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PRODUCTION AND SEQUENCING PROCESSES**

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Cooperation and Competition in Linear Production and Sequencing Processes

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Abstract

We analyze applications of biform games to linear production (LP) and sequencing processes. Biform games, as introduced by Brandenburger and Stuart (2007), apply to problems in which strategic decisions are followed by some cooperative game, where the specific environment of the cooperative game that is played, is in turn determined by these strategic decisions. We extend the work on LP-processes by allowing firms to compete for resources that are scarce or hard to produce, rather than assuming these resource bundles are simply given. With strategy dependent resource bundles that can be obtained from two locations, we show that the induced strategic game has a (pure) Nash equilibrium, using the Owen set or any game-theoretic solution concept that satisfies anonymity to solve the cooperative LP-game. To analyze competition in sequencing processes, we no longer assume an initial processing order is given. Instead, this initial order is strategically determined. Solving the second-stage cooperative sequencing game using a gain splitting rule, we fully determine the set of Nash equilibria of the induced strategic game.

Keywords: biform games, pure Nash equilibria, linear production, Owen set, sequencing, gain splitting rule

JEL classification: C70

1 Introduction

Cooperative and non-cooperative game theory are often presented as two opposing branches of the same field, where players either cooperate and form coalitions, or do not cooperate and

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decide only on their own strategies. In this paper, we analyze hybrid models that combine elements from these two branches of game theory in one two-stage model. In particular, we incorporate strategy dependence into linear production (LP) processes and sequencing processes.

Cooperative game theory studies situations in which groups of players can cooperate by signing a contract or establishing a joint plan of action. It is assumed that these agreements between players are binding. In cooperative games with transferable utility, one then assigns a (joint) value to every possible coalition, thereby defining the cooperative game. By ‘solving’ this game, we find allocations of the total joint revenue among the set of players. This stands in contrast to non-cooperative game theory, where binding agreements are not allowed. In a non-cooperative game, strategic players are interested in maximizing their individual payoffs, taking into account the strategic behavior of other players. The main topic of interest is often (the existence of) Nash equilibria (Nash, 1951). Combining cooperative and non-cooperative game theory has numerous advantages. Among other things, such a combination is able to incorporate externalities, by having the value created by a coalition in the cooperative stage also depend on the strategic choices of players who are not in that coalition, through their strategies in the non-cooperative stage.

Brandenburger and Stuart (2007) create a two-stage model called a *biform game* to analyze strategic moves in business, where a non-cooperative first stage is followed by a cooperative second stage. The non-cooperative stage concerns a strategic decision like whether to invest in innovation, and these strategic decisions made by the players then determine the competitive environment in which some cooperative game is played. They use a cooperative solution procedure based on the core to find a unique allocation in the cooperative phase for all possible strategy combinations. These allocations per strategy combination are used as the payoff vectors for the so-called induced non-cooperative game. This induced game can then be analyzed as a standard strategic game, for which the existence of pure Nash equilibria is investigated. We restrict our analysis to pure Nash equilibria, henceforth simply referred to as Nash equilibria.

Although certainly inspired by the work of Brandenburger and Stuart (2007), this paper deviates from the original biform games both in model and in application. Rather than studying business strategy, we define biform games to analyze two well-investigated operations research (OR) problems. We introduce a strategic component to cooperative ‘OR-games’ (see Borm et al. (2001) for a survey) corresponding to LP-processes and sequencing processes. In particular, we

analyze the influence of a strategy dependent resource bundle and a strategy dependent initial processing order, respectively. Further, we will not use the core to determine the allocation in the cooperative game. The core is a powerful solution concept that is very suitable for the analysis of cooperative games in a competitive environment, but a disadvantage of the core is that it usually does not prescribe a unique allocation vector, where a unique payoff vector is required for the induced strategic game. Brandenburger and Stuart (2007) solve this problem by taking a weighted average of the extreme points of the core. The weights are determined by so-called confidence indices, reflecting the degree of confidence each player has in their performance in the cooperative game. Finding these confidence indices is somewhat arbitrary, perhaps even more so in an OR-game setting. Therefore, we choose payoff vectors based on the Owen set (Owen, 1975) and (a generalization of) the equal gain splitting rule (as introduced by Curiel et al. (1989b) and generalized by Hamers et al. (1996)), respectively. Though examples can be constructed for which the Owen set contains infinitely many allocation vectors, it generally prescribes a unique allocation vector. A gain splitting rule always leads to a unique payoff vector. Note that the Owen set is a subset of the core in LP-games, and any non-negative gain splitting rule yields a core element in sequencing games, so we certainly do not disregard or disqualify the core as a solution concept. By choosing these context specific core selectors, however, we aim to let the allocation of value in the cooperative game more aptly reflect the specific problem at hand.

Before introducing both models in more detail, it is important to discuss the general purpose of this paper, also considering the existing literature. Two-stage (or multi-stage) models are not uncommon in the game-theoretic literature, but the two-stages are often both cooperative or both non-cooperative. Combining cooperative and non-cooperative game theory into the same model, is still far less common. Before the aforementioned paper on biform games, Hart and Moore (1990) consider a two-stage model to compare transactions within firms to those between firms (with a focus on the optimal assignment of assets), in which a non-cooperative stage is followed by a cooperative game. Instead of the core, they use the Shapley value (Shapley, 1953) to solve the cooperative game. Further, this specific type of two-stage game has been used to analyze, among other things, supply network formation (Hennet and Mahjoub, 2010), smart grid communications (Kim, 2012), stochastic programming with recourse (Summerfield and Dror, 2013), and the impact of surplus division on investment incentives (Feess and Thun, 2014). Nevertheless, the existing literature is relatively limited in quantity, compared to some other branches of game theory. We believe that the conceptual idea of biform games can be

applied to a great variety of problems, and much work still needs to be done. We extend the existing literature by analyzing two applications of biform games. Despite the fact that we find several results concerning (the existence of) Nash equilibria in the two biform games we consider, the objective of this paper is not to give a fully exhaustive theoretical analysis of either of the models. By analyzing and reflecting on both models, we also aim to inspire others to continue in this field of research.

An LP-process in a general setting, as presented in Owen (1975), can be used to model situations in which a set of players is able to pool a set of resources used in the manufacturing of a set of products. How much of each resource players need to manufacture a product is described by a linear technology matrix, the availability of resources per player is determined by resource bundles, and the prices of products by some price vector. From this, a cooperative game can be derived, called a linear production game (LP-game). Without the need to formally define the game, the Owen set derives a tailor-made set of payoff allocations directly from the LP-process on the basis of duality arguments from linear programming. Van Gellekom et al. (2000) provide a detailed axiomatic characterization of the Owen set. Granot (1986) generalizes LP-processes such that the resource bundle of a set of players need not equal the sum of the resource bundles of the individual players. Curiel et al. (1989a) consider LP-processes with committee control. Feltkamp et al. (1993) generalize LP-processes to allow for the existence of multiple production facilities. In general, these models do not incorporate strategy dependence in the LP-process. Hennet and Mahjoub (2010) use a biform model in the context of LP-processes, but do this through a strategy dependent price vector, and analyze this in the context of the role of a player in a supply chain. We extend the existing literature by adapting the original definition of LP-processes such that the resource bundles of individual players are determined strategically.

It is often assumed that resources are owned completely by the players (firms) at the start of an LP-process. This assumption is quite restrictive, since in practice the firms are often dependent on their supply chain to obtain these resources. This is a situation that lends itself well for analysis with a biform model. Starting with the non-cooperative first stage, players compete to obtain resources. One might think of a situation in which firms can obtain a scarce or hard to produce resource, like fossil fuels or complicated electronics, from different sources. There may be significant costs and preparation time involved with settling on a source, for example due to a need for lobbying to access some scarce resource in another country, or to train or financially support manufacturers of some hard to produce resource. Therefore, firms can only

settle on one source. The resource bundle available at a source is often restricted, meaning this bundle has to be split between firms if multiple firms decide to settle on the same location. The competition for resource bundles gives rise to the first-stage strategic game that determines the exact LP-process in which the firms end up. Once each firm has obtained a resource bundle, it may be of interest to the firms to cooperate, as they might have a surplus in one resource and a deficit in another resource needed in the manufacturing process. This is modeled in the second-stage LP-process, which is solved using a payoff vector based on the Owen set. We refer to this model as a *biform linear production (BLP) process*.

The induced strategic games that arise from BLP-processes are shown to exhibit some interesting properties. The existence of a (pure) Nash equilibrium is guaranteed in the ‘standard case’: a finite number of players, two locations of sources, and players who choose the same source split the corresponding resource bundle equally. The payoff based on the Owen set is then contrasted with one-point game-theoretic solution concepts like the Shapley value that explicitly make use of the corresponding LP-games with respect to each strategy combination. This approach is somewhat more similar to the aforementioned model of Hart and Moore (1990), albeit applied to an entirely different setting. We show that the existence of a Nash equilibrium is still guaranteed, provided that the solution concept based on the Owen set is replaced by an anonymous game-theoretic solution concept of the corresponding LP-games. Finally, we discuss the effect of changes in the structure of the BLP-process on the existence of Nash equilibria. We consider modifications with unequal resource bundle splitting, or with more than two locations.

Next to the analysis of LP-processes, we also apply the biform framework to sequencing processes. A sequencing problem involves determining in what order a finite number of jobs should be lined up in front of one or more machines to minimize the joint costs incurred by the set of jobs as a whole. These costs are often a linear function of the completion times of the jobs, where different jobs generally have different cost parameters and processing times. This creates a measure of ‘urgency’ for the jobs. For linear cost functions, the sequencing problem is optimized by processing the jobs in (weakly) decreasing order of urgency, also defined as a Smith order (Smith, 1956). If we are given some initial order of jobs in the queue (often said to be representing initial processing rights), this order is generally not optimal. Solving the sequencing problem then yields a rearrangement of the jobs leading to maximal joint cost savings. A natural question is how to allocate these cost savings over the various jobs, which is particularly relevant when different jobs belong to different agents (e.g. different companies processing jobs

on the same machines). By treating jobs as agents, or players, this question can be answered using game theory.

The first so-called sequencing game was developed by Curiel et al. (1989b) for the deterministic one-machine sequencing problem. The work on sequencing games has been extended in several ways (see also Curiel et al. (2002) for a survey), including, but not limited to sequencing games with ready times (Hamers et al., 1995), due dates (Borm et al., 2002), setup times (Grundel et al., 2013), precedence relations (Hamers et al., 2005), and externalities (Yang et al., 2019). In the vast majority of the literature on sequencing games, an initial order is assumed to be given. An exception to this is Klijn and Sánchez (2006), who analyze sequencing games without an initial order. As a consequence, they no longer look at cost savings, but simply at costs. They define two different cooperative games, a ‘tail game’ in which each coalition assumes it forms the tail of an artificial initial order, and a ‘pessimistic game’ in which the worst case (high initial cost and few cooperation possibilities) is considered by each coalition. A somewhat less pessimistic alternative is proposed by Hall and Liu (2016), who use a model where each agent has a probability that it is processed first and last, and define the value of a coalition using a corresponding ‘head-tail allocation’. Still, the games defined here are cooperative games with a single stage. Multi-stage sequencing situations were proposed by Curiel (2010), in which each stage corresponds to a cooperative game, where the order arrived at after the first stage becomes the initial order of the second stage, and so on. In this multi-stage sequencing situation, however, an initial order is still assumed to be given for the first stage.

There are many scenarios in which the initial processing rights are not naturally fully determined, but can somehow be strategically influenced by the players. For example, consider a small business that receives jobs overnight, where the shopkeeper does not know what job arrived first. In this case the shopkeeper can simply decide on an initial processing order based on some other reason (e.g. processing time due to complexity), unless some player explicitly requests to be processed first. Players might also be able to influence their position in the initial processing order by incurring some costs, consider e.g. payment for priority service in an airport or for some delivery service. Therefore, we do not assume an initial order is simply given, nor do we create a cooperative game based on artificial orders. In our model, the initial order is determined strategically, where all players have the opportunity to request their preferred position in the order. If two or more players request the same position, a tie-breaking rule is used to determine who is processed first. This tie-breaking rule can be based on e.g. the urgency of players.

In this way, we create a *biform sequencing (BS) process*. The first stage corresponds to a strategic game that determines the initial order, where the second stage is a (cooperative) sequencing process in which we assume that the cost savings of rearranging the initial order to an optimal order are allocated using a so-called gain splitting rule. We consider biform sequencing processes with and without additional costs associated with the strategic decision. Most prominently, we fully specify the set of (pure) Nash equilibria in biform sequencing processes without strategy dependent additional costs. We also discuss biform sequencing processes with such costs. Players incur additional costs based only on their obtained or requested position in the initial order. We still find a Nash equilibrium if the costs are associated with the *obtained* position in the initial order, and provide an example of the absence of Nash equilibria in case these costs are associated with the *requested* position instead.

Section 2 analyzes biform linear production processes and Section 3 treats biform sequencing processes. Both sections are completed by a subsection in which we briefly discuss possible extensions of the models.

2 Biform Linear Production Processes

Biform linear production (BLP) processes are an extension of standard LP-processes, in which the resource bundles of players are strategically determined. As mentioned in the introduction, the motivation behind this model is to analyze an LP-process where firms compete for resources that are scarce or hard to manufacture. An LP-processes is described by the tuple

$$L = (N, R, P, A, \{b^i\}_{i \in N}, c),$$

where N represents the finite set of players, R the finite set of resources, P the finite set of products, A the $|R| \times |P|$ linear technology matrix of which the cell in the r -th row and p -th column corresponds to the number of units of resource r needed to manufacture one unit of product p , $b^i \in \mathbb{R}_+^R$ represents the resource bundle of player $i \in N$, and $c \in \mathbb{R}_+^P$ represents the market prices for a unit of each product.

Let 2^N denote the collection of subsets of N . These subsets are referred to as coalitions, and N is called the grand coalition. A transferable utility game (TU-game) is a tuple (N, v) , where $v : 2^N \rightarrow \mathbb{R}$ with $v(\emptyset) = 0$ is referred to as the characteristic function. The number $v(T)$ in principle provides the highest total monetary value a coalition $T \in 2^N$ can jointly generate

without the help of the players in $N \setminus T$. The class of TU-games with player set N is denoted by TU^N .

Let $L = (N, R, P, A, \{b^i\}_{i \in N}, c)$ be an LP-process. Then, the corresponding transferable utility LP-game $v_L \in TU^N$ is defined such that the value of coalition $T \in 2^N \setminus \{\emptyset\}$ is the solution of the following maximization problem

$$v_L(T) = \max_{y \in \mathbb{R}^P} c^T y \quad \text{subject to } Ay \leq \sum_{i \in T} b^i, \quad y \geq 0.$$

In words, the value of a coalition is the maximum revenue generated by the sale of products, where production is restricted by the sum of resource bundles available to the coalition. The value $v_L(T)$ of a coalition T can also be found by solving the dual program instead. For a coalition $T \in 2^N \setminus \{\emptyset\}$,

$$v_L(T) = \min_{z \in \mathbb{R}^R} z^T \sum_{i \in T} b^i \quad \text{subject to } z^T A \geq c^T, \quad z \geq 0.$$

For any $z \in \mathbb{R}^R$ that solves the dual program, z_r is the shadow price of resource $r \in R$ corresponding to this solution. Note that in the dual programs, the feasible region does not depend on the coalition T at hand. We denote the corresponding feasible region by F , formally defined as

$$F = \{z \in \mathbb{R}^R \mid z^T A \geq c^T, z \geq 0\}.$$

The Owen set is a solution concept that exploits the unique structure of an LP-process to find an allocation vector without the need to explicitly derive the LP-game. The Owen set is a polytope (i.e., a convex hull of finitely many vectors) that is based on the shadow prices that solve the dual linear programming problem for the grand coalition. Formally, the Owen set (Owen, 1975) is defined as

$$Owen(L) = \{(z^T b^i)_{i \in N} \in \mathbb{R}^N \mid z \in F, v_L(N) = z^T \sum_{i \in N} b^i\}.$$

The Owen set of an LP-process is a subset of the core of the corresponding LP-game. The elements of the Owen set are called Owen vectors. We define the optimal region O as the set of shadow prizes within the feasible region that solve the minimization problem for the grand coalition, i.e.,

$$O = \{z \in F \mid v_L(N) = z^T \sum_{i \in N} b^i\}.$$

Note that O is a polytope. Though the Owen set generally consists of a single vector, it need not always prescribe a unique allocation vector. This is due to the fact that the optimal region O

can contain more than one element. For any given set of resource bundles, we therefore use the mean of all extreme points of O , denoted by $\bar{z}$, as ‘the’ vector of shadow prices of the resources to select one specific Owen vector $(\bar{z}^T b^i)_{i \in N}$.¹

Our model introduces a strategic element to standard LP-processes by letting players compete for resources, rather than assuming each player owns some resource bundle beforehand. We assume that resources can be obtained from two locations, sources 1 and 2, with resource bundles $l_1 \in \mathbb{R}_+^R$ and $l_2 \in \mathbb{R}_+^R$, respectively. The strategic choice of the players will be to settle on source 1 or on source 2.

To model this strategic phase, we need some notation on general strategic games. We only consider finite *strategic games* $G = (\{X^i\}_{i \in N}, \{\pi_i\}_{i \in N})$, for which the finite strategy set of player i is denoted by X^i for all $i \in N$, and the set of all strategy combinations is $X = \prod_{i \in N} X^i$. A strategy player i chooses is denoted by $x^i \in X^i$. A strategy combination chosen by all other players in $N \setminus \{i\}$ is denoted by $x^{-i} \in X^{-i}$, with $X^{-i} = \prod_{j \in N \setminus \{i\}} X^j$. For any player i , $\pi_i : X \rightarrow \mathbb{R}$ is the payoff function of this player.

A strategy combination $x \in X$ is a Nash equilibrium of the strategic game G if $\pi_i(x) \geq \pi_i(\tilde{x}^i, x^{-i})$ for all $\tilde{x}^i \in X^i$ and all $i \in N$. In words, a strategy combination is a Nash equilibrium if no player has a reason to unilaterally deviate (i.e., change strategy, given the strategies of all other players). The set of all Nash equilibria of G is denoted by $E(G)$.

We are now ready to formally define a BLP-process and corresponding induced strategic games.

Definition 2.1

A BLP-process is a tuple

$$\mathcal{L} = (N, \{X^i\}_{i \in N}, l_1, l_2, \{L(x)\}_{x \in X}),$$

in which for all $i \in N$ we have $X^i = \{1, 2\}$, l_1 and l_2 are the respective resource bundles at the two locations, and for any $x \in X$,

$$L(x) = (N, R, P, A, \{b^i(x)\}_{i \in N}, c)$$

is a corresponding LP-process, with

$$b^i(x) = \begin{cases} \frac{1}{|S_1(x)|} l_1 & \text{if } x^i = 1, \\ \frac{1}{|S_2(x)|} l_2 & \text{if } x^i = 2, \end{cases}$$

¹Note that the Owen vector based on $\bar{z}$ in fact equals the mean of all extreme points of the Owen set.

where $S_1(x) = \{i \in N \mid x^i = 1\}$ and $S_2(x) = \{i \in N \mid x^i = 2\}$.

Using the notation $L(x)$, we emphasize that an LP-process is strategy dependent. We explicitly show what part of the LP-process (indirectly) becomes strategy dependent in our notation as well, using e.g. $b^i(x)$ for the strategically determined resource bundle of player $i \in N$. If a set of players chooses the same source, the resource bundle available at this location is divided using an ‘equal bundle splitting rule’, i.e., each player gets an equal fraction of the available resource bundle. For this, we let $S_1(x)$ and $S_2(x)$ denote the set of all players who choose location 1 and 2, respectively.

The next step is to determine the payoff vectors associated with a BLP-process, thereby defining the induced strategic game. In the following two subsections, the payoff vectors of an induced strategic game are determined using ‘the’ Owen vector or some one-point game-theoretic solution concept that satisfies anonymity. In both cases, we are able to guarantee the existence of a Nash equilibrium in the induced strategic game.

2.1 BLP-Processes Using the Owen Set

In this section, the strategic game induced by a BLP-process has payoffs based on the Owen set of the corresponding LP-processes. Let $\mathcal{L} = (N, \{X^i\}_{i \in N}, l_1, l_2, \{L(x)\}_{x \in X})$ be a BLP-process. For any given $x \in X$, we define $\bar{z}(x)$ to be the average of all extreme points of the optimal region $O(x) = \{z(x) \in F \mid v_{L(x)}(N) = z(x)^T \sum_{i \in N} b^i(x)\}$, where $v_{L(x)}$ is the LP-game corresponding to LP-process $L(x) = (N, R, P, A, \{b^i(x)\}_{i \in N}, c)$. Moreover, let $\omega(x) \in \mathbb{R}^N$ be the Owen vector corresponding to $\bar{z}(x)$, given by

$$\omega_i(x) = \bar{z}(x)^T b^i(x)$$

for all $i \in N$. The induced strategic game $G^{\mathcal{L}, Owen} = (\{X^i\}_{i \in N}, \{\pi_i^{\mathcal{L}, Owen}\}_{i \in N})$ that follows from this BLP-process $\mathcal{L}$ is now defined by setting

$$\pi_i^{\mathcal{L}, Owen}(x) = \omega_i(x)$$

for any $x \in X$ and all $i \in N$.

Importantly, there are only three possibilities for the average shadow prices of a BLP-process. The key observation is that $\sum_{i \in N} b^i(x)$ is the only strategy dependent factor that influences $\bar{z}(x)$ for any $x \in X$. If all players choose the same location as their source, then only the resource bundle at that location will be available to the grand coalition. We define $\bar{z}^1$ as the (average) vector of shadow prices corresponding to the strategy combination $x \in X$ for which

$S_1(x) = N$, and $\bar{z}^2$ is defined similarly for x such that $S_2(x) = N$. For all remaining strategy combinations, note that each location is chosen by at least one player, so that the corresponding total resource bundle is the sum of l_1 and l_2 . All such strategy combinations then lead to the same (average) price vector, denoted by $\bar{z}^{1,2}$. We formalize this in the following lemma.

Lemma 2.2

Let $\mathcal{L} = (N, \{X^i\}_{i \in N}, l_1, l_2, \{L(x)\}_{x \in X})$ be a BLP-process. Let $x \in X$. Then,

(i)

$$\bar{z}(x) = \begin{cases} \bar{z}^1 & \text{if } S_1(x) = N, \\ \bar{z}^2 & \text{if } S_2(x) = N, \\ \bar{z}^{1,2} & \text{otherwise.} \end{cases}$$

(ii) For all $i \in N$,

$$\pi_i^{\mathcal{L}, Owen}(x) = \begin{cases} (\bar{z}^1)^T \frac{1}{|N|} l_1 & \text{if } S_1(x) = N, \\ (\bar{z}^2)^T \frac{1}{|N|} l_2 & \text{if } S_2(x) = N, \\ (\bar{z}^{1,2})^T \frac{1}{|S_1(x)|} l_1 & \text{if } i \in S_1(x) \text{ and } S_2(x) \neq \emptyset, \\ (\bar{z}^{1,2})^T \frac{1}{|S_2(x)|} l_2 & \text{if } i \in S_2(x) \text{ and } S_1(x) \neq \emptyset. \end{cases}$$

The following example illustrates a BLP-process and its corresponding induced game.

Example 2.1

Consider a BLP-process $\mathcal{L} = (N, \{X^i\}_{i \in N}, l_1, l_2, \{L(x)\}_{x \in X})$ with

$$N = \{1, 2, 3\}, \quad X^i = \{1, 2\} \text{ for all } i \in N, \quad l_1 = \begin{bmatrix} 18 \\ 48 \end{bmatrix} \text{ and } l_2 = \begin{bmatrix} 90 \\ 12 \end{bmatrix},$$

where for any $x \in X$, we have $L(x) = (N, R, P, A, \{b^i(x)\}_{i \in N}, c)$ with

$$R = \{r_1, r_2\}, \quad P = \{p_1, p_2\}, \quad A = \begin{bmatrix} 1 & 4 \\ 2 & 2 \end{bmatrix} \text{ and } c = \begin{bmatrix} 3 \\ 6 \end{bmatrix}.$$

The feasible region is given by

$$F = \{z \in \mathbb{R}^2 \mid z_1 + 2z_2 \geq 3, 4z_1 + 2z_2 \geq 6, z \geq 0\},$$

with extreme points $[0 \ 3]^T$, $[1 \ 1]^T$ and $[3 \ 0]^T$.

Consider $x = (1, 1, 1)$, so that the players (equally) divide only resource bundle l_1 . In this case, the objective function for the dual program of N is given by $z^T \sum_{i \in N} b^i(x) = 18z_1 + 48z_2$

for all $z \in F$. The unique extreme point of the feasible region that minimizes this objective function, is $[3 \ 0]^T$. Since the optimal region $O(x)$ consists of a single vector, we simply get $\bar{z}(x) = \bar{z}^1 = [3 \ 0]^T$. For these shadow prices, the corresponding payoff in the induced strategic game is $\pi_i^{\mathcal{L}, Owen}(x) = (\bar{z}^1)^T b^i(x) = (\bar{z}^1)^T \frac{1}{3} l_1 = [3 \ 0][6 \ 16]^T = 18$ for all $i \in N$.

Similarly, $x = (2, 2, 2)$ leads to $\bar{z}(x) = \bar{z}^2 = [0 \ 3]^T$ and $\pi_i^{\mathcal{L}, Owen}(x) = (\bar{z}^2)^T \frac{1}{3} l_2 = 12$ for all $i \in N$.

For any other strategy combination x , the total resource bundle available to the grand coalition N becomes $l_1 + l_2 = [108 \ 60]^T$. Solving the corresponding minimization problem $\min_{z \in F} 108z_1 + 60z_2$ gives $\bar{z}(x) = \bar{z}^{1,2} = [1 \ 1]^T$. With $x = (1, 1, 2)$, we find

$$\pi^{\mathcal{L}, Owen}(x) = \left([1 \ 1] \begin{bmatrix} \frac{18}{2} \\ \frac{48}{2} \end{bmatrix}, [1 \ 1] \begin{bmatrix} \frac{18}{2} \\ \frac{48}{2} \end{bmatrix}, [1 \ 1] \begin{bmatrix} 90 \\ 12 \end{bmatrix} \right) = (33, 33, 102).$$

For each strategy combination, the resulting payoffs determined by the Owen set of the corresponding LP-process are given in Table 1. For such tables, we always let the row represent the strategy of player 1, where player 2 chooses a column, and the matrix is determined by the choice of player 3.

	1	2		1	2	
1	(18,18,18)	(33,102,33)		1	(33,33,102)	(66,51,51)
2	(102,33,33)	(51,51,66)		2	(51,66,51)	(12,12,12)
	1			2		

Table 1: The strategic game $G^{\mathcal{L}, Owen}$ induced by the BLP-process of Example 2.1

The set of Nash equilibria of the induced game $G^{\mathcal{L}, Owen}$ is

$$E(G^{\mathcal{L}, Owen}) = \{(1, 2, 2), (2, 1, 2), (2, 2, 1)\}. \quad \triangle$$

Next, we show that any strategic game $G^{\mathcal{L}, Owen}$ induced by a BLP-process $\mathcal{L}$, has a Nash equilibrium. We do so using a comprehensive result from Konishi et al. (1997). In particular, they show that any strategic game satisfying four properties has a Nash equilibrium. These properties are presented in the general context of ‘congestion games’, in which players compete by choosing to use a certain facility in a facility set. The first property states that this facility set is finite. Second, ‘independence of irrelevant choices’ requires that the payoff of a player is not affected by any change in the strategy combination, as long as the set of players who choose the same facility as this player is not altered. Third, ‘anonymity’ implies that only the number of players choosing each facility impacts the payoffs, their identities do not. Finally, ‘partial rivalry’ means that if some player chose the same facility as player $i \in N$, but then deviates

to a different facility, this will never decrease the payoff of i . After formalizing the result in Proposition 2.3, we prove Theorem 2.4 by showing that $G^{\mathcal{L}, Owen}$ satisfies all four properties.

Proposition 2.3 [Konishi et al. (1997)]

Let $G = (\{X^i\}_{i \in N}, \{\pi_i\}_{i \in N})$ be a strategic game that satisfies properties (P1), (P2), (P3) and (P4), with

- (P1) there exists a finite set K such that $X^i = K$ for all $i \in N$;
- (P2) $\pi_i(x^j, x^{-j}) = \pi_i(\tilde{x}^j, x^{-j})$ for any $i, j \in N$ and any $x \in X$ and $\tilde{x}^j \in X^j$ such that $x^i \neq x^j$ and $x^i \neq \tilde{x}^j$;
- (P3) $\pi_i(x) = \pi_i(y)$ for any $i \in N$ and any $x, y \in X$ such that $x^i = y^i$ and $|\{j \in N \mid x^j = k\}| = |\{j \in N \mid y^j = k\}|$ for any $k \in K$;
- (P4) $\pi_i(x^j, x^{-j}) \leq \pi_i(\tilde{x}^j, x^{-j})$ for any $i, j \in N$, $i \neq j$, and any $x \in X$ and $\tilde{x}^j \in X^j$ such that $x^i = x^j$ and $x^i \neq \tilde{x}^j$.

Then, $E(G) \neq \emptyset$.

Theorem 2.4

Let $\mathcal{L}$ be a BLP-process and let $G^{\mathcal{L}, Owen}$ be the induced strategic game. Then, $E(G^{\mathcal{L}, Owen}) \neq \emptyset$.

Proof. We prove the theorem by showing that $G^{\mathcal{L}, Owen}$ satisfies (P1)-(P4) from Proposition 2.3. For $G^{\mathcal{L}, Owen}$, we have $X^i = \{1, 2\}$ for all $i \in N$, so (P1) is clearly satisfied.

Next, let $i, j \in N$, $x \in X$ and $\tilde{x}^j \in X^j$ such that $x^i \neq x^j$ and $x^i \neq \tilde{x}^j$. Note that since $X^i = \{1, 2\}$ for all $i \in N$, if $x^i \neq x^j$ and $x^i \neq \tilde{x}^j$, then $x^j = \tilde{x}^j$, meaning (P2) is satisfied as well.

For (P3), let $i \in N$, $x, y \in X$ such that $x^i = y^i$ and $|S_k(x)| = |S_k(y)|$ for all $k \in \{1, 2\}$. Note that $x^i = y^i$ implies that $i \in S_k(x) \Leftrightarrow i \in S_k(y)$ for any $k \in \{1, 2\}$. Using Lemma 2.2(ii), it follows that $\pi_i^{\mathcal{L}, Owen}(x) = \pi_i^{\mathcal{L}, Owen}(y)$.

To show (P4) is satisfied, let $i, j \in N$, $i \neq j$, let $k \in \{1, 2\}$, and let $x \in X$ such that $x^i = x^j = k$ and $\tilde{x}^j \neq k$. Since, in x , at least two distinct players choose k , we must have either $S_k(x) = N$, or $1 < |S_k(x)| < |N|$. For $S_k(x) = N$, we have

$$\pi_i^{\mathcal{L}, Owen}(x^j, x^{-j}) = (\bar{z}^k)^T \frac{1}{|N|} l_k \leq (\bar{z}^{1,2})^T \frac{1}{|N|} l_k \leq (\bar{z}^{1,2})^T \frac{1}{|N| - 1} l_k = \pi_i^{\mathcal{L}, Owen}(\tilde{x}^j, x^{-j}),$$

where the final equality follows from the fact that $|S_k(\tilde{x}^j, x^{-j})| = |S_k(x)| - 1$. We also use this for $1 < |S_k(x)| < |N|$, which yields

$$\pi_i^{\mathcal{L}, Owen}(x^j, x^{-j}) = (\bar{z}^{1,2})^T \frac{1}{|S_k(x)|} l_k \leq (\bar{z}^{1,2})^T \frac{1}{|S_k(x)| - 1} l_k = \pi_i^{\mathcal{L}, Owen}(\tilde{x}^j, x^{-j}).$$

This shows that (P4) is satisfied as well, which completes the proof. $\square$

2.2 BLP-Processes Using an Anonymous Solution

In this subsection, we focus on a modification of Theorem 2.4 using an anonymous one-point game-theoretic solution concept (from now on referred to as an *anonymous solution*) instead of the Owen set to determine the payoffs in the induced strategic game. A game-theoretic solution concept $f : TU^N \rightarrow \mathbb{R}^N$ satisfies anonymity if for every game $v \in TU^N$, any bijection $\sigma : N \rightarrow N$, and all $i \in N$, we have $f_{\sigma(i)}(v) = f_i(v^\sigma)$, where $v^\sigma(T) = v(\sigma(T))$ for all $T \subseteq N \setminus \{\emptyset\}$.² In the context of LP-games, this means that any difference in payoffs between players is explained by a difference in their resource bundles, not by their identities. A direct implication of the anonymity of f is that for any $v \in TU^N$ and all $i, j \in N$ with i and j symmetric in v , i.e., with $v(T \cup \{i\}) = v(T \cup \{j\})$ for any $T \subseteq N \setminus \{i, j\}$, it holds that $f_i(v) = f_j(v)$. Examples of prominent anonymous solutions include the Shapley value (Shapley, 1953) and the nucleolus³ (Schmeidler, 1969).

Given a BLP-process $\mathcal{L} = (N, \{X^i\}_{i \in N}, l_1, l_2, \{L(x)\}_{x \in X})$ and an anonymous solution f , we define the corresponding induced strategic game as $G^{\mathcal{L}, f} = (\{X^i\}_{i \in N}, \{\pi_i^{\mathcal{L}, f}\}_{i \in N})$, where the payoff of player $i \in N$ equals

$$\pi_i^{\mathcal{L}, f}(x) = f_i(v_{L(x)}).$$

Lemma 2.5 states that, in $G^{\mathcal{L}, f}$, for any strategy combination in which two players choose the same location, these players have the same payoff. Further, the effect of unilaterally deviating from this specific location to the other location is the same for each player.

Lemma 2.5

Let $\mathcal{L} = (N, \{X^i\}_{i \in N}, l_1, l_2, \{L(x)\}_{x \in X})$ be a BLP-process, let f be an anonymous solution and let $G^{\mathcal{L}, f} = (\{X^i\}_{i \in N}, \{\pi_i^{\mathcal{L}, f}\}_{i \in N})$ be the induced strategic game. Then,

- (i) $\pi_i^{\mathcal{L}, f}(x) = \pi_j^{\mathcal{L}, f}(x)$ for any $i, j \in N$ and any $x \in X$ such that $x^i = x^j$.
- (ii) $\pi_i^{\mathcal{L}, f}(\tilde{x}^i, x^{-i}) = \pi_j^{\mathcal{L}, f}(\tilde{x}^j, x^{-j})$ for any $i, j \in N$, any $x \in X$, and any $\tilde{x}^i \in X^i$ and $\tilde{x}^j \in X^j$ such that $x^i = x^j$, $\tilde{x}^i = \tilde{x}^j$ and $x^i \neq \tilde{x}^i$.

Proof.

- (i) Let $i, j \in N$ and $x \in X$ such that $x^i = x^j$. Since this implies that $b^i(x) = b^j(x)$, it readily follows that i and j are symmetric in $v_{L(x)}$, so that $\pi_i^{\mathcal{L}, f}(x) = \pi_j^{\mathcal{L}, f}(x)$ by anonymity of f .

²Here, $\sigma(i) = j$ implies player j in v is ‘named’ i in v^σ , and $\sigma(T)$ is the set of players in v corresponding to coalition T in v^σ , i.e., $\sigma(T) = \{j \in N \mid \exists i \in T \text{ with } \sigma(i) = j\}$.

³Although the nucleolus is formally not a solution on TU^N , it is well-defined for any LP-game.

(ii) Let $i, j \in N$, $x \in X$, $\tilde{x}^i \in X^i$ and $\tilde{x}^j \in X^j$ such that $x^i = x^j$, $\tilde{x}^i = \tilde{x}^j$ and $x^i \neq \tilde{x}^i$. Then, $b^i(\tilde{x}^i, x^{-i}) = b^j(\tilde{x}^j, x^{-j})$ and $b^i(\tilde{x}^j, x^{-j}) = b^j(\tilde{x}^i, x^{-i})$. Consider the LP-games $v_{L(\tilde{x}^i, x^{-i})}$ and $v_{L(\tilde{x}^j, x^{-j})}$, and consider the bijection $\sigma : N \rightarrow N$ such that $\sigma(i) = j$, $\sigma(j) = i$, and $\sigma(k) = k$ for all $k \in N \setminus \{i, j\}$. Note that $v_{L(\tilde{x}^i, x^{-i})}^\sigma = v_{L(\tilde{x}^j, x^{-j})}$. Hence, applying anonymity of f (in the second equality), we get

$$f_i(v_{L(\tilde{x}^i, x^{-i})}) = f_{\sigma(j)}(v_{L(\tilde{x}^i, x^{-i})}) = f_j(v_{L(\tilde{x}^i, x^{-i})}^\sigma) = f_j(v_{L(\tilde{x}^j, x^{-j})}). \quad \square$$

In particular, this lemma implies that if a player $i \in N$ can deviate profitably in $G^{\mathcal{L},f}$ from some $x \in X$, then all players $j \in N$ with $x^i = x^j$ can deviate profitably.

Corollary 2.6

Let $\mathcal{L} = (N, \{X^i\}_{i \in N}, l_1, l_2, \{L(x)\}_{x \in X})$ be a BLP-process, let f be an anonymous solution and let $G^{\mathcal{L},f} = (\{X^i\}_{i \in N}, \{\pi_i^{\mathcal{L},f}\}_{i \in N})$ be the induced strategic game. Let $i, j \in N$, $x \in X$, $\tilde{x}^i \in X^i$ and $\tilde{x}^j \in X^j$ such that $x^i = x^j$ and $\tilde{x}^i = \tilde{x}^j$. If $\pi_i^{\mathcal{L},f}(x) < \pi_i^{\mathcal{L},f}(\tilde{x}^i, x^{-i})$, then $\pi_j^{\mathcal{L},f}(x) < \pi_j^{\mathcal{L},f}(\tilde{x}^j, x^{-j})$.

The following example illustrates that the Nash equilibria of $G^{\mathcal{L},Owen}$ and $G^{\mathcal{L},f}$ induced by the same BLP-process $\mathcal{L}$ need not coincide. Hence, the existence of a Nash equilibrium in $G^{\mathcal{L},f}$ does not follow from Theorem 2.4. However, using Corollary 2.6, its existence is still guaranteed, as formalized in Theorem 2.7.

Example 2.2

Consider a BLP-process $\mathcal{L} = (N, \{X^i\}_{i \in N}, l_1, l_2, \{L(x)\}_{x \in X})$ with

$$N = \{1, 2\}, \quad X^1 = X^2 = \{1, 2\}, \quad l_1 = \begin{bmatrix} 100 \\ 200 \end{bmatrix} \quad \text{and} \quad l_2 = \begin{bmatrix} 300 \\ 50 \end{bmatrix},$$

where for any $x \in X$, we have $L(x) = (N, R, P, A, \{b^i(x)\}_{i \in N}, c)$ with

$$R = \{r_1, r_2\}, \quad P = \{p_1, p_2\}, \quad A = \begin{bmatrix} 5 & 5 \\ 6 & 6 \end{bmatrix} \quad \text{and} \quad c = \begin{bmatrix} 9 \\ 9 \end{bmatrix}.$$

The feasible region F has two extreme points, leading to $\bar{z}^1 = [9/5 \ 0]^T$ and $\bar{z}^2 = \bar{z}^{1,2} = [0 \ 3/2]^T$.

The LP-games corresponding to each strategy combination are given in Table 2.

T	$\{1\}$	$\{2\}$	N
$v_{L(1,1)}(T)$	90	90	180
$v_{L(1,2)}(T)$	180	75	375
$v_{L(2,1)}(T)$	75	180	375
$v_{L(2,2)}(T)$	37.5	37.5	75

Table 2: The LP-game $v_{L(x)}$ for each $x \in X$ in Example 2.2

The induced strategic game $G^{\mathcal{L}, Owen}$, for which the payoffs can be calculated using Lemma 2.2(ii), is depicted in Table 3.

	1	2
1	(90,90)	(300,75)
2	(75,300)	(37.5,37.5)

Table 3: The strategic game $G^{\mathcal{L}, Owen}$ induced by the BLP-process of Example 2.2

To derive $G^{\mathcal{L}, \Phi}$, given in Table 4, the Shapley value $\Phi(v_{L(x)})$ can be straightforwardly calculated using the fact that for any $v \in TU^N$ with $N = \{1, 2\}$, we have $\Phi_i(v) = v(\{i\}) + \frac{v(N) - v(\{1\}) - v(\{2\})}{2}$ for any $i \in N$.

	1	2
1	(90,90)	(240,135)
2	(135,240)	(37.5,37.5)

Table 4: The strategic game $G^{\mathcal{L}, \Phi}$ induced by the BLP-process of Example 2.2

Note that $E(G^{\mathcal{L}, Owen}) = \{(1, 1)\}$, whereas $E(G^{\mathcal{L}, \Phi}) = \{(1, 2), (2, 1)\}$. $\triangle$

Theorem 2.7

Let $\mathcal{L}$ be a BLP-process, let f be an anonymous solution and let $G^{\mathcal{L}, f}$ be the induced strategic game. Then, $E(G^{\mathcal{L}, f}) \neq \emptyset$.

Proof. Denote $\mathcal{L} = (N, \{X^i\}_{i \in N}, l_1, l_2, \{L(x)\}_{x \in X})$ and suppose $E(G^{\mathcal{L}, f}) = \emptyset$. Set $x^0 = (1, 1, \dots, 1)$ and $x^{|N|} = (2, 2, \dots, 2)$. The proof is based on the following idea: using the fact that there are no equilibria, and starting from strategy x^0 , we show that for all strategy combinations $x \in X$ with $S_1(x) \neq \emptyset$ it is beneficial to unilaterally deviate for all players in $S_1(x)$, but not for the players in $S_2(x)$. The argumentation is recursive in the sense that we consider strategy combinations in a stepwise manner, where the number of players who choose location 2 increases by one in each step. This reasoning, however, would lead to the conclusion that $x^{|N|}$ is a Nash equilibrium, yielding a contraction.

As $\mathcal{L}$ and f are fixed, we use π instead of $\pi^{\mathcal{L},f}$ in the proof. Since $x^0 \notin E(G^{\mathcal{L},f})$, Corollary 2.6 implies that

$$\pi_i(x^1) > \pi_i(x^0) \quad (1)$$

for all $x^1 \in X$ with $|S_2(x^1)| = 1$ and $i \in S_2(x^1)$. For an arbitrary $x^1 \in X$ with $|S_2(x^1)| = 1$, (1) clearly implies that there is no profitable unilateral deviation for the player in $S_2(x^1)$. Since $E(G^{\mathcal{L},f}) = \emptyset$, there must therefore be a strictly profitable deviation for a player in $S_1(x^1)$. Corollary 2.6 then implies that

$$\pi_i(x^2) > \pi_i(x^1)$$

for all $x^1, x^2 \in X$ with $|S_2(x^1)| = 1$, $|S_2(x^2)| = 2$, $S_2(x^1) \subset S_2(x^2)$ and $i \in S_2(x^2) \setminus S_2(x^1)$. Note that i is the player who unilaterally deviates from x^1 by choosing location 2 instead of 1.

Repeating a similar argument another $|N| - 2$ times, we find that

$$\pi_i(x^{|N|}) > \pi_i(x^{|N|-1}) \quad (2)$$

for all $x^{|N|-1} \in X$ with $|S_2(x^{|N|-1})| = |N| - 1$ and $i \in N \setminus S_2(x^{|N|-1})$.

However, (2) would imply that $x^{|N|} = (2, 2, \dots, 2) \in E(G^{\mathcal{L},f})$, a contradiction. $\square$

For the sake of completeness, we remark that a proof along similar lines as the proof of Theorem 2.7 can also be used to prove Theorem 2.4. However, the result of Konishi et al. (1997) as stated in Proposition 2.3, used to prove Theorem 2.4, *cannot*⁴ be used to prove Theorem 2.7.

2.3 Discussion

In this section, we analyzed biform linear production processes, in which the resource bundles of players are the result of strategic choices. For a BLP-process with two locations at which players can compete for resources, we guarantee the existence of a Nash equilibrium using a solution concept based on the Owen set or using any anonymous one-point game-theoretic solution concept for the associated LP-game.

In BLP-processes, we use an equal bundle splitting rule, where players who choose the same

⁴In particular, property (P4) need not be satisfied. Consider the BLP-process from Example 2.2, using the anonymous solution f given by $f_i(v_{L(x)}) = \frac{3}{|N|}v_{L(x)}(N) - 2\Phi_i(v_{L(x)})$ for all $x \in X$ and all $i \in N$. This would yield $\pi_1^{\mathcal{L},f}(1, 1) = 90 > 82.5 = \pi_1^{\mathcal{L},f}(1, 2)$.

source all get an equal fraction of the resource bundle. This rule is contingent on the firms holding equal amounts of sway over the sources. In practice, power dynamics may be such that one firm will obtain a larger part of the resource bundle if firms are forced to compete at a source. Using an alternative splitting rule, where each firm gets some non-negative fraction of the resource bundle that is not necessarily equal to the fraction of the other players, affects the results. This effect does not yet come into play when there are two players: when using an unequal bundle splitting rule, Theorem 2.4 can still be generalized for $|N| = 2$. However, allowing for arbitrary ways of splitting, the set of Nash equilibria of the induced strategic game can become empty for $|N| > 2$.

Alternatively, maintaining an equal bundle splitting rule, one can generalize the BLP-process using a model with three locations. Even for $|N| = 2$, the set of Nash equilibria of the induced strategic game can become empty.

3 Biform Sequencing Processes

In this section, we introduce a strategic component to sequencing processes. We first recall the definition of a standard (cooperative) sequencing process, where the initial order is given. Such a process will form the second stage of a biform sequencing process, at which we arrive after the initial order is strategically determined in the first stage.

3.1 Sequencing Processes

A sequencing process is summarized by the tuple

$$S = (N, \sigma_0, p, \alpha).$$

Here, N is the finite player set, where each player represents a job (these two terms will be used interchangeably). From the start, all jobs are lined up to be processed sequentially on a single machine. The initial processing order is denoted by σ_0 , where any order is described by a bijection $\sigma : \{1, 2, \dots, |N|\} \rightarrow N$ and the collection of all such orders is denoted by $\Pi(N)$. In particular, $\sigma(k) = i$ indicates that job $i \in N$ is on the k -th place in the processing order. The vectors $p = (p_i)_{i \in N} > 0$ and $\alpha = (\alpha_i)_{i \in N} > 0$ denote the processing times and cost parameters of the players, respectively. The cost parameter of a player determines the linear relationship between this player's costs and the completion time of the corresponding job.

Let $S = (N, \sigma_0, p, \alpha)$ be a sequencing process. Then, for any processing order $\sigma \in \Pi(N)$,

the completion time of a job $i \in N$ is denoted by $C_i(\sigma)$, with

$$C_i(\sigma) = \sum_{j \in N: \sigma^{-1}(j) \leq \sigma^{-1}(i)} p_j.$$

The corresponding individual costs of player i w.r.t. σ are given by $\alpha_i C_i(\sigma)$.

Clearly, an individual player's costs are lower when this player is closer to the head of the queue in the initial order, since fewer players are processed before this player then. To determine an optimal order, however, we are interested in minimizing the total costs $\sum_{i \in N} \alpha_i C_i(\sigma)$ of the set of jobs as a whole over all orders $\sigma \in \Pi(N)$. Intuitively, it is clear that jobs with a high cost parameter should be processed before those with low cost parameter, unless the processing time of the former is substantially higher than that of the latter. This creates a measure of urgency for each player, denoted by $u = (u_i)_{i \in N}$, where the urgency of player $i \in N$ is defined as $u_i = \frac{\alpha_i}{p_i}$. Smith (1956) proved that any optimal processing order $\hat{\sigma} \in \Pi(N)$ is such that the players are ordered in weakly decreasing order of urgency, i.e., for all $k \in \{1, 2, \dots, |N| - 1\}$, we have

$$\frac{\alpha_{\hat{\sigma}(k)}}{p_{\hat{\sigma}(k)}} \geq \frac{\alpha_{\hat{\sigma}(k+1)}}{p_{\hat{\sigma}(k+1)}}.$$

The set of misplaced pairs $MP(\sigma_0)$ of the given initial order $\sigma_0 \in \Pi(N)$ comprises all pairs of players who are not ordered according to a Smith order, formally defined as

$$MP(\sigma_0) = \{(i, j) \in N \times N \mid \sigma_0^{-1}(i) < \sigma_0^{-1}(j) \text{ and } u_i < u_j\}.$$

By rearranging all misplaced pairs, one arrives at a Smith order. If two neighboring players $i, j \in N$ with $\sigma_0^{-1}(i) = \sigma_0^{-1}(j) - 1$ are misplaced, i.e., $(i, j) \in MP(\sigma_0)$, these players can save costs by switching places. Specifically, the cost savings of such a pair of neighboring players are

$$g_{ij} = \alpha_j p_i - \alpha_i p_j > 0.$$

Importantly, note that these cost savings are independent of the exact position of the neighbors in the queue.

The maximal cost savings the grand coalition can make, are achieved by rearranging the initial order by consecutively switching misplaced neighbor pairs until the jobs are arranged in a Smith order. The corresponding maximal cost savings obtained by a Smith order $\hat{\sigma} \in \Pi(N)$ therefore equal

$$\sum_{i \in N} \alpha_i (C_i(\sigma_0) - C_i(\hat{\sigma})) = \sum_{(i, j) \in MP(\sigma_0)} g_{ij}.$$

Next, we need to determine how to allocate these cost savings over the players. For this, we use so-called gain splitting rules. The concept of such a rule is that whenever a pair of neighboring players makes a gain by switching places, this gain is only divided among these players. Curiel et al. (1989b) propose the Equal Gain Splitting (*EGS*) rule, that divides each such gain equally over the two players involved. Now, we consider gain splitting rules GS^λ defined by⁵

$$GS^\lambda(S) = \sum_{(i,j) \in MP(\sigma_0)} (\lambda_{ij}e^{\{i\}} + (1 - \lambda_{ij})e^{\{j\}})g_{ij}$$

for any $\lambda \in \Lambda$. Here, $\Lambda = \{\lambda : N \times N \rightarrow [0, 1] \mid \lambda(r, s) + \lambda(s, r) = 1 \text{ for all } r, s \in N, r \neq s, \text{ and } \lambda(r, r) = 1\}$. With minor abuse of notation, we write λ_{ij} instead of $\lambda(i, j)$. For the equal gain splitting rule we have $EGS(S) = GS^\lambda(S)$ for $\lambda \in \Lambda$ such that $\lambda_{rs} = \frac{1}{2}$ for all $r, s \in N, r \neq s$. The set $\{GS^\lambda(S) \mid \lambda \in \Lambda\}$ of all allocation vectors corresponding to a gain splitting rule is called the split core of S in Hamers et al. (1996).

For an arbitrary gain splitting rule GS^λ , the corresponding net profit of player $i \in N$ in the sequencing process $S = (N, \sigma_0, p, \alpha)$ is defined by

$$\pi_i^\lambda(S) = GS_i^\lambda(S) - IC_i(S),$$

where the initial individual costs $IC_i(S)$ of player i w.r.t. σ_0 are given by $IC_i(S) = \alpha_i C_i(\sigma_0)$.

Example 3.1

Let $N = \{1, 2, 3\}$, $p = (4, 3, 4)$ and $\alpha = (2, 3, 6)$. In this example, we use the equal gain splitting rule and we determine the net profit vectors $\pi^{EGS}(N, \sigma_0, p, \alpha)$ corresponding to each of the six possible initial orders. First, note that there is a unique Smith order, given by $\hat{\sigma} = (3, 2, 1)$. Next, consider $\sigma_0 = (1, 2, 3)$. Then, the initial cost vector is given by $(8, 21, 66)$. There are three misplaced pairs, $(1, 2)$, $(1, 3)$ and $(2, 3)$, with $g_{12} = g_{23} = 6$ and $g_{13} = 16$, so $EGS(N, (1, 2, 3), p, \alpha) = (11, 6, 11)$. Hence,

$$\pi^{EGS}(N, (1, 2, 3), p, \alpha) = (11, 6, 11) - (8, 21, 66) = (3, -15, -55).$$

Similarly, we get

$$\begin{aligned} \pi^{EGS}(N, (1, 3, 2), p, \alpha) &= (11, 3, 8) - (8, 33, 48) = (3, -30, -40), \\ \pi^{EGS}(N, (2, 1, 3), p, \alpha) &= (8, 3, 11) - (14, 9, 66) = (-6, -6, -55), \\ \pi^{EGS}(N, (2, 3, 1), p, \alpha) &= (0, 3, 3) - (22, 9, 42) = (-22, -6, -39), \\ \pi^{EGS}(N, (3, 1, 2), p, \alpha) &= (3, 3, 0) - (16, 33, 24) = (-13, -30, -24), \\ \pi^{EGS}(N, (3, 2, 1), p, \alpha) &= (0, 0, 0) - (22, 21, 24) = (-22, -21, -24). \end{aligned}$$

⁵For any $i \in N$, $e^{\{i\}}$ denotes the unit vector of size $|N|$ of which the i -th entry is 1.

The net profit vectors clearly depend on the initial order. In particular, note that if a player $i \in N$ is closer to the head of the queue in the initial processing order, this leads to a strict improvement of the net profit of i . $\triangle$

For any player $i \in N$, being placed one position closer to the head of the queue in the initial order $\sigma_0 \in \Pi(N)$ is always strictly better than achieving this by switching positions afterwards, if the gains of such position switches are allocated using a gain splitting rule. When getting an earlier position in σ_0 , the full initial individual cost decrease of player i is kept by i , and the (strictly positive) cost increase of the player who is then processed later is not taken into account. Using a gain splitting rule to divide the cost savings in the sequencing process, however, the gain is first decreased to account for the increased cost of this other player in the switch, and then divided over the two players. Further, note that rearranging the players in front of i in σ_0 does not influence i 's completion time or payoff. It follows that if any (set of) player(s) is 'removed' from the set of players before i in σ_0 , the payoff of i increases, as formalized in Lemma 3.1.

Lemma 3.1

Let $S = (N, \sigma_0, p, \alpha)$ and $\tilde{S} = (N, \tilde{\sigma}_0, p, \alpha)$ be sequencing processes and let $i \in N$ be such that⁶ $P(\sigma_0, i) \subsetneq P(\tilde{\sigma}_0, i)$. Then, for all $\lambda \in \Lambda$, we have

$$\pi_i^\lambda(S) > \pi_i^\lambda(\tilde{S}).^7$$

Proof. Starting from $\tilde{\sigma}_0$, note that σ_0 can be reached by first adequately rearranging i 's predecessors, then i 's successors (i.e., all players for which i is a predecessor) and finally by some switches between i and its neighboring predecessor. All initial orders $\sigma'_0 \in \Pi(N)$ derived from $\tilde{\sigma}_0$ by using rearrangements of the first two types will lead to a corresponding sequencing process S' with the same net payoff $\pi_i^\lambda(S') = \pi_i^\lambda(\tilde{S})$, since $C_i(\sigma'_0) = C_i(\tilde{\sigma}_0)$, and $(i, j) \in MP(\sigma'_0) \Leftrightarrow (i, j) \in MP(\tilde{\sigma}_0)$ and $(j, i) \in MP(\sigma'_0) \Leftrightarrow (j, i) \in MP(\tilde{\sigma}_0)$ for any $j \in N \setminus \{i\}$. It therefore suffices to show that any sequencing process for which the initial order is derived from switches of the third and final type, leads to a strictly higher net profit of i . So, we can

⁶The set of predecessors $P(\sigma, i)$ of player $i \in N$ with respect to $\sigma \in \Pi(N)$ is defined by $P(\sigma, i) = \{j \in N \mid \sigma^{-1}(j) < \sigma^{-1}(i)\}$.

⁷This result holds specifically for payoff vectors defined using a gain splitting rule. If instead we allocate the cost savings using the Shapley value of the corresponding cooperative sequencing game, it can happen that a player's net profit becomes lower if the player 'moves' towards the head of the queue in the initial processing order. Intuitively, the reason why this can happen, is that when e.g. $\sigma_0 = (1, 2, 3)$, player 2 needs to cooperate (i.e., be a member of the coalition) to make switching positions admissible for players 1 and 3 in the sequencing game. Because player 2 is needed to 'enable' the gain g_{13} , player 2 gets a third of this gain in the Shapley value. If this gain is relatively very high, it can outweigh the decrease in the initial costs of player 2 for $\sigma_0 = (2, 1, 3)$. Such an effect does not occur if the gains are allocated using a gain splitting rule.

restrict to the situation where $\sigma_0, \tilde{\sigma}_0$ and i are such that for some $l \in \{1, \dots, |N| - 1\}$ and $j \in N \setminus \{i\}$, we have $\sigma_0(l) = \tilde{\sigma}_0(l + 1) = i$, $\sigma_0(l + 1) = \tilde{\sigma}_0(l) = j$, and $\tilde{\sigma}_0(k) = \sigma_0(k)$ for all $k \in \{1, \dots, |N|\} \setminus \{l, l + 1\}$.

First, if $u_i < u_j$, then $MP(\sigma_0) = MP(\tilde{\sigma}_0) \cup \{(i, j)\}$ and, consequently,

$$\begin{aligned} \pi_i^\lambda(S) - \pi_i^\lambda(\tilde{S}) &= GS_i^\lambda(S) - IC_i(S) - GS_i^\lambda(\tilde{S}) + IC_i(\tilde{S}) \\ &= \alpha_i (C_i(\tilde{\sigma}_0) - C_i(\sigma_0)) + \lambda_{ij} g_{ij} \\ &\geq \alpha_i (C_i(\tilde{\sigma}_0) - C_i(\sigma_0)) \\ &> 0. \end{aligned}$$

Next, for $u_i = u_j$, note that $MP(\sigma_0) = MP(\tilde{\sigma}_0)$ and $GS_i^\lambda(S) = GS_i^\lambda(\tilde{S})$, so that

$$\pi_i^\lambda(S) - \pi_i^\lambda(\tilde{S}) = GS_i^\lambda(S) - IC_i(S) - GS_i^\lambda(\tilde{S}) + IC_i(\tilde{S}) = \alpha_i (C_i(\tilde{\sigma}_0) - C_i(\sigma_0)) > 0.$$

Finally, for $u_j < u_i$, $MP(\tilde{\sigma}_0) = MP(\sigma_0) \cup \{(j, i)\}$ and, consequently,

$$\begin{aligned} \pi_i^\lambda(S) - \pi_i^\lambda(\tilde{S}) &= GS_i^\lambda(S) - IC_i(S) - GS_i^\lambda(\tilde{S}) + IC_i(\tilde{S}) \\ &= \alpha_i (C_i(\tilde{\sigma}_0) - C_i(\sigma_0)) - (1 - \lambda_{ji}) g_{ji} \\ &= \alpha_i (C_i(\tilde{\sigma}_0) - C_i(\sigma_0)) - (1 - \lambda_{ji}) (\alpha_i p_j - \alpha_j p_i) \\ &= \alpha_i (C_i(\tilde{\sigma}_0) - C_i(\sigma_0)) - (1 - \lambda_{ji}) (\alpha_i (C_i(\tilde{\sigma}_0) - C_i(\sigma_0)) - \alpha_j p_i) \\ &= \lambda_{ji} \alpha_i (C_i(\tilde{\sigma}_0) - C_i(\sigma_0)) + (1 - \lambda_{ji}) \alpha_j p_i \\ &\geq \lambda_{ji} \alpha_i (C_i(\tilde{\sigma}_0) - C_i(\sigma_0)) \\ &\geq 0. \end{aligned}$$

Note that the first inequality is strict unless $\lambda_{ji} = 1$, where the second inequality is strict unless $\lambda_{ji} = 0$. Hence, we may conclude that $\pi_i^\lambda(S) - \pi_i^\lambda(\tilde{S}) > 0$. $\square$

3.2 Biform Sequencing Processes

Our model introduces a strategic element to the sequencing processes described in the previous section. We no longer assume an initial order is given. Instead, strategic individual choices for a position in the first stage of the biform sequencing (BS) process determine an initial order that is used in the cooperative sequencing process of the second stage. The finite strategy set of any player $i \in N$ is given by $X^i = \{1, 2, \dots, |N|\}$. For example, $x^i = 1$ indicates that player i requests to be processed first.

As it can happen that several players request to be in the same position, we need a tie-breaking rule. We interpret a tie-breaking rule as a decision made by some entity that determines the initial order for any given strategy combination. This entity first simply assigns every position that is requested by exactly one player to this player. An unassigned position is called *empty* if it is not requested by any player and called *undecided* if it is requested by at least two players. Then, recursively, starting from the earliest undecided position, all players who requested this position are assigned to either this position or an empty one. At every step in the recursion, the players are assigned such that a player with a higher ‘priority’ is assigned to an earlier position among the empty ones and the requested one. This recursive process is illustrated in Example 3.2.

Given a strategy combination $x \in X$ and tie-breaking rule $\tau : X \rightarrow \Pi(N)$, the strategically determined initial processing order is denoted by $\sigma_0^\tau(x)$. We assume a tie-breaking rule τ is *priority order based* (POB). This means that the way players within a subgroup are prioritized can be directly induced from the unique *priority order* $\bar{\sigma}_0^\tau$ on all players, needed when all players would request exactly the same position.

Example 3.2

Consider a sequencing process with $N = \{1, 2, 3, 4\}$, $\alpha = (2, 4, 4, 3)$ and $p = (2, 1, 2, 1)$, so that $u = (1, 4, 2, 3)$. An initial order will be strategically determined using the priority order based tie-breaking rule τ , fully determined by the priority order $\bar{\sigma}_0^\tau = (2, 4, 3, 1)$. Note that this priority order fits with the HUCF-principle: highest urgency comes first. By definition, this means that if e.g. $x = (2, 2, 2, 2)$, then $\sigma_0^\tau(x) = (2, 4, 3, 1)$.

If $x = (2, 1, 3, 1)$, the first step is to assign players to the positions that were requested exactly once. After assigning position 2 to player 1 and position 3 to player 3, τ is used to break the tie between players 2 and 4, who compete for the undecided position 1 and the empty position 4. Player 2 has priority over player 4, which leads to $\sigma_0^\tau(x) = (2, 1, 3, 4)$.

If $x = (1, 1, 2, 2)$, τ is first used to break the tie between players 1 and 2, who can be assigned to undecided position 1 and empty positions 3 and 4. Since player 2 has the highest priority, we get $\sigma_0^\tau(x)(1) = 2$, after which player 1 is placed in the best remaining available position, $\sigma_0^\tau(x)(3) = 1$. The next undecided position is position 2, with position 4 empty. Since player 4 has priority over player 3, player 4 is assigned to position 2 and player 3 to position 4. Hence, $\sigma_0^\tau(x) = (2, 4, 1, 3)$.

If $x = (3, 3, 4, 4)$, players 1 and 2 first compete for empty positions 1 and 2 and undecided

position 3. After assigning 2 to position 1 and 1 to position 2, players 3 and 4 compete for empty position 3 and undecided position 4. As a consequence, $\sigma_0^\tau(x) = (2, 1, 4, 3)$. $\triangle$

Now we are able to formally define a biform sequencing process and a corresponding induced strategic game based on a gain splitting rule GS^λ .

Definition 3.2

A biform sequencing (BS) process is a tuple

$$\mathcal{S} = (N, \{X^i\}_{i \in N}, \tau, \{S(x)\}_{x \in X}),$$

in which for all $i \in N$ we have $X^i = \{1, 2, \dots, |N|\}$, $\tau : X \rightarrow \Pi(N)$ is a POB tie-breaking rule, and for any $x \in X$,

$$S(x) = (N, \sigma_0^\tau(x), p, \alpha)$$

is a corresponding sequencing process with initial order $\sigma_0^\tau(x)$. Given such a BS-process $\mathcal{S}$ and a gain splitting rule GS^λ with $\lambda \in \Lambda$, the corresponding induced strategic game is given by $G^{\mathcal{S}, \lambda} = (\{X^i\}_{i \in N}, \{\pi_i^{\mathcal{S}, \lambda}\}_{i \in N})$, where for any $x \in X$ and all $i \in N$ we set

$$\pi_i^{\mathcal{S}, \lambda}(x) = \pi_i^\lambda(N, \sigma_0^\tau(x), p, \alpha).^8$$

Example 3.3

Reconsider the 3-player sequencing process of Example 3.1. We now consider the BS-process $\mathcal{S} = (\{1, 2, 3\}, \{X^i\}_{i \in N}, \tau, \{S(x)\}_{x \in X})$ using the POB tie-breaking rule τ with priority order $\bar{\sigma}_0^\tau = (2, 3, 1)$. For all $x \in X$, $\sigma_0^\tau(x)$ is presented in Table 5.

	1	2	3		1	2	3		1	2	3
1	(2,3,1)	(3,2,1)	(3,1,2)	1	(2,3,1)	(1,2,3)	(1,3,2)	1	(2,1,3)	(1,2,3)	(1,2,3)
2	(2,1,3)	(3,2,1)	(3,1,2)	2	(2,3,1)	(2,3,1)	(3,1,2)	2	(2,1,3)	(2,1,3)	(2,1,3)
3	(2,3,1)	(3,2,1)	(3,2,1)	3	(2,3,1)	(2,3,1)	(2,3,1)	3	(2,3,1)	(3,2,1)	(2,3,1)
	1				2				3		

Table 5: The strategy dependent initial processing orders $\sigma_0^\tau(x)$ in Example 3.3

Next, we analyze the induced strategic game $G^{\mathcal{S}, EGS} = (\{X^i\}_{i \in \{1,2,3\}}, \{\pi_i^{\mathcal{S}, EGS}\}_{i \in \{1,2,3\}})$ based on the equal gain splitting rule. For each of the six possible initial orders, the corresponding net profits are given in Example 3.1. The induced game $G^{\mathcal{S}, EGS}$ is given in Table 6.

⁸Note that if $x, y \in X$ are such that $\sigma_0^\tau(x) = \sigma_0^\tau(y)$, then $\pi_i^{\mathcal{S}, \lambda}(x) = \pi_i^{\mathcal{S}, \lambda}(y)$ for all $i \in N$.

	1	2	3		1	2	3
1	(-22,-6,-39)	(-22,-21,-24)	(-13,-30,-24)	1	(-22,-6,-39)	(3,-15,-55)	(3,-30,-40)
2	(-6,-6,-55)	(-22,-21,-24)	(-13,-30,-24)	2	(-22,-6,-39)	(-22,-6,-39)	(-13,-30,-24)
3	(-22,-6,-39)	(-22,-21,-24)	(-22,-21,-24)	3	(-22,-6,-39)	(-22,-6,-39)	(-22,-6,-39)
		1				2	
		1	2			3	
		1	(-6,-6,-55)	(3,-15,-55)	(3,-15,-55)		
		2	(-6,-6,-55)	(-6,-6,-55)	(-6,-6,-55)		
		3	(-22,-6,-39)	(-22,-21,-24)	(-22,-6,-39)		
				3			

Table 6: The induced strategic game $G^{S,EGS}$ in Example 3.3

It follows that

$$E(G^{S,EGS}) = \{(1, 1, 2), (2, 1, 2), (3, 1, 2)\}.$$

Note that for all $x \in E(G^{S,EGS})$ we have $\sigma_0^\tau(x) = \bar{\sigma}_0^\tau = (2, 3, 1)$, but there are (many) $x \in X$ such that $\sigma_0^\tau(x) = (2, 3, 1)$ with $x \notin E(G^{S,EGS})$. The equilibrium $(3, 1, 2)$ is quite special, since it corresponds to the strategy combination in which all players request the position they are entitled to according to the priority order $\bar{\sigma}_0^\tau$. $\triangle$

The ‘special’ type of equilibrium found in Example 3.3 exists for any BS-process.

Theorem 3.3

Let $\mathcal{S} = (N, \{X^i\}_{i \in N}, \tau, \{S(x)\}_{x \in X})$ be a BS-process, let GS^λ be a gain splitting rule with $\lambda \in \Lambda$, and let $G^{S,\lambda} = (\{X^i\}_{i \in N}, \{\pi_i^{S,\lambda}\}_{i \in N})$ be the induced strategic game. Let $x \in X$ be such that $x^i = (\bar{\sigma}_0^\tau)^{-1}(i)$ for all $i \in N$. Then, $x \in E(G^{S,\lambda})$.

Proof. Let $i \in N$ and set $k = (\bar{\sigma}_0^\tau)^{-1}(i)$, so that $x^i = k$. Consider $\tilde{x}^i = l$ with $l \neq k$. With respect to the strategy combination $(\tilde{x}^i, x^{-i})$, only position l is undecided and only position k is empty. In particular, position l is requested by players i and $j = \bar{\sigma}_0^\tau(l)$. If $l < k$, the underlying priority order ranks j before i , so j is assigned to the earlier (undecided) position l , while i is assigned to the later (empty) position k . If $l > k$, the underlying priority order ranks i before j , so i is assigned to the earlier position k and j to the later position l . In both cases, $\sigma_0^\tau(\tilde{x}^i, x^{-i}) = \sigma_0^\tau(x)$ and therefore $\pi_i^{S,\lambda}(\tilde{x}^i, x^{-i}) = \pi_i^{S,\lambda}(x)$. Consequently, $x \in E(G)$. $\square$

Next, we show that any induced strategic game $G^{S,\lambda}$ has exactly $|N|$ Nash equilibria. In particular, these equilibria are such that the player with the lowest priority (as determined by the underlying priority order) can request any position. All other players should request the

position they are entitled to according to the priority order.

Theorem 3.4

Let $\mathcal{S} = (N, \{X^i\}_{i \in N}, \tau, \{S(x)\}_{x \in X})$ be a BS-process, let GS^λ be a gain splitting rule with $\lambda \in \Lambda$, and let $G^{\mathcal{S}, \lambda} = (\{X^i\}_{i \in N}, \{\pi_i^{\mathcal{S}, \lambda}\}_{i \in N})$ be the induced strategic game. Then, $|E(G^{\mathcal{S}, \lambda})| = |N|$.

Proof. For ease of notation and without loss of generality, let $N = \{1, 2, \dots, n\}$ and let the priority order underlying the tie-breaking rule τ be given by $\bar{\sigma}_0^\tau = (1, 2, \dots, n)$, meaning the players are ‘numbered’ in decreasing order of priority. It suffices to prove the following two claims. Claim 1 shows there are at least n equilibria, and Claim 2 shows there are at most n equilibria.

Claim 1 Let $x \in X$ be such that $x^j = j$ for all $j \in \{1, 2, \dots, n-1\}$. Then, $x \in E(G^{\mathcal{S}, \lambda})$.

Claim 2 Let $x \in E(G^{\mathcal{S}, \lambda})$. Then, $x^j = j$ for all $j \in \{1, 2, \dots, n-1\}$.

Proof Claim 1 Let $i \in N$ and $\tilde{x}^i \in X^i$. We will show that either $\sigma_0^\tau(\tilde{x}^i, x^{-i}) = \sigma_0^\tau(x) = \bar{\sigma}_0^\tau$, or $P(\sigma_0^\tau(x), i) \subsetneq P(\sigma_0^\tau(\tilde{x}^i, x^{-i}), i)$. In both cases, this implies that $\pi_i^{\mathcal{S}, \lambda}(x) \geq \pi_i^{\mathcal{S}, \lambda}(\tilde{x}^i, x^{-i})$. Note that in the latter case, this is a consequence of Lemma 3.1.

For $i = n$, we have that $\sigma_0^\tau(\tilde{x}^n, x^{-n}) = \sigma_0^\tau(x)$ for all $\tilde{x}^n \in \{1, 2, \dots, n\}$. This is obvious if $\tilde{x}^n = n$. If $\tilde{x}^n = k$, $k \neq n$, note that there is a unique undecided position k and a unique empty position n w.r.t. $(\tilde{x}^n, x^{-n})$. Due to the fact that player n has a lower priority than player k , the tie-breaking rule assigns position k to player k and position n to n .

Next, let $i \in N \setminus \{n\}$, and let $\tilde{x}^i = k$ with $k \neq i$ be a possible unilateral deviation from x for player i . Denote $x^n = l$. We distinguish several cases. In these cases, we assume without loss of generality that $l < n$, as Theorem 3.3 states that $x \in E(G^{\mathcal{S}, \lambda})$ for $x^n = n$.

Case 1: $k = n$. Then, player i is the only player who requests position n , so player i is assigned to position n . It is clear that $P(\sigma_0^\tau(x), i) \subsetneq P(\sigma_0^\tau(\tilde{x}^i, x^{-i}), i)$.

Case 2: $l = i$, $k < n$. In this case, position k is undecided and position n is empty. If $k < i$, player i has lower priority than player k , so player i loses the tie and is assigned to position n . If $k > i$, player i has higher priority than player k , so player i is assigned to position k and player k is assigned to position n . Either way, note that $P(\sigma_0^\tau(x), i) \subseteq P(\sigma_0^\tau(\tilde{x}^i, x^{-i}), i)$ and $n \in P(\sigma_0^\tau(\tilde{x}^i, x^{-i}), i) \setminus P(\sigma_0^\tau(x), i)$, so that $P(\sigma_0^\tau(x), i) \subsetneq P(\sigma_0^\tau(\tilde{x}^i, x^{-i}), i)$.

Case 3: $l = k$, $k < n$. Then, we have a three-way tie, where position k is undecided and

positions i and n are empty. Player n always has the lowest priority and therefore always ends up in position n . Player i loses the tie against player k if $k < i$ and wins the tie if $k > i$. Either way, player i is assigned to position i and player k to position k , so that $\sigma_0^\tau(\tilde{x}^i, x^{-i}) = \sigma_0^\tau(x)$.

Case 4: $l \neq i, l > k, k < n$. Then, positions k and l are undecided and positions i and n are empty. The tie-breaking rule first assigns the players k and i , who request the earliest undecided position k . Similar to the previous case, players k and i are always assigned to positions k and i , respectively. Player n always loses the tie for position l . Again, we get $\sigma_0^\tau(\tilde{x}^i, x^{-i}) = \sigma_0^\tau(x)$.

Case 5: $l \neq i, l < k, k < n$. Again, positions k and l are undecided and positions i and n are empty. However, the tie-breaking rule now first assigns the players l and n , who request the earliest undecided position l . Player l is assigned to position l and player n to position i if $l < i$, and vice versa if $l > i$. Player i is then assigned to empty position n if $k < i$ and to undecided position k if $k > i$. Similar to the second case, we get $P(\sigma_0^\tau(x), i) \subsetneq P(\sigma_0^\tau(\tilde{x}^i, x^{-i}), i)$.

Proof Claim 2 Let $x \in E(G^{\mathcal{S}, \lambda})$ and assume towards contradiction that $x^j \neq j$ for some $j \in \{1, 2, \dots, n-1\}$. Let i be the smallest index for which this is true, i.e., $x^1 = 1, \dots, x^{i-1} = i-1$, and $x^i = k$ with $k \neq i$. Note that player i can still be assigned to position i through tie breaking, but only if position i is empty. So, if $\sigma_0^\tau(x)^{-1}(i) = i$, then $x^j \neq i$ for all $j \in N$. In that case, player n can deviate and be assigned to $\tilde{x}^n = i$. Alternatively, if $\sigma_0^\tau(x)^{-1}(i) > i$, player i can deviate and be assigned to $\tilde{x}^i = i$, as player i will then be the player with the highest priority who requests position i . In both cases, the deviating player makes sure to be assigned to position i , instead of some position strictly later than i . Using the fact that the set of players assigned to the first $i-1$ positions is fixed, we can apply Lemma 3.1 to argue that the corresponding deviation is strictly profitable, contradicting $x \in E(G^{\mathcal{S}, \lambda})$. $\square$

3.3 Biform Sequencing Processes with Additional Costs

As an extension of the previous model, one could analyze the influence of associating costs $\gamma \in \mathbb{R}^N$ with the strategic choice for a certain position in the initial order. First, we consider a cost function that assigns fixed costs to each position in the strategically determined initial order. Here, players do not pay for their requested position, but for the position in which they actually end up. More formally, for any $x \in X$ and $k \in \{1, 2, \dots, |N|\}$, player $i \in N$ incurs fixed costs $\gamma(k)$ if $\sigma_0^\tau(x)^{-1}(i) = k$.

Example 3.4

Reconsider the BS-process $\mathcal{S} = (\{1, 2, 3\}, \{X^i\}_{i \in N}, \tau, \{S(x)\}_{x \in X})$ of Example 3.3 using the POB tie-breaking rule τ with priority order $\bar{\sigma}_0^\tau = (2, 3, 1)$, where for each $x \in X$, $S(x) =$

$(\{1, 2, 3\}, \sigma_0^T(x), p, \alpha)$ is a corresponding sequencing process for which $\sigma_0^T(x)$ is given in Table 5, $p = (4, 3, 4)$ and $\alpha = (2, 3, 6)$. Introducing costs, the net profits as specified by $\pi^{S,EGS}$ in the corresponding induced strategic game $G^{S,EGS}$ will change. We emphasize that the costs are associated with the *obtained* position in the strategically determined initial processing order here, meaning that the net profit vectors of two different strategy combinations that lead to the same initial order are still equal.

Let $\gamma = (20, 10, 0)$, i.e., the fixed costs of being first, second or third in the resulting initial processing order are 20, 10 or 0, respectively. Consider $\sigma_0^T(x) = (3, 2, 1)$, for which we saw in Example 3.1 that the net profit vector is $(-22, -21, -24)$ without fixed costs. By assigning the additional costs to each position in the initial processing order, this net profit vector becomes $(-22, -31, -44)$. In a similar way, we find the new net profits for each strategy combination. The new induced strategic game is given in Table 7.

	1	2	3		1	2	3
1	(-22,-26,-49)	(-22,-31,-44)	(-23,-30,-44)	1	(-22,-26,-49)	(-17,-25,-55)	(-17,-30,-50)
2	(-16,-26,-55)	(-22,-31,-44)	(-23,-30,-44)	2	(-22,-26,-49)	(-22,-26,-49)	(-23,-30,-44)
3	(-22,-26,-49)	(-22,-31,-44)	(-22,-31,-44)	3	(-22,-26,-49)	(-22,-26,-49)	(-22,-26,-49)
	1				2		
		1	2	3			
		(-16,-26,-55)	(-17,-25,-55)	(-17,-25,-55)			
		(-16,-26,-55)	(-16,-26,-55)	(-16,-26,-55)			
		(-22,-26,-49)	(-22,-31,-44)	(-22,-26,-49)			
		3					

Table 7: *The induced strategic game of Example 3.4 with fixed costs $\gamma = (20, 10, 0)$ associated with obtained positions*

For this game, the set of equilibria is given by $\{(2, 1, 2), (3, 1, 2)\}$. $\triangle$

Example 3.4 shows that Theorem 3.4 cannot be generalized to this new setting of BS-processes with additional costs. The reason for this is that Lemma 3.1 cannot be generalized: the fixed costs may outweigh the benefits of obtaining an earlier position in the initial order. However, it can be shown that the existence of the specific equilibrium, in which all players request the position they are entitled to according to the underlying priority order, is still guaranteed. So, Theorem 3.3 can be generalized to this setting.

As an alternative option, one can associate costs with the strategic choice itself. In this case,

players pay a fixed amount depending on their *requested* position. The position in which a player actually ends up in the initial processing order does not play a role here. Formally, for any $x \in X$ and $k \in \{1, 2, \dots, |N|\}$, player $i \in N$ incurs fixed costs $\gamma(k)$ if $x^i = k$. Note that different strategy combinations that lead to the same initial processing order can now have different net profit vectors. In fact, the existence of an equilibrium is no longer guaranteed, as illustrated by Example 3.5.

Example 3.5

Reconsider the 3-player biform sequencing process with fixed costs presented in Example 3.4, with one key difference: the costs $\gamma = (20, 10, 0)$ are now associated with the requested position rather than the obtained position in the initial order. The resulting induced strategic game is given in Table 8.

	1	2	3		1	2	3
1	(-42,-26,-59)	(-42,-31,-44)	(-33,-30,-44)	1	(-42,-26,-49)	(-17,-25,-65)	(-17,-30,-50)
2	(-16,-26,-75)	(-32,-31,-44)	(-23,-30,-44)	2	(-32,-26,-49)	(-32,-16,-49)	(-23,-30,-34)
3	(-22,-26,-59)	(-22,-31,-44)	(-22,-21,-44)	3	(-22,-26,-49)	(-22,-16,-49)	(-22,-6,-49)
		1				2	
		1	2	3			
		1	(-26,-26,-55)	(-17,-25,-55)	(-17,-15,-55)		
		2	(-16,-26,-55)	(-16,-16,-55)	(-16,-6,-55)		
		3	(-22,-26,-39)	(-22,-31,-24)	(-22,-6,-39)		
			3				

Table 8: *The induced strategic game of Example 3.5 with fixed costs $\gamma = (20, 10, 0)$ associated with requested positions*

Note that the set of Nash equilibria is empty. $\triangle$

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