 0\} + \kappa(m, d, 0)|_{2m \geq \hat{z}^I + \hat{z}^P, d \geq \tilde{d}(m)}, \\ l(m, d, 0)|_{2m \geq \hat{z}^I + \hat{z}^P, d \geq \tilde{d}(m)} - \kappa(m, d, 0)|_{2m \geq \hat{z}^I + \hat{z}^P, d \geq \tilde{d}(m)} \end{array} \right\},$$

where $\tilde{d}(m) = 0$ when $m \geq \hat{z}^I$ and $\tilde{d}(m) > 0$ when $m < \hat{z}^I$.

We can then write welfare as:

$$\mathcal{W}(m, d, 0) \Big|_{\substack{2m \geq \hat{z}^I + \hat{z}^P \\ d \geq \tilde{d}(m)}} = \pi^I(1 - \chi) \left[1 - \eta^I \frac{\alpha[1 - \theta^I(m)]\delta^I + [\alpha\theta^I(m) + (1 - \alpha)]\delta^P}{\alpha\delta^I + (1 - \alpha)\delta^P} \right] \frac{\mathcal{L}^I(m)}{\theta^I(m)} + \mathcal{C},$$

where $\mathcal{C}$ is a term that does not depend on m and d when $m \geq (\hat{z}^I + \hat{z}^P)$ and $d > \tilde{d}(m)$. Applying exactly the same reasoning as in the proof of Proposition 1, completes the proof of the current proposition. Q.E.D.

D.9 Proof of Proposition 4

When we start from a situation in which $\eta^I < \tilde{\eta}_\chi^I$ the Friedman rule is optimal. An infinitesimally small increase in ω , which means η^I increases, will therefore have no effect on optimal policy and welfare.

When we start from a situation in which $\eta^I > \tilde{\eta}_\chi^I$, then optimal policy does not implement the Friedman rule. To understand welfare effects of an increase in ω , suppose: policy was initially set optimally, the supply of notes does not respond to the change in ω , and γ changes in such a way that real money balances remain unaffected. Using equation (27), we find $d\mathcal{W}/(d\omega) = \mathcal{F}(m, d, 0) + \mathcal{E}(m, d, 0)$. If this expression is strictly positive, it must be that welfare increases following an increase in ω . By supposition, this is the case. Why? Because if initially the Friedman rule was not optimal in an economy with only notes and money, the welfare contribution of financial market activity must have been strictly positive to compensate for suppressed DGM activity. Q.E.D.

D.10 Proof of Proposition 5

Suppose there are no externalities from OTC trade, i.e. $\theta^i(z) = 1$ for all z and $i \in \{I, P\}$. Recall that steady state flow welfare in an economy with only notes and money can be written as $\mathcal{W}(m, d, 0) = (1 - \chi)\tilde{\mathcal{W}}(m, g_1(m, d, 0)) + \chi\tilde{\mathcal{W}}(m + d, g_2(m, d, 0))$. First, observe that when $g < 0$, we have $\text{sgn}(l(z, g)) = \text{sgn}(-\hat{l}(z))$. It follows that

$$\begin{aligned} \mathcal{W}(z, g) &= \pi^I \mathcal{L}^I(z) + \pi^P \mathcal{L}^P(z) + \frac{\delta^P[\mathcal{L}^I(l(z, g) + z) - \mathcal{L}^I(z)] + \delta^I[\mathcal{L}^P(z - l(z, g)) - \mathcal{L}^P(z)]}{\alpha\delta^I + (1 - \alpha)\delta^P} \\ &\leq \pi^I \mathcal{L}^I(z) + \pi^P \mathcal{L}^P(z) \\ &\leq \pi^I \mathcal{L}^I(\hat{z}^I) + \pi^P \mathcal{L}^P(\hat{z}^P) \equiv \mathcal{W}^{fr} \end{aligned}$$

where $\mathcal{W}^{fr}$ is flow welfare when the Friedman rule is implemented. Therefore, we have that $\tilde{\mathcal{W}}(m, g_1), \tilde{\mathcal{W}}(m + d, g_2) \leq \max_{\{z, g\} \in \mathbb{R}_+^2} \tilde{\mathcal{W}}(z, g)$. Second, because the Friedman rule can always be implemented we have $\max_{\{m, d\} \in \mathbb{R}_+^2} \mathcal{W}(m, d, 0) \geq \mathcal{W}^{fr}$. Suppose $m^* \geq 0$ and $d^* \geq 0$ are such that $\max_{\{m, d\} \in \mathbb{R}_+^2} \mathcal{W}(m, d, 0) = \mathcal{W}(m^*, d^*, 0)$. Let $l^* = l(m^*, g_1(m^*, d^*, 0))$ and $\kappa^* = l(m^* + d^*, g_2(m^*, d^*, 0))$, with l given by Equation (21). Exploiting the characterization of OTC match surplus, we obtain

$$\begin{aligned}
\mathcal{W}^{fr} &\leq \mathcal{W}(m^*, d^*, 0) \\
&\leq \chi \left\{ \pi^I [\eta^I \mathcal{L}^I(\kappa^* + m^* + d^*) + (1 - \eta^I) \mathcal{L}^I(m^* + d^*)] \right. \\
&\quad \left. + \pi^P [\eta^P \mathcal{L}^P(m^* + d^* - \kappa^*) + (1 - \eta^P) \mathcal{L}^P(m^* + d^*)] \right\} \\
&\quad + (1 - \chi) \left\{ \pi^I [\eta^I \mathcal{L}^I(l^* + m^*) + (1 - \eta^I) \mathcal{L}^I(m^*)] \right. \\
&\quad \left. + \pi^P [\eta^P \mathcal{L}^P(m^* - l^*) + (1 - \eta^P) \mathcal{L}^P(m^*)] \right\} - \omega\beta(\delta^P - \delta^I)\kappa^* \\
&\leq \pi^I \mathcal{L}^I(\hat{z}^I) + \pi^P \mathcal{L}^P(\hat{z}^P) - \omega\beta(\delta^P - \delta^I)\kappa^* \\
&= \mathcal{W}^{fr} - \omega\beta(\delta^P - \delta^I)\kappa^*,
\end{aligned}$$

which implies $\kappa^* \leq 0$. Equation (21) implies $a(z, g) - l(z, g) \geq 0 \Leftrightarrow l(z, b) \geq 0$, so $a(m^* + d^*, g_2(m^*, d^*, 0)) - l(m^* + d^*, g_2(m^*, d^*, 0)) \leq 0$. As a result

$$\begin{aligned}
\tilde{\mathcal{W}}(m^* + d^*, g_2(z^*, d^*, 0)) &= \pi^I [\eta^I \mathcal{L}^I(\kappa^* + m^* + d^*) + (1 - \eta^I) \mathcal{L}^I(m^* + d^*)] \\
&\quad + \pi^P [\eta^P \mathcal{L}^P(m^* + d^* - \kappa^*) + (1 - \eta^P) \mathcal{L}^P(m^* + d^*)] \\
&\quad + \omega\beta(\delta^P - \delta^I) \begin{bmatrix} a(m^* + d^*, g_2(m^*, d^*, 0)) \\ -l(m^* + d^*, g_2(m^*, d^*, 0)) \end{bmatrix} \\
&\leq \pi^I \mathcal{L}^I(\hat{z}^I) + \pi^P \mathcal{L}^P(\hat{z}^P).
\end{aligned}$$

We conclude that $\max_{\{m, d\} \in \mathbb{R}_+^2} \mathcal{W}(m, d, 0) \leq (1 - \chi) \left[\max_{\{z, g\} \in \mathbb{R}_+^2} \tilde{\mathcal{W}}(z, g) \right] + \chi [\pi^I \mathcal{L}^I(\hat{z}^I) + \pi^P \mathcal{L}^P(\hat{z}^P)]$. Now, let $z' \geq 0$ and $g' \geq 0$ satisfy $\max_{\{z, g\} \in \mathbb{R}_+^2} \tilde{\mathcal{W}}(z, g) = \tilde{\mathcal{W}}(z', g')$. Because there are no externalities from OTC trade, $g' \geq \hat{a}(z')$, and $\tilde{\mathcal{W}}(z', g') = \tilde{\mathcal{W}}(z', g'')$ for all $g'' \geq \hat{a}(z')$. Suppose we choose $m' = z'$ and $d' = \max\{\hat{z}^I + \hat{a}(z') - \hat{l}(z'), \hat{z}^P + \hat{l}(z') - \hat{a}(z'), \hat{a}(z')\}$. It follows $l(m', g_1(m', d', 0)) = \hat{l}(m')$ and $l(m' + d', g_2(m', d', 0)) = \hat{l}(m') - \hat{a}(m')$, so that

$$\begin{aligned}
\mathcal{W}(m', d', 0) &= (1 - \chi) \tilde{\mathcal{W}}(m', g_1(m', d', 0)) + \chi \tilde{\mathcal{W}}(m' + d', g_2(m', d', 0)) \\
&= (1 - \chi) \tilde{\mathcal{W}}(z', g') + \chi [\pi^I \mathcal{L}^I(\hat{z}^I) + \pi^P \mathcal{L}^P(\hat{z}^P)] \\
&= (1 - \chi) \left[\max_{\{z, g\} \in \mathbb{R}_+^2} \tilde{\mathcal{W}}(z, g) \right] + \chi [\pi^I \mathcal{L}^I(\hat{z}^I) + \pi^P \mathcal{L}^P(\hat{z}^P)] \\
&\geq \max_{\{m, d\} \in \mathbb{R}_+^2} \mathcal{W}(m, d, 0)
\end{aligned}$$

$$\geq \mathcal{W}(m', d', 0).$$

So, in an economy without externalities from OTC trade, we can conclude $\mathcal{W}(m', d', 0) = \max_{\{m, d\} \in \mathbb{R}_+^2} \mathcal{W}(m, d, 0) = (1 - \chi) \left[\max_{\{z, g\} \in \mathbb{R}_+^2} \tilde{\mathcal{W}}(z, g) \right] + \chi [\pi^I \mathcal{L}^I(\hat{x}^I) + \pi^P \mathcal{L}^P(\hat{x}^P)]$. Optimized welfare in such an economy is linear in χ , and strictly decreasing in χ when $\eta^I > \tilde{\eta}^I$.

Next, consider an economy with externalities from OTC trade. Suppose a social planner can choose OTC trades subject to proportional bargaining. I.e., he/she solves

$$\begin{aligned} \hat{\mathcal{F}}^s(m, d, b) = \max_{l, \kappa} & \left\{ \chi \left[\frac{\mathcal{L}^I(\kappa + m + d)}{\theta^I(\kappa + m + d)} - \frac{\mathcal{L}^I(m + d)}{\theta^I(m + d)} + \frac{\mathcal{L}^P(m + d - \kappa)}{\theta^P(m + d - \kappa)} - \frac{\mathcal{L}^P(m + d)}{\theta^P(m + d)} \right] \right. \\ & + (1 - \chi) \left[\frac{\mathcal{L}^I(l + m)}{\theta^I(l + m)} - \frac{\mathcal{L}^I(m)}{\theta^I(m)} + \frac{\mathcal{L}^P(m - l)}{\theta^P(m - l)} - \frac{\mathcal{L}^P(m)}{\theta^P(m)} \right] \\ & \left. + \beta(\delta^P - \delta^I)(a - \kappa) \right\}, \\ \text{s.t. } & -m \leq l \leq m, \quad -d \leq \kappa - l \leq d, \quad \text{and} \quad -b \leq a \leq b, \end{aligned}$$

where a is given by Equation (D.3). Let $\widehat{\mathcal{W}}(m, d, b) = \chi[\pi^I \mathcal{L}^I(m + d)/\theta^I(m + d) + \pi^P \mathcal{L}^P(m + d)/\theta^P(m + d)] + (1 - \chi)[\pi^I \mathcal{L}^I(m)/\theta^I(m) + \pi^P \mathcal{L}^P(m)/\theta^P(m)] + \omega \hat{\mathcal{F}}^s(m, d, b)$. Using a similar procedure as in Appendix D.3, it can be shown there exist $\hat{g}_1(m, d, b)$ and $\hat{g}_2(m, d, b)$ such that $\widehat{\mathcal{W}} = (1 - \chi)\widehat{\mathcal{W}}(m, \hat{g}_1(m, d, b)) + \chi\widehat{\mathcal{W}}(m + d, \hat{g}_2(m, d, b))$, where $\widehat{\mathcal{W}}(z, g) = \pi^I \mathcal{L}^I(z) + \pi^P \mathcal{L}^P(z) + \omega \hat{\mathcal{F}}^s(z, g)$ and

$$\hat{\mathcal{F}}^s(z, g) = \max_{l \in [-z, z]} \left\{ \frac{\mathcal{L}^I(l + z)}{\theta^I(l + z)} - \frac{\mathcal{L}^I(z)}{\theta^I(z)} + \frac{\mathcal{L}^P(z - l)}{\theta^P(z - l)} - \frac{\mathcal{L}^P(z)}{\theta^P(z)} + \beta(\delta^P - \delta^I)(a - l) \right\}$$

subject to $-g \leq a \leq g$, where

$$a - l = \frac{1}{\beta} \frac{\alpha[\mathcal{L}^I(l + z) - \mathcal{L}^I(z)] - (1 - \alpha)[\mathcal{L}^P(z - l) - \mathcal{L}^P(z)]}{\alpha\delta^I + (1 - \alpha)\delta^P}. \quad (\text{D.13})$$

The analysis of an economy without externalities from OTC trade can then be used to find

$$\max_{\{m, d\} \in \mathbb{R}_+^2} \widehat{\mathcal{W}}(m, d, 0) = \max_{\{z, g\} \in \mathbb{R}_+^2} \{\widehat{\mathcal{W}}(z, g)\} + \chi[\pi^I \mathcal{L}^I(\hat{z}^I)/\theta^I(\hat{z}^I) + \pi^P \mathcal{L}^P(\hat{z}^P)/\theta^P(\hat{z}^P)].$$

Also, let $\tilde{\mathcal{W}}(z^*, g^*) = \max_{\{z, g\} \in \mathbb{R}_+^2} \tilde{\mathcal{W}}(z, g)$, with $z^* \geq 0$ and $g^* \geq 0$ and observe $\hat{l}(z^*) > 0$ when Condition 1 or 2 holds.

Suppose $\hat{l}(z^*) \leq \hat{l}^s(z^*)$, where

$$\hat{l}^s(z) = \arg \max_{l \in [-z, z]} \left\{ \frac{\mathcal{L}^I(l + z)}{\theta^I(l + z)} - \frac{\mathcal{L}^I(z)}{\theta^I(z)} + \frac{\mathcal{L}^P(z - l)}{\theta^P(z - l)} - \frac{\mathcal{L}^P(z)}{\theta^P(z)} + \beta(\delta^P - \delta^I)(a - l) \right\}$$

with a given by Equation (D.13). It follows $g^* \geq \hat{a}(z^*)$ and $\tilde{\mathcal{W}}(z^*, g^{**}) = \tilde{\mathcal{W}}(z^*, g^*) \Leftrightarrow g^{**} \geq \hat{a}(z^*)$. Let $m' = z^*$ and $d' = \max\{\hat{z}^I + \hat{a}(m') - \hat{l}(m'), \hat{z}^P + \hat{l}(m') - \hat{a}(m'), \hat{a}(m')\}$. Suppose there exist $\{m'', d''\} \in \mathbb{R}_+^2$ such that $\mathcal{W}(m'', d'', 0) > \mathcal{W}(m', d', 0)$. Let $\kappa'' = l(m'' + d'', g_2(m'', d'', 0))$ and $l'' = l(m'', g_1(m'', d'', 0))$, with l given by Equation (21). We can then perform the following manipulations

$$\begin{aligned}
& -\omega\beta(\delta^P - \delta^I)\kappa'' + \pi^I \mathcal{L}^I(\hat{z}^I)/\theta^I(\hat{z}^I) + \pi^P \mathcal{L}^P(\hat{z}^P)/\theta^P(\hat{z}^P) \\
& \geq \chi \left\{ \pi^I \left[\eta^I \frac{\mathcal{L}^I(\kappa'' + m'' + d'')}{\theta^I(\kappa'' + m'' + d'')} + (1 - \eta^I) \frac{\mathcal{L}^I(m'' + d'')}{\theta^I(m'' + d'')} \right] \right. \\
& \quad \left. + \pi^P \left[\eta^P \frac{\mathcal{L}^P(m'' + d'' - \kappa'')}{\theta^P(m'' + d'' - \kappa'')} + (1 - \eta^P) \frac{\mathcal{L}^P(m'' + d'')}{\theta^P(m'' + d'')} \right] \right\} \\
& \quad + (1 - \chi) \left\{ \pi^I \left[\eta^I \frac{\mathcal{L}^I(l'' + m'')}{\theta^I(l'' + m'')} + (1 - \eta^I) \frac{\mathcal{L}^I(m'')}{\theta^I(m'')} \right] \right. \\
& \quad \left. + \pi^P \left[\eta^P \frac{\mathcal{L}^P(m'' - l'')}{\theta^P(m'' - l'')} + (1 - \eta^P) \frac{\mathcal{L}^P(m'')}{\theta^P(m'')} \right] \right\} \\
& \quad - \omega\beta(\delta^P - \delta^I)\kappa'' \\
& = \mathcal{W}(m'', d'', 0) \\
& > \mathcal{W}(m', d', 0) \\
& = (1 - \chi) \max_{\{z, g\} \in \mathbb{R}_+^2} \{\tilde{\mathcal{W}}(z, g)\} + \chi [\pi^I \mathcal{L}^I(\hat{z}^I) + \pi^P \mathcal{L}^P(\hat{z}^P)] \\
& \geq \pi^I \mathcal{L}^I(\hat{z}^I)/\theta^I(\hat{z}^I) + \pi^P \mathcal{L}^P(\hat{z}^P)/\theta^P(\hat{z}^P),
\end{aligned}$$

to find that $l(m'' + d'', g_2(m'', d'', 0)) < 0$. In turn, that implies $a'' - \kappa'' < 0$ where $a'' = a(m'' + d'', g_2(m'', d'', 0))$ and a given by Equation (21). We then obtain

$$\begin{aligned}
\tilde{\mathcal{W}}(m'' + d'', g_1(m'', d'', 0)) & = \omega\beta(\delta^P - \delta^I)(a'' - \kappa'') \\
& \quad + \pi^I \left[\eta^I \frac{\mathcal{L}^I(\kappa'' + m'' + d'')}{\theta^I(\kappa'' + m'' + d'')} + (1 - \eta^I) \frac{\mathcal{L}^I(m'' + d'')}{\theta^I(m'' + d'')} \right] \\
& \quad + \pi^P \left[\eta^P \frac{\mathcal{L}^P(m'' + d'' - \kappa'')}{\theta^P(m'' + d'' - \kappa'')} + (1 - \eta^P) \frac{\mathcal{L}^P(m'' + d'')}{\theta^P(m'' + d'')} \right] \\
& < \pi^I \mathcal{L}^I(\hat{z}^I)/\theta^I(\hat{z}^I) + \pi^P \mathcal{L}^P(\hat{z}^P)/\theta^P(\hat{z}^P)
\end{aligned}$$

At the same time, with Condition 1 or 2 satisfied, it follows from the analysis in Section 6.1 that $g < 0 \Rightarrow \mathcal{W}(z, g) < \pi^I \mathcal{L}^I(\hat{z}^I)/\theta^I(\hat{z}^I) + \pi^P \mathcal{L}^P(\hat{z}^P)/\theta^P(\hat{z}^P)$. It follows that

$$\begin{aligned}
\mathcal{W}(m'', d'', 0) & = (1 - \chi) \tilde{\mathcal{W}}(m'', g_1(m'', d'', 0)) + \chi \tilde{\mathcal{W}}(m'' + d'', g_2(m'', d'', 0)) \\
& < (1 - \chi) \left[\max_{\{z, g\} \in \mathbb{R}_+^2} \tilde{\mathcal{W}}(z, g) \right] + \chi [\pi^I \mathcal{L}^I(\hat{z}^I)/\theta^I(\hat{z}^I) + \pi^P \mathcal{L}^P(\hat{z}^P)/\theta^P(\hat{z}^P)] \\
& = \mathcal{W}(m', d', 0),
\end{aligned}$$

so we obtain a contradiction. We can conclude that when $\hat{l}(z^*) \leq \hat{l}^s(z^*)$, we have

$$\max_{\{m,d\} \in \mathbb{R}_+^2} \mathcal{W}(m, d, 0) = (1 - \chi) \left[\max_{\{z,g\} \in \mathbb{R}_+^2} \tilde{\mathcal{W}}(z, g) \right] + \chi [\pi^I \mathcal{L}^I(\hat{z}^I)/\theta^I(\hat{z}^I) + \pi^P \mathcal{L}^P(\hat{z}^P)/\theta^P(\hat{z}^P)].$$

Optimized welfare is then linear in χ , and strictly decreasing in χ if and only if $\eta^I > \tilde{\eta}^I$.

Next, suppose that $\hat{l}(z^*) > \hat{l}^s(z^*)$ and note that $\hat{l}^s(z^*) > 0$. We have

$$\max_{\{m,d\} \in \mathbb{R}_+^2} \widehat{\mathcal{W}}(m, d, 0) = (1 - \chi) \tilde{\mathcal{W}}(z^*, g^*) + \chi [\pi^I \mathcal{L}^I(\hat{z}^I)/\theta^I(\hat{z}^I) + \pi^I \mathcal{L}^P(\hat{z}^P)/\theta^P(\hat{z}^P)]$$

and

$$g^* - \hat{l}^s(z^*) = \frac{1}{\beta} \frac{\alpha [\mathcal{L}^I(\hat{l}^s(z^*) + z^*) - \mathcal{L}^I(z^*)] - (1 - \alpha) [\mathcal{L}^P(z^* - \hat{l}^s(z^*)) - \mathcal{L}^P(z^*)]}{\alpha \delta^I + (1 - \alpha) \delta^P}.$$

When $\hat{z}^I \leq \min\{z^* + g^*, z^* + \hat{l}^s(z^*)\} = z^* + \hat{l}^s(z^*)$ and $\hat{z}^P \leq \min\{z^* + g^*, z^* - \hat{l}^s(z^*) + 2g^*\} = z^* + g^*$, then $\mathcal{W}(z^*, g^*, 0) = (1 - \chi) \tilde{\mathcal{W}}(z^*, g^*) + \chi [\pi^I \mathcal{L}^I(\hat{z}^I)/\theta^I(\hat{z}^I) + \pi^I \mathcal{L}^P(\hat{z}^P)/\theta^P(\hat{z}^P)]$, so

$$\max_{\{m,d\} \in \mathbb{R}_+^2} \mathcal{W}(m, d, 0) = (1 - \chi) \left[\max_{\{z,g\} \in \mathbb{R}_+^2} \tilde{\mathcal{W}}(z, g) \right] + \chi [\pi^I \mathcal{L}^I(\hat{z}^I)/\theta^I(\hat{z}^I) + \pi^I \mathcal{L}^P(\hat{z}^P)/\theta^P(\hat{z}^P)].$$

Optimized welfare is then linear in χ , and strictly decreasing in χ if and only if $\eta^I > \tilde{\eta}^I$. When $\hat{z}^I > \min\{z^* + g^*, z^* + \hat{l}^s(z^*)\} = z^* + \hat{l}^s(z^*)$ or $\hat{z}^P > \min\{z^* + g^*, z^* - \hat{l}^s(z^*) + 2g^*\} = z^* + g^*$, suppose a financial transaction tax can be levied during the OTC. Taxing only inefficiently high transfers of money, i.e. $\max\{l - \hat{l}^s(z), 0\}$, at a sufficiently high rate, implies that with $m' = z^*$ and $d' \geq \max\{\hat{x}^I + g^* - \hat{l}^s(z^*), \hat{x}^P + \hat{l}^s(z^*) - g^*\}$, we have $l^{tax}(m', g_1(m', d')) = \hat{l}^s(m')$. In turn, this implies

$$\begin{aligned} \mathcal{W}^{tax}(m', d', 0) &= (1 - \chi) \left[\max_{\{z,g\} \in \mathbb{R}_+^2} \tilde{\mathcal{W}}(z, g) \right] + \chi [\pi^I \mathcal{L}^I(\hat{z}^I)/\theta^I(\hat{z}^I) + \pi^I \mathcal{L}^P(\hat{z}^P)/\theta^P(\hat{z}^P)] \\ &= \max_{\{m,d\} \in \mathbb{R}_+^2} \widehat{\mathcal{W}}(m, d, 0) \end{aligned}$$

so we again have that optimized welfare is linear in χ , and strictly decreasing in χ if and only if $\eta^I > \tilde{\eta}^I$.

From the above we can also conclude that if OTC trade can only be affected by changing the supply of real money balances and notes, i.e. we cannot levy a financial transaction tax, then

$$\pi^I \mathcal{L}^I(\hat{z}^I)/\theta^I(\hat{z}^I) + \pi^P \mathcal{L}^P(\hat{z}^P)/\theta^P(\hat{z}^P) \leq \max_{\{m,d\} \in \mathbb{R}_+^2} \mathcal{W}(m, d, 0)$$

$$\begin{aligned} &\leq (1 - \chi) \max_{\{z, g\} \in \mathbb{R}_+^2} \{\tilde{\mathcal{W}}(z, g)\} \\ &\quad + \chi [\pi^I \mathcal{L}^I(\hat{z}^I)/\theta^I(\hat{z}^I) + \pi^P \mathcal{L}^P(\hat{z}^P)/\theta^P(\hat{z}^P)]. \end{aligned}$$

Note as well that $\max_{\{z, g\} \in \mathbb{R}_+^2} \tilde{\mathcal{W}}(z, g) \geq \pi^I \mathcal{L}^I(\hat{z}^I)/\theta^I(\hat{z}^I) + \pi^P \mathcal{L}^P(\hat{z}^P)/\theta^P(\hat{z}^P)$, with strict inequality if and only if $\eta^I > \tilde{\eta}^I$. Therefore when $\eta^I > \tilde{\eta}^I$ optimized welfare as a function of χ attains a maximum when $\chi \rightarrow 1$ and a minimum when $\chi \rightarrow 0$, while if $\eta^I \leq \tilde{\eta}^I$ optimized welfare is independent of χ . Q.E.D.

D.11 Proof of Lemma 3

Households enter WFM t with money worth m_t CM $t + 1$ goods and bonds worth b_t CM $t + 1$ goods. Let $\mathcal{M}^i$ be the inverse of $\mathcal{L}_z^i$, which is well defined on the domain $(0, \lim_{m \rightarrow 0} \mathcal{L}_z^i(m))$. We find the following schedule for l_t^i , which is the net value of money that an agent with $\delta^i \in \{\delta^I, \delta^P\}$ acquires when able to trade during WFM t :

$$l_t^i \begin{cases} = -m_t & \text{if } \lim_{m \rightarrow 0} \mathcal{L}_z^i(m) \leq \iota_t \beta \delta^i \\ = \min\{\mathcal{M}^i(\iota_t \beta \delta^i) - m_t, b_t/(1 + \iota_t)\} & \text{if } 0 < \iota_t \beta \delta^i \lim_{m \rightarrow 0} \mathcal{L}_z^i(m) < \iota_t \beta \delta^i \\ \in [\min\{\hat{z}^i - m_t, b_t\}, b_t] & \text{if } \iota_t \beta \delta^i = 0 \\ = b_t/(1 + \iota_t) & \text{if } \iota_t \beta \delta^i < 0 \end{cases}. \quad (\text{D.14})$$

Clearing WFM t requires $\eta^I \pi^I l_t^I + \eta^P \pi^P l_t^P = 0$. Observe that $\lim_{\iota_t \rightarrow \infty} l_t^i = -m_t$ and that $l_t^i > 0$ when $\iota_t < 0$. Because l_t^i is increasing in ι_t , it follows that there exists a unique $\bar{\iota}_t \geq 0$ such that $\eta^I \pi^I l_t^I + \eta^P \pi^P l_t^P = 0$. By continuity of $\mathcal{L}_z^i$, $\bar{\iota}_t$ is continuous in m_t and b_t . When $\eta^I \pi^I = \eta^P \pi^P = \omega$, then $\bar{\iota}_t$ does not depend on ω . Also, $\iota_t > 0 \Leftrightarrow \eta^I \pi^I \min\{\hat{z}^I - m, b\} + \eta^P \pi^P \min\{\hat{z}^P - m, b\} > 0$ and $[\eta^I \pi^I + \eta^P \pi^P] m_t > \eta^I \pi^I \hat{z}^I + \eta^P \pi^P \hat{z}^P \Rightarrow \bar{\iota}_t = 0$. Q.E.D.

D.12 Proof of Lemma 4

Like SSE of the baseline economy, b_t is fixed by policy at b , $m_t = m$, and $\beta i_t^f = \gamma - \beta$. It follows directly from Definition 2, Equation (34) and $\iota_t = \bar{\iota}(m_t, b_t)$ that symmetric equilibria with a positive value for fiat money satisfy

$$\gamma - \beta = \sum_{i \in \{I, P\}} \pi \left[\eta^i \max\{\mathcal{L}_z^i(b/(1 + \bar{\iota}(m, b)) + m), \beta \delta^i \bar{\iota}(m, b)\} + (1 - \eta^i) \mathcal{L}_z^i(m) \right]. \quad (\text{D.15})$$

Any $m > 0$ that solves Equation (D.15) then sufficiently describes an SSE. To see why, note that b is given by policy, that $\iota_t = \bar{\iota}(m, b)$, and that the interest rate on bonds can

be found by using (m, b) in Equation (35) to yield

$$i^b = \frac{\gamma[1 + \bar{\iota}(m, b)]}{\sum_{i \in \{I, P\}} \pi^i \eta^i \max\{\mathcal{L}_z^i(b/(1 + \bar{\iota}(m, b)) + m) - \beta \delta^i \bar{\iota}(m, b), 0\} + \beta[1 + \bar{\iota}(m, b)]} - 1.$$

Given an exogenously specified initial supply of money M_0 , the sequence $\{\phi_t, \psi_t\}_{t=0}^\infty$ is given by

$$\phi_t = m/[\gamma^{t-1} M_0] \quad \text{and} \quad \psi_t = \phi_t(1 + i^b).$$

Finally, using $\iota_t = \bar{\iota}(m, b)$ in Equation (D.14) yields l^I and l^P . It follows that the obtain sequence of portfolios choices, interest rates, WFM trades, and prices meets the conditions specified in Definition 1. Hence, we have an equilibrium.

The next step is to show that $m \geq \hat{m} \Leftrightarrow \gamma = \beta$ and $m < \hat{m} \Leftrightarrow \gamma > \beta$, where

$$\hat{m} = \begin{cases} \hat{z}^I & \text{if } \eta^I < 1 \\ \max\{\hat{z}^I - b, (\hat{z}^I + \hat{z}^P)/2\} & \text{if } \eta^I = 1 \end{cases}.$$

Suppose first that $\eta^I = 1$. When $m \geq \max\{\hat{z}^I - b, (\hat{z}^I + \hat{z}^P)/2\}$, then $m \geq (\hat{z}^I + \hat{z}^P)/2$ so $\bar{\iota}(m, b) = 0$ (see Lemma 3), $m > \hat{z}^P$ and $\mathcal{L}_z^I(b + m) = \mathcal{L}_z^P(b + m) = 0$. Equation (D.15) then implies $\gamma = \beta$. When $m < \max\{\hat{z}^I - b, (\hat{z}^I + \hat{z}^P)/2\}$, then we must have $\bar{\iota}(m, b) > 0$ and/or $\mathcal{L}_z^I(b + m) > 0$. To see this, note that when $m < \hat{z}^I - b$ we have $\mathcal{L}_z^I(b + m) > 0$ and when $\hat{z}^I - b \leq m < (\hat{z}^I + \hat{z}^P)/2$, we cannot have $\bar{\iota}(m, b) \leq 0$ as this would imply $l^I + l^P > 0$ in Equation (D.14). So $\gamma > \beta$ according to Equation (D.15). Then, consider $\eta^I < 1$. When $m \geq \hat{z}^I$ then $\bar{\iota}(m, b) = 0$ (see Lemma 3) and $\mathcal{L}_z^I(b + m) = \mathcal{L}_z^P(b + m) = 0$, so Equation (D.15) implies $\gamma = \beta$. When $m < \hat{z}^I$, then $(1 - \eta^I)\mathcal{L}_z^I(m) > 0$ and because $\bar{\iota}(m, b) \geq 0$ (see Lemma 3), it follows from Equation (D.15) that $\gamma > \beta$.

Next, consider continuity of γ . First, $\iota(m, b)$ is continuous in m and b . Second, $\mathcal{L}_z^i(m)$ is continuous in m . Hence, the right-hand side of Equation (D.15) is continuous in m and b , which immediately implies that γ is continuous in m and b .

The last step is to conclude that by continuity, $\lim_{\gamma \downarrow \beta} m = \hat{m}$.

Q.E.D.

D.13 Proof of Proposition 6

When $m \geq (\hat{z}^P + \hat{z}^I)/2$, Lemma 3 implies that $\bar{\iota}(m, b) = 0$. Moreover, Equation (D.14) implies that $l^P \geq \hat{z}^P - m$ and that, without loss of generality, $l^P = \max\{0, \min\{m + b, \hat{z}^I\} - m\}$. Flow welfare (36) can then be written as

$$\tilde{\mathcal{W}}(m, b)|_{2m \geq \hat{z}^I + \hat{z}^P} = \pi^I \left[\eta^I \frac{\mathcal{L}^I(\min\{m + b, \hat{z}^I\})}{\theta^I(\min\{m + b, \hat{z}^I\})} + (1 - \eta^I) \frac{\mathcal{L}^I(m)}{\theta^I(m)} \right] + \mathcal{C}, \quad (\text{D.16})$$

where $\mathcal{C}$ is a constant that does not depend on (m, b) . When $\eta^I < 1$, it follows from Lemma 4 that $m \geq \hat{z}^I \Leftrightarrow \gamma = \beta$, $m < \hat{z}^I \Leftrightarrow \gamma > \beta$ and that γ is continuous in m . It follows from Equation (D.16) that small deviations from the Friedman rule reduce welfare when $\eta^I < 1$.

Now, consider $\eta^I = 1$. At the Friedman rule, we have $m \geq (\hat{z}^I + \hat{z}^P)/2$ when households are unconstrained by their bond holdings in the WFM. When unconstrained by their bond holdings, $l_t^I = \hat{l}(m)$ with $\hat{l}(m)$ similar as in the baseline economy. WFM equilibrium then satisfies

$$\bar{v}(m, b) = \max\{\mathcal{L}_z^I(\hat{l}(m) + m)/(\beta\delta^I), \mathcal{L}_z^P(-\hat{l}(m) + m)/(\beta\delta^P)\}.$$

Defining $\mathcal{S}^i(m) = \mathcal{L}^i(m)/\theta^i(m)$, the welfare effect of a small reduction in m when evaluated at the Friedman rule is:

$$-\frac{\partial \mathcal{W}(m, b)}{\partial m} \Big|_{\substack{2m = \hat{z}^I + \hat{z}^P \\ 2b > \hat{z}^I - \hat{z}^P}} = -\pi^I \lim_{\substack{m^I \uparrow \hat{z}^I \\ m^P \uparrow \hat{z}^P}} \left[2 \frac{\delta^I \mathcal{S}_z^I(m^I) \mathcal{L}_{zz}^P(m^P) + \delta^P \mathcal{S}_z(m^P) \mathcal{L}_{zz}^I(m^I)}{\delta^P \mathcal{L}_{zz}^I(m^I) + \delta^I \mathcal{L}_{zz}^P(m^P)} + \frac{\mathcal{L}_{zz}^I(m^I) \mathcal{L}_{zz}^P(m^P) (\delta^P - \delta^I) (m^I - m^P)}{\delta^P \mathcal{L}_{zz}^I(m^I) + \delta^I \mathcal{L}_{zz}^P(m^P)} \right],$$

Note that $\mathcal{L}^i(m)$ attains its maximum for $m \geq \hat{z}^i$, so $\lim_{m^i \uparrow \hat{z}^i} \mathcal{L}_{zz}^i(m^i) < 0$. The second line between the square brackets is negative since $\delta^I < \delta^P$. The first line between square brackets is non-negative, and strictly positive when either $\lim_{m^I \uparrow \hat{z}^I} \mathcal{S}_z^I(m^I) > 0$ or $\lim_{m^P \uparrow \hat{z}^P} \mathcal{S}_z^P(m^P) > 0$. That means, the social benefits of DGM trade are not maximized when households are effectively unconstrained by their liquid asset holdings in the DGM, which occurs when there exist $ij \in \{I, P\}^2$ such that $\sigma_{ij}(q_{ij}^* - \hat{q}_{ij}) > 0$. It follows that the Friedman rule can still be optimal, even when $\delta^I < \delta^P$ and $\omega = \max\{\pi^I, \pi^P\}$. This occurs when δ^I is sufficiently close to δ^P and either $\lim_{m^I \uparrow \hat{z}^I} \mathcal{S}_z^I(m^I) > 0$ or $\lim_{m^P \uparrow \hat{z}^P} \mathcal{S}_z^P(m^P) > 0$.

With a DGM characterized by a Walrasian pricing or random matching with proportional bargaining, we have that $\theta^i(m)$ is a constant so $\mathcal{L}^i(m) = \theta \mathcal{S}^i(m)$. Moreover $\lim_{m^I \uparrow \hat{z}^I} \mathcal{S}_z^I(m^I) = \lim_{m^P \uparrow \hat{z}^P} \mathcal{S}_z^P(m^P) = 0$. It follows from continuity that deviations from the Friedman rule improve welfare for η^I sufficiently close to one. Because $\tilde{\mathcal{F}}(m, b)$ and $\tilde{\mathcal{E}}(m, b)$ in Equation (36) are independent of ω , it follows from a similar argument as in the proof of Proposition 1 that when the Friedman rule is not optimal for some $\eta^I < 1$, there must exist a critical threshold $\tilde{\eta}^I$ to rationalize deviations from the Friedman rule. Because optimal deviations from the Friedman rule must imply that $\bar{v}(m, b) > 0$ (see Equation (37)), it follows that coexistence of interest-bearing bonds and money arises endogenously in an optimal policy regime when $\eta^I > \tilde{\eta}^I$. Q.E.D.

D.14 Proof of Lemma C.1

From the characterization of $\mathcal{L}^i(z)$ and $\mathcal{S}_{ij}(z)$ it follows that:

$$\mathcal{L}_z^i(z) = \sigma\theta\beta \sum_{j \in \{I, P\}} \pi^j \max \left\{ \frac{\delta^j [u' \circ v_{ij}^{-1}(z)] - \delta^i [c' \circ v_{ij}^{-1}(z)]}{(1 - \theta) [u' \circ v_{ij}^{-1}(z)] + \theta [c' \circ v_{ij}^{-1}(z)]}, 0 \right\}$$

and $[u' \circ v_{ij}^{-1}(z)]\delta^j - [c' \circ v_{ij}^{-1}(z)]\delta^i > 0 \Leftrightarrow z < v_{ij}(q_{ij}^*)$. Hence, $\hat{z}^i = \max\{v_{iI}(q_{iI}^*), v_{iP}(q_{iP}^*)\}$.

First, I demonstrate that $\hat{z}^I > \hat{z}^P$. It suffices to show that $v_{II}(q_{II}^*) > v_{Pj}(q_{Pj}^*)$ for all $j \in \{I, P\}$. This requires

$$[(1 - \theta)u(q_{II}^*) + \theta c(q_{II}^*)]/\delta^I > [(1 - \theta)u(q_{Pj}^*) + \theta c(q_{Pj}^*)]/[(1 - \theta)\delta^P + \theta\delta^j],$$

which is satisfied for all $j \in \{I, P\}$ because $\delta^I < \delta^P$ and $q_{II}^* = q_{PP}^* > q_{PI}^*$.

Next, I show that $\mathcal{L}_z^I(z) \geq \mathcal{L}_z^P(z)$ with strict inequality if and only if $0 < z < \hat{z}^I$. For $z \geq \hat{z}^I$ we have by construction that $\mathcal{L}_z^I(z) = \mathcal{L}_z^P(z) = 0$. It remains to consider $z < \hat{z}^I$. Consider the following inequality:

$$\begin{aligned} \max \left\{ \frac{\delta^j [u' \circ v_{Ij}^{-1}(z)] - \delta^I [c' \circ v_{Ij}^{-1}(z)]}{(1 - \theta)[u' \circ v_{Ij}^{-1}(z)] + \theta[c' \circ v_{Ij}^{-1}(z)]}, 0 \right\} \\ \geq \max \left\{ \frac{\delta^j [u' \circ v_{PI}^{-1}(z)] - \delta^P [c' \circ v_{PI}^{-1}(z)]}{(1 - \theta)[u' \circ v_{PI}^{-1}(z)] + \theta[c' \circ v_{PI}^{-1}(z)]}, 0 \right\}, \end{aligned}$$

with $j \in \{I, P\}$. First, $v_{Ij}(q_{Ij}^*) > v_{Pj}(q_{Pj}^*)$. Therefore, if $z \geq v_{Pj}(q_{Pj}^*)$ the above is satisfied trivially. If $z < v_{Pj}(q_{Pj}^*)$, then the inequality above reduces to

$$\frac{\delta^j [u' \circ v_{Ij}^{-1}(z)] - \delta^I [c' \circ v_{Ij}^{-1}(z)]}{(1 - \theta)[u' \circ v_{Ij}^{-1}(z)] + \theta[c' \circ v_{Ij}^{-1}(z)]} \geq \frac{\delta^j [u' \circ v_{PI}^{-1}(z)] - \delta^P [c' \circ v_{PI}^{-1}(z)]}{(1 - \theta)[u' \circ v_{PI}^{-1}(z)] + \theta[c' \circ v_{PI}^{-1}(z)]},$$

which holds with strict inequality when $z > 0$ because $\delta^I < \delta^P$, $[u'(q) - c'(q)]/[(1 - \theta)u'(q) + \theta c'(q)]$ is decreasing in q , and $v_{Ij}^{-1}(z) < v_{PI}^{-1}(z)$ for all $z > 0$ and $j \in \{I, P\}$. From the characterization of $\mathcal{L}_z^i$ it follows that for $0 < z < \hat{z}^I$, we have $\mathcal{L}_z^I(z) > \mathcal{L}_z^P(z)$. Finally, $\lim_{z \rightarrow 0} \mathcal{L}_z^i(z) = \sigma\beta\theta/(1 - \theta)$. Therefore $\mathcal{L}_z^I(z) \geq \mathcal{L}_z^P(z)$, with strict inequality if and only if $0 < z < \hat{z}^I$. Q.E.D.

D.15 Proof of Lemma C.2

From the characterization of $\mathcal{L}^i(z)$ it follows that

$$\mathcal{L}_z^i(z) = \sigma\theta\beta \sum_{j \in \{I, J\}} \pi^j \max \left\{ \delta^j [u' \circ v_{ij}^{-1}(z)]/[c' \circ v_{ij}^{-1}(z)] - \delta^i, 0 \right\}$$

and $[u' \circ v_{ij}^{-1}(z)]\delta^j - [c' \circ v_{ij}^{-1}(z)]\delta^i > 0 \Leftrightarrow z < v(q_{ij}^*)$. Hence, $\hat{z}^i = \max\{v(q_{iI}^*), v(q_{iP}^*)\}$.

First, I demonstrate that $\hat{z}^I > \hat{z}^P$. For this, it suffices to show that $v_{II}(q_{II}^*) \geq v_{Pj}(q_{Pj}^*)$ for all $j \in \{I, P\}$. Because $q_{II}^* = q_{PP}^* > q_{PI}^*$ and $\delta^I < \delta^P$, we find for all $j \in \{I, P\}$

$$\begin{aligned} v_{II}(q_{II}^*) &= \int_0^{q_{II}^*} \frac{1}{\beta} \frac{u'(r)c'(r)}{\delta^I \theta u'(r) + \delta^I (1-\theta)c'(r)} dr \geq \int_0^{q_{II}^*} \frac{1}{\beta} \frac{u'(r)c'(r)}{\delta^j \theta u'(r) + \delta^P (1-\theta)c'(r)} dr \\ &> \int_0^{q_{Pj}^*} \frac{1}{\beta} \frac{u'(r)c'(r)}{\delta^j \theta u'(r) + \delta^P (1-\theta)c'(r)} dr = v_{Pj}(q_{Pj}^*). \end{aligned}$$

Next, I show $\mathcal{L}_z^I(z) \geq \mathcal{L}_z^P(z)$ with strict inequality if $z < \hat{z}^I$. For $z \geq \hat{z}^I$ we have by construction that $\mathcal{L}_z^I(z) = \mathcal{L}_z^P(z) = 0$. It remains to consider $z < \hat{z}^I$. Consider the following inequality

$$\max\{\delta^j[u' \circ v_{Ij}^{-1}(z)]/[c' \circ v_{Ij}^{-1}(z)] - \delta^I, 0\} \geq \max\{\delta^j[u' \circ v_{Pj}^{-1}(z)]/[c' \circ v_{Pj}^{-1}(z)] - \delta^P, 0\},$$

with $j \in \{I, P\}$. First, $v_{Ij}(q_{Ij}^*) > v_{Pj}(q_{Pj}^*)$. Therefore, if $z \geq v_{Pj}(q_{Pj}^*)$ the above is satisfied trivially. If $z < v_{Pj}(q_{Pj}^*)$, then the inequality above reduces to

$$\delta^j[u' \circ v_{Ij}^{-1}(z)]/[c' \circ v_{Ij}^{-1}(z)] - \delta^I \geq \delta^j[u' \circ v_{Pj}^{-1}(z)]/[c' \circ v_{Pj}^{-1}(z)] - \delta^P.$$

Since $\delta^I < \delta^P$, it suffices to show that $[u' \circ v_{Ij}^{-1}(z)]/[c' \circ v_{Ij}^{-1}(z)] \geq [u' \circ v_{Pj}^{-1}(z)]/[c' \circ v_{Pj}^{-1}(z)]$. Observe $u'(q)/c'(q)$ is decreasing in q , so it suffices to check whether $v_{Ij}^{-1}(z) \leq v_{Pj}^{-1}(z)$. Observe that

$$z = \int_0^{v_{Ij}^{-1}(z)} \frac{u'(r)c'(r)}{\delta^j \theta u'(r) + \delta^I (1-\theta)c'(r)} dr \quad \text{and} \quad z = \int_0^{v_{Pj}^{-1}(z)} \frac{u'(r)c'(r)}{\delta^j \theta u'(r) + \delta^P (1-\theta)c'(r)} dr$$

so $v_{Ij}^{-1}(z) \leq v_{Pj}^{-1}(z)$, with strict inequality if and only if $z > 0$. It then follows from the characterization of $\mathcal{L}_z^i(x)$ that $\mathcal{L}_z^I(z) > \mathcal{L}_z^P(z)$ for $z < \hat{z}^I$. Q.E.D.

D.16 Proof of Lemma C.3

It follows from the characterization $\mathcal{L}^i(z)$ that:

$$\mathcal{L}_z^i(z) = \sigma\beta\theta\delta^i \sum_{j \in \{I, P\}} \pi^j \max \left\{ \frac{\delta^j[u' \circ v_{ij}^{-1}(z)] - \delta^i[c' \circ v_{ij}^{-1}(z)]}{\delta^j(1-\theta)[u' \circ v_{ij}^{-1}(z)] + \delta^i\theta[c' \circ v_{ij}^{-1}(z)]}, 0 \right\}.$$

It follows that $\delta^j[u' \circ v_{ij}^{-1}(z)] - \delta^i[c' \circ v_{ij}^{-1}(z)] > 0 \Leftrightarrow z < v_{ij}(q_{ij}^*)$. Therefore, $\hat{z}^i = \max\{v_{iI}(q_{iI}^*), v_{iP}(q_{iP}^*)\}$.

To demonstrate that $\hat{z}^I > \hat{z}^P$, it suffices to show that $v_{II}(q_{II}^*) > v_{Pj}(q_{Pj}^*)$ for all $j \in \{I, P\}$. In turn, this implies that we need

$$[(1 - \theta)u(q_{II}^*) + \theta c(q_{II}^*)]/\delta^I > [\delta^j(1 - \theta)u(q_{Pj}^*) + \delta^P \theta c(q_{Pj}^*)]/[\delta^I \delta^j] \quad \text{for all } j \in \{I, P\}.$$

Given that $\delta_I < \delta_P$ and that $q_{II}^* = q_{PP}^* > q_{PI}^*$, the inequality above is indeed satisfied for all $j \in \{I, P\}$.

Finally, because $\lim_{z \rightarrow 0} \mathcal{L}_z^i(z) = \sigma \beta \delta^i \theta / (1 - \theta)$ and $\delta^I < \delta^P$, we have $\lim_{z \rightarrow 0} \mathcal{L}_z^I(z) < \lim_{z \rightarrow 0} \mathcal{L}_z^P(z)$. Q.E.D.

D.17 Proof of Lemma C.4

Because we have $\lim_{\theta \rightarrow 1} v_{ij}(q) = c(q)/(\beta \delta^j)$ for both generalized Nash Bargaining and proportional bargaining, it follows from the proof of Lemma C.1 that with generalized Nash bargaining, $\lim_{\theta \rightarrow 1} \hat{z}^I > \lim_{\theta \rightarrow 1} \hat{z}^P$ and $\lim_{\theta \rightarrow 1} \mathcal{L}_z^I(z) \geq \lim_{\theta \rightarrow 1} \mathcal{L}_z^P(z)$, with strict inequality if and only if $z < \hat{z}^I$ for $z > 0$. It follows from continuity that with generalized Nash bargaining there exists a $\underline{\theta} < 1$ such that when $\theta > \underline{\theta}$ we have $\hat{z}^I > \hat{z}^P$ and $\mathcal{L}_z^I(z) \geq \mathcal{L}_z^P(z)$, with strict inequality if and only if $z < \hat{z}^I$. Q.E.D.

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