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Optimal pricing and inventory policies: Centralized and decentralized decision making

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Abstract

The paper studies an optimal control problem of pricing and inventory replenishment in a system with serial inventories. Consumer demand for a specific product at a retail outlet depends on price as well as the in-store stock of the product. The hypothesis is that for certain consumer products, a large volume of displayed goods leads consumers to buy more than if the stock is small. In addition to the stock that is on display in the store, there is an inventory of the product in a central warehouse. First we consider a setup in which management of the two stocks is decentralized such that pricing decisions are made by the store manager who also decides on the level of in-store inventory. The warehouse manager makes the replenishment decisions concerning the stock in the warehouse. Next we study the problem where stock management and pricing decisions are centralized. Optimal trajectories for inventories, replenishment rates, and retail price are derived by using phase diagrams and a formal synthesizing procedure. © 2002 Elsevier Science B.V. All rights reserved.

Keywords: Marketing; Retailing; Purchasing; Inventory; Optimal control

1. Introduction

The paper studies a problem of pricing and inventory replenishment for a particular product. Our hypothesis is that demand at a retail outlet depends on the consumer price as well as the in-store stock of the product. In particular, demand decreases with price but increases with the displayed inventory of the product. The idea is that a large stock generates more demand and leads consumers to buy more than if the inventory is small. This phenomenon is not a general one, but it may pertain to certain product categories.

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One may think of used automobiles that are sold from an open yard of a car dealer, or displays of large quantities of soft drinks, detergents, and canned food sometimes found in supermarkets.¹

The idea of letting the demand rate depend on the level of the inventory was first used in Wolfe (1968), Schary and Becker (1972) and Levin et al. (1972); the problem of determining optimal ordering and inventory policies under stock-dependent demand was addressed by, e.g., Baker and Urban (1988), Goh (1992, 1994), Urban (1992, 1995) and Gerchak and Wang (1994). These contributions use an economic order quantity (EOQ) approach and determine an optimal lot size, ordered and delivered at the beginning of each order cycle. An implicit assumption of this literature is that the whole stock of the product is on display and hence is capable of affecting demand. This paper studies a different situation and supposes that there are *two stocks* of the product. One is displayed in the retail store and the size of this stock affects sales. Another stock is stored in a central warehouse (distribution center). From this stock, the retail store's inventory is replenished, but clearly the size of the stock in the warehouse cannot affect consumer demand. One may think of a supermarket chain having a distribution center from which the retail outlets are supplied. For simplicity, we assume that, upon arrival at a retail store, all goods are put on display.

We first consider a setup in which management is *decentralized*. The store manager decides how his inventory should be replenished. Moreover, she sets the consumer price. The central warehouse manager makes the replenishment decisions concerning that stock. Each decision maker has her own objective function which she seeks to maximize. We also consider an organizational structure in which decisions are *centralized*. Here, a single decision maker (e.g., the headquarter of the supermarket chain) makes all inventory replenishment and pricing decisions and seeks to optimize the total payoff of the distribution system.

As regards the organization of the inventory system, our paper departs from the EOQ literature which considers a single inventory and hence is unable to address the organizational problem. There are other differences, too. In contrast to the EOQ models, we use review policies where inventory levels are monitored continuously and a manager can place an order at any instant of time (although there may be periods of time during which no ordering takes place). We do not allow the EOQ-type of ordering where the stock is replenished instantaneously at discrete instants of time.² Moreover, our model includes price as a second decision variable that influences consumer demand, where the EOQ literature often has assumed a constant sales price. Our reason for including price is that marketing studies have found that the impact on sales of a large inventory may be enhanced by price cuts (cf. Berman, 1996).

Research on production and inventory planning using continuous-time modeling started with the seminal paper of Holt et al. (1960). Pekelman (1974) introduced price as a decision variable in a production-inventory model. Later contributions, using optimal control theory or differential games are Thompson et al. (1984), Feichtinger and Hartl (1986), Jørgensen (1986), Abad (1987), Gaimon (1998) and Jørgensen et al. (1999).

The contributions of this paper are the following. EOQ models with inventory-dependent demand are extended to a serial, two-stock setting in which one of the inventories affects demand, and price is a decision variable. Optimal control models of inventory and pricing are extended with an additional determinant of demand, viz., the level of displayed inventory. In the setup of a retail distribution system, we assess the policy implications of having a stock-dependent demand and identify the negative cost effects of decentralized decision making. The paper proceeds as follows. Section 2 presents an optimal control model of

¹ The argument can be reversed. For other products, the existence of a large stock may decrease demand if the customers believe that since a large stock is left, the product must be a bad one.

² In control theoretic terms this would imply the use of impulsive controls and there would be jumps in the state variables (the stocks).

pricing, ordering, and inventory management. Sections 3 and 4 solve the decentralized and the centralized problems, respectively. Section 5 identifies the policy impacts of having a stock-dependent demand and compares the outcomes of decentralized and centralized decision making. Section 6 provides our concluding comments.

2. The dynamic model

Since we do not wish to address issues of category management, interbrand competition, or the effects of retailer competition/coordination, we confine our interest to a *single-product, single-retail store* problem. A multiple-store, multiple-goods EOQ model is studied in Gallego and Queyranne (1995) who use heuristic methods to solve the problem. Needless to say, an optimal control model for such a complex structure is doomed to be analytically intractable.

In our setup, the central warehouse orders the product from a supplier (a manufacturer). Upon arrival at the warehouse, the goods are stocked until being ordered by the retail store. All goods shipped from the warehouse to the store are put on display. We consider a fixed and finite planning period $[0, T]$ and the problem is to determine optimal policies for the retail price, the ordering rate of the warehouse from the manufacturer, the ordering rate of the retail store from the warehouse, as well as the inventories kept in the store and in the warehouse.

Let $I_D(t)$ (D for “displayed”) represent the retail inventory level by time $t \in [0, T]$. The stock in the warehouse is denoted by $I_N(t)$ (N for “not” displayed). The stock I_D is decreased by the demand rate, but can be replenished from the stock I_N . Denote by $q_D(t)$ the replenishment rate of the in-store inventory. The stock in the warehouse is replenished by ordering from the manufacturer. Denote this ordering rate by $q_N(t)$. Letting $x(t)$ represent the rate of consumer demand at time t , the inventory system is a serial, two-stage system (Axsäter, 1985, 1993) with the dynamics

$$\dot{I}_D(t) = q_D(t) - x(t), \quad I_D(0) = I_{D0} > 0, \quad (1a)$$

$$\dot{I}_N(t) = q_N(t) - q_D(t), \quad I_N(0) = I_{N0} > 0, \quad (1b)$$

where we have assumed zero lead times in the ordering processes.

The demand rate depends on $p(t)$, the consumer price at time t , as well as the level of displayed stock. We have chosen a linear specification

$$x(t) = f(p(t), I_D(t)) = [\alpha - \beta p(t)]kI_D(t) \quad (2)$$

in which α , β , k are positive constants. The specification (2) allows for interaction between the two factors that influence demand, viz., the consumer price and the displayed stock. Here there is a parallel to marketing science models in which price and advertising interact in affecting demand. Feichtinger and Hartl (1986) have shown that positive advertising effects should be accompanied by price cuts. Thus, the two marketing instruments work in the same direction. In our setup, the displayed stock can be seen as an advertisement for the product and one should then expect a similar synergistic effect.³ (Advertising optimal control literature is surveyed in Feichtinger et al., 1994.)

Admissible decisions $q_D(t)$, $q_N(t)$, $p(t)$ must satisfy

$$q_j(t) \geq 0, \quad j \in \{D, N\}, \quad 0 \leq p(t) \leq \alpha/\beta \quad \forall t \in [0, T]. \quad (3)$$

³ We are grateful to an anonymous reviewer for suggesting this interpretation.

Using (3) shows the demand function in (2) has the following properties:

$$\begin{aligned} f(p, I_D) &\geq 0, \quad f(p, 0) = 0, \\ \partial f / \partial p &< 0, \quad \partial^2 f / \partial p^2 = 0, \\ \partial f / \partial I_D &> 0, \quad \partial^2 f / \partial I_D^2 = 0, \quad \partial^2 f / \partial p \partial I_D < 0. \end{aligned} \quad (4)$$

The second assumption in the first row of (4) states that if the displayed inventory is zero, the demand rate is zero (“no stock, no demand”). The demand rate increases linearly with the displayed stock level which means that a marginal increase in the inventory always has the same impact on demand (see also Mandal and Phaujdar, 1989). However, many of the EOQ studies suppose that demand depends on inventory in a concave way.⁴ Finally, negativity of the mixed second-order partial derivative means that the marginal impact of the displayed stock on demand increases when price decreases. This hypothesis reflects a finding of marketing literature saying that the impact on demand of a large inventory may be enhanced by a low price (Berman, 1996).

The state of the system at any time t is described by the stock levels $I_N(t)$, $I_D(t)$. Backlogging is not allowed, i.e., both stocks must be nonnegative at any instant of time.⁵ Using (1a) yields $\dot{I}_D(t)|_{I_D(t)=0} = q_D(t) - f(p(t), 0) \geq 0$. Exploiting the “no stock, no demand” assumption, and $I_{D0} > 0$, $q_D(t) \geq 0$, shows that the inventory $I_D(t)$ is positive for all t . Satisfaction of the other nonnegativity constraint,

$$I_N(t) \geq 0 \quad \forall t \in [0, T], \quad (5)$$

is not guaranteed.

The cost of replenishing the stock I_N , by ordering from the manufacturer, is given by

$$C_N(q_N) = c_N q_N, \quad \text{where } c_N = \text{constant} > 0, \quad (6)$$

assuming that there are no quantity discounts or fixed ordering costs. The cost of replenishing the retail stock I_D from the warehouse is quadratic

$$C_D(q_D) = \frac{c_D}{2} q_D^2, \quad \text{where } c_D = \text{constant} > 0 \quad (7)$$

and includes expenses incurred by shipping goods from the warehouse to the retail shop and processing goods before being put on display. The assumption in (7) is that there are specific organizational costs involved in the internal replenishment process and these costs increase more than proportional with the replenishment rate. The convex cost in (7) can be also justified by the following argument. The demand rate in (2) is linear in the stock I_D but more realistically it should be concave in I_D (to reflect decreasing marginal returns). In this case one could have a linear replenishment cost function in (7), because the concavity of demand in I_D would penalize the use of large replenishment rates q_D . In our setup, demand increases linearly in I_D , but large replenishment rates are penalized by the convex cost function. See also Feichtinger and Hartl (1986, p. 318).

Supposing that there are no warehousing constraints, we can assume (cf. Pekelman, 1974; Axsäter, 1993; Jørgensen et al., 1999) that inventory holding costs are linear

$$S_j(I_j) = s_j I_j, \quad \text{where } s_j = \text{constant} > 0, \quad j \in \{D, N\}. \quad (8)$$

⁴ Our demand function is linear in the inventory and hence does not exhibit those decreasing returns implied by a concave function. We comment briefly on this in Section 6.

⁵ The no-backlogging constraint is necessary to be consistent with the “no stock, no demand” assumption; cf. Urban (1992) and Goh (1994). In Mandal and Phaujdar (1989) and Urban (1995) stockouts are allowed (at a cost) and even during a stockout period, some demand occurs.

Fig. 1. Structure of the model.

The overall undiscounted profit of the system is given by

$$J = \int_0^T \left[p(t)f(p(t), I_D(t)) - c_N q_N(t) - \frac{c_D}{2} q_D(t)^2 - s_D I_D(t) - s_N I_N(t) \right] dt. \quad (9a)$$

We consider a relatively short planning period since what we have in mind is a problem of operational/tactical planning. In that case, introducing a discount factor would not change the results very much.

In a decentralized system, we represent the store manager by the letter “D” and the warehouse manager by “N”. Manager D solves an optimal control problem with the objective of maximizing J_D , whereas N solves an optimal control problem to maximize J_N . The objective functions are given by

$$J_D = \int_0^T \left[p(t)f(p(t), I_D(t)) - \frac{c_D}{2} q_D(t)^2 - s_D I_D(t) \right] dt, \quad J_N = - \int_0^T [c_N q_N(t) + s_N I_N(t)] dt \quad (9b)$$

such that $J_D + J_N = J$. For the reader's convenience we have summarized the main features of the model in Fig. 1.

Notice that the decentralized problem can be viewed as a noncooperative game. One player, the store manager, determines the retail price and his ordering rate to maximize the local profits J_D . Another player, the warehouse manager, determines the ordering rate to minimize the local costs J_N . Goods are transferred between the players at the rate q_D , but the transfer price is zero. (In Jørgensen (1986) the transfer price is a decision variable of the warehouse manager.) In Section 3, we identify solutions to the optimal control problems of the two players when they determine their optimal decisions as functions of time. This amounts to finding an open-loop equilibrium of the dynamic game. We shall see, however, that this game is degenerate. The reason is that the optimal control problem of the store manager is unaffected by the actions of the warehouse manager. The centralized problem, on the other hand, has a single decision maker who makes all decisions with the objective of maximizing J . This can be seen as a cooperative game in which the store and warehouse managers agree to coordinate their actions to achieve maximal total profits in the system.

3. Optimal policies in a decentralized organization

This section provides the solutions to the optimal control problems faced by two independent managers. The store manager maximizes J_D in (9b), subject to (1a) and (1b) and the constraints on p and q_D stated in

(3). The warehouse manager maximizes J_N in (9b), subject to (1a) and (1b) and the constraints on q_N and I_N in (3) and (5). In a decentralized setup, the warehouse manager is responsible for the satisfaction of the no-backlogging constraint (5).

Fig. 1 seems to indicate that the optimal control problems of decision makers D and N are related only through D 's control q_D which appears in both state equations. Each of the Hamiltonian functions will, however, involve two costate variables that reflect that each manager a priori takes both state equations into account. Thus, a further interdependence is introduced through costate variables. However, in Appendix A we show that the “mixed” costates, representing the shadow price of stock $I_D[I_N]$, as assessed by manager $N[D]$, are identically zero. The implication is that the I_D -dynamics in (1a) (I_N -dynamics in (1b)) play no role for N 's problem (D 's problem). The reason is that $I_D[I_N]$ is not payoff-relevant for the problem of $N[D]$, and that D does not care about the satisfaction of the nonnegativity constraint imposed on I_N (cf. (5)). We solve problem N in Section 3.1; Section 3.2 solves problem D .

3.1. Solution of problem N

For any admissible trajectory $q_D(t)$, $t \in [0, T]$, define

$$S(t) = \int_0^t q_D(\zeta) d\zeta,$$

which represents accumulated order volume at the warehouse. Let τ be given by

$$S(\tau) = \int_0^\tau q_D(t) dt = I_{N0}. \quad (10)$$

Clearly, τ is the instant of time at which the initial stock I_{N0} is exhausted if no replenishments are made, that is, if $q_N(t) = 0$ for $t \in [0, \tau]$. The solution of problem N is stated in Propositions 1 and 2. All proofs have been relegated to appendices.

Proposition 1. *If $I_{N0} \geq S(T) = \int_0^T q_D(t) dt$, the optimal replenishment rate of problem N is $q_N^*(t) = 0$ for $t \in [0, T]$.*

The intuition of Proposition 1 is clear: if the initial stock suffices to satisfy the accumulated orders from the store for the entire planning period, no replenishment of the warehouse inventory should be made (because this would imply unnecessary procurement and holding costs). The warehouse inventory remains positive, except possibly at the horizon date.

Proposition 2. *If $I_{N0} < S(T)$, the optimal replenishment rate of problem N is*

$$\begin{aligned} q_N^*(t) &= 0, & I_N(t) &= I_{N0} - S(t) & \text{for } t \in [0, \tau], \\ q_N^*(t) &= q_D(t), & I_N(t) &= 0 & \text{for } t \in (\tau, T]. \end{aligned} \quad (11)$$

In order to minimize holding costs, the warehouse manager starts out by decreasing the stock as fast as possible. This is most efficiently done by having a zero replenishment rate until the initial stock is exhausted. Thus, the manager escapes the holding costs as fast as possible and avoids any replenishment cost. Thereafter, replenishment is done at the “demand rate” $q_D(t)$ for the rest of the planning period. This makes the inventory zero and no holding costs are incurred. (The optimal policy satisfies $I_N(t)q_N(t) = 0$ which is a well-known condition in optimal inventory planning.)

Propositions 1 and 2 show that the warehouse manager only needs to react to the ordering policy of the store manager, given by $q_D(t)$. The pricing policy has no direct influence on the warehouse manager's policy.

3.2. Solution of problem D

Define the Hamiltonian

$$H^D = pf(p, I_D) - \frac{c_D}{2} q_D^2 - s_D I_D + \lambda_D [q_D - f(p, I_D)] \quad (12)$$

in which λ_D is the costate variable associated with state variable I_D . Using (12), and the fact that state variable $I_D(t)$ is always positive, shows that an optimal pricing policy is given by

$$\begin{aligned} p^* &= 0 & \text{iff } \lambda_D &\leq -\alpha/\beta, \\ p^* &= [\alpha + \beta\lambda_D]/2\beta > 0 & \text{iff } -\alpha/\beta < \lambda_D < \alpha/\beta, \\ p^* &= \alpha/\beta & \text{iff } \lambda_D &\geq \alpha/\beta. \end{aligned} \quad (13)$$

Here, and in the sequel, we omit the time argument when no confusion can arise. The intuition of the pricing policy is as follows. If the shadow price λ_D is sufficiently low, it is not worthwhile to carry an inventory. To decrease the initial stock rapidly, the manager sets a low price. On the other hand, it pays to conserve the stock if the shadow price is sufficiently high. This is accomplished by quoting a high price in order to reduce demand rapidly.

Using (12) shows that an optimal replenishment rate q_D^* is

$$\begin{aligned} q_D^* &= 0 & \text{iff } \lambda_D &\leq 0, \\ q_D^* &= \lambda_D/c_D > 0 & \text{iff } \lambda_D > 0. \end{aligned} \quad (14)$$

If the shadow price λ_D is nonpositive, it is clearly not worthwhile to replenish. On the other hand, the replenishment rate should be increased with the shadow price whenever the latter is positive. Note that if price and replenishment rate constraints are not binding, (13) and (14) yield $dq_D/dp = 2/c_D$. This shows that price and replenishment have a synergistic effect with respect to changes in the inventory level. Suppose, for example, that the manager increases the replenishment rate in order to raise the inventory level. This policy is supported by an increasing price which tends to reduce demand.

In view of (13) and (14) it is useful to distinguish four regimes:

$$\begin{aligned} \text{Regime 1 (R1): } & \lambda_D \leq -\alpha/\beta, \quad p^* = q_D^* = 0, \\ \text{Regime 2 (R2): } & -\alpha/\beta < \lambda_D \leq 0, \quad p^* = [\alpha + \beta\lambda_D]/2\beta > 0, \quad q_D^* = 0, \\ \text{Regime 3 (R3): } & 0 < \lambda_D < \alpha/\beta, \quad p^* = [\alpha + \beta\lambda_D]/2\beta > 0, \quad q_D^* = \lambda_D/c_D > 0, \\ \text{Regime 4 (R4): } & \lambda_D \geq \alpha/\beta, \quad p^* = \alpha/\beta, \quad q_D^* = \lambda_D/c_D > 0. \end{aligned}$$

Then we are ready to state the solution of problem D.

Proposition 3. *The optimal policy of problem D is as follows.*

$$\begin{aligned} \text{Case A: } & k\alpha^2/4\beta < s_D < k\alpha^2/\beta: \quad \text{R2 for all } t \in [0, T], \\ \text{Case B: } & s_D > k\alpha^2/\beta: \quad \text{R1 for } t \in [0, \bar{t}), \text{ R2 for } t \in [\bar{t}, T], \\ \text{Case C: } & s_D < k\alpha^2/4\beta: \quad \text{R3 for } t \in [0, T), \text{ R2 for } t = T. \end{aligned} \quad (15)$$

In Appendix A we prove Proposition 3 by using phase diagrams. The diagrams for Cases A and B are qualitatively the same. We only illustrate Case B which is done in Fig. 2. Fig. 3 shows Case C. In the figures,

Fig. 2. Solution of store manager's problem with large holding cost.

Fig. 3. Solution of store manager's problem with small holding cost.

variables wearing a “hat” are steady-states. Regime 4 is never used in an optimal solution. This result is driven by the assumption of a zero salvage value, but one should also note that Regime 4 is an extreme one; price is chosen to make demand vanish. In a situation where inventory stimulates demand, it seems less sensible to choose a price that makes demand vanish.

Proposition 3 shows that the size of the holding cost s_D , relative to the parameters k , α , β determines the optimal ordering and pricing policies. Recall that the larger the parameter k , the larger the demand rate (given the level of the in-store inventory). In Case B (Fig. 2), the holding cost, s_D , is “large” and the demand-stimulating effect, k , is “small”. The implication is that inventory replenishment is not worthwhile at all. To reduce the inventory as fast as possible, the manager selects the minimal price. Notice that price increases over time which tends to reinforce the negative effect of the decreasing inventory on demand. This policy conforms with the finding in marketing literature, that the sales impact of a large inventory may be enhanced by price cuts (Berman, 1996). Case C (Fig. 3) occurs if the holding cost s_D is “small” and the efficiency parameter k is “large”. Then it is not costly to have a large inventory, and the inventory level has a sizeable impact on demand. Then it is worthwhile to have a positive replenishment rate. Fig. 3 shows that the inventory level is steadily decreasing (if it is large initially), or it first increases and then decreases (if it is small initially). This result may seem counterintuitive as one might expect an increasing inventory when holding cost s_D is small and efficiency k is large. The primary explanation for the result is the zero salvage

value of the inventory, which induces the overall decrease of the shadow price.⁶ Since the inventory becomes less valuable over time, it makes sense to reduce it, even if it carries a low holding cost and has a sizeable impact on demand. To take advantage of the large value of k , price is steadily decreased over time. This tends to counterbalance the effect on demand of a decreasing inventory. Again, we see that a demand-stimulating effect of an inventory can be enhanced by cutting the price.

3.3. Combining the solutions of problems N and D

Inserting the policies $q_D^*(t)$ from Proposition 3 into problem N one obtains the following. In Cases A and B it holds that $q_D^*(t) \equiv 0$ and we are in the situation dealt with in Proposition 1. The implication is $q_N^*(t) \equiv 0$, which makes the warehouse inventory constant over time. In Case C we have Regime 3, where $q_D^*(t) = \lambda_D(t)/c_D$ on the interval $[0, T)$. Regime 2 only applies at the horizon date at which $q_D^*(T) = 0$. From (12) one gets the costate equation valid in Regime 3:

$$\dot{\lambda}_D = s_D - k[\alpha - \beta\lambda_D]^2/4\beta. \quad (16)$$

The solution of the Riccati equation (16), with transversality condition $\lambda_D(T) = 0$, is

$$\lambda_D(t) = \frac{1 - \exp\left((k/2)(\beta\hat{\lambda}^L - \alpha)(t - T)\right)}{\frac{\beta}{2\alpha - \beta\hat{\lambda}^L} - \frac{1}{\hat{\lambda}^L} \exp\left((k/2)(\beta\hat{\lambda}^L - \alpha)(t - T)\right)}, \quad (17)$$

where $\hat{\lambda}^L = \alpha/\beta - \beta^{-1}\sqrt{4s_D\beta/k} > 0$ is the steady-state value depicted in Fig. 3.

Finally, to see if we are in the situation of Proposition 1, or in that of Proposition 2, one calculates

$$S(T) = \int_0^T q_D^*(t) dt = \frac{1}{c_D} \int_0^T \lambda_D(t) dt \quad (18)$$

in which $\lambda_D(t)$ is given by (17). Finally one compares $S(T)$ and I_{N0} . However, as long as the parameter values are unspecified, this will not provide a definitive answer. Hence, the problem of finding a “best” balance between a central and a local stock (Axsäter, 1993) cannot be resolved.

4. Optimal policies in a centralized organization

This section solves the optimal control problem of a single decision maker who chooses the decision variables q_N , q_D , and p to maximize the system profits in (9a), subject to (1a), (1b), (3) and (5). For the optimal warehouse replenishment policy q_N^* , Propositions 1 and 2 remain valid. In problem D it is useful to make a subdivision of Case C in (15) to get

$$\begin{aligned} \text{Case A : } & k\alpha^2/4\beta < s_D < k\alpha^2/\beta, \\ \text{Case B : } & s_D > k\alpha^2/\beta, \\ \text{Case C1 : } & s_D < k(\beta c_N - \alpha)^2/4\beta, \\ \text{Case C2 : } & k(\beta c_N - \alpha)^2/4\beta < s_D < k\alpha^2/4\beta. \end{aligned} \quad (19)$$

We start by determining the replenishment and pricing policies if the scenario of Proposition 2 obtains.

⁶ If there were a positive salvage value, being larger than the steady-state value $\hat{\lambda}^L$, the inventory could be increasing all the time if it initially were small.

4.1. The case of Proposition 2

Recall that there is no replenishment of the warehouse inventory during the interval $[0, \tau]$. It is convenient to introduce the following notation:

Regime *ia* (*Ria*), defined for $t \in [t_a^1, t_a^2] \subseteq [0, \tau]$,

Regime *ib* (*Rib*), defined for $t \in [t_b^1, t_b^2] \subseteq (\tau, T]$, $i = 1, 2, 3, 4$,

which provides eight possible regimes. We make the plausible assumption $c_N < \alpha/\beta$ which says that the cost of ordering one unit to the warehouse does not exceed the maximal consumer price. Letting λ_N represent the costate of the warehouse inventory I_N we define

R1a: $\lambda_D \leq -\alpha/\beta$, $p^* = 0$, $q_D^* = 0$ iff $\lambda_D \leq \lambda_N$,

R1b: $\lambda_D \leq -\alpha/\beta$, $p^* = q_D^* = q_N^* = 0$,

R2a: $-\alpha/\beta < \lambda_D \leq c_N$, $p^* = [\alpha + \beta\lambda_D]/2\beta > 0$, $q_D^* = 0$ iff $\lambda_D \leq \lambda_N$,

R2b: $-\alpha/\beta < \lambda_D \leq c_N$, $p^* = [\alpha + \beta\lambda_D]/2\beta > 0$, $q_D^* = q_N^* = 0$,

R3a: $c_N < \lambda_D \leq \alpha/\beta$, $p^* = [\alpha + \beta\lambda_D]/2\beta > 0$, $q_D^* = [\lambda_D - \lambda_N]/c_D > 0$,

R3b: $c_N < \lambda_D \leq \alpha/\beta$, $p^* = [\alpha + \beta\lambda_D]/2\beta > 0$, $q_D^* = [\lambda_D - c_N]/c_D = q_N^* > 0$,

R4a: $\lambda_D > \alpha/\beta$, $p^* = \alpha/\beta$, $q_D^* = [\lambda_D - \lambda_N]/c_D > 0$,

R4b: $\lambda_D > \alpha/\beta$, $p^* = \alpha/\beta$, $q_D^* = [\lambda_D - c_N]/c_D = q_N^* > 0$.

Now we can state the solution of the centralized problem for the scenario in which Proposition 2 applies.

Proposition 4. Suppose that $I_{N0} < S(T)$. There is an optimal solution of the centralized problem if and only if the assumption of Case C1 is satisfied. The solution is one of the following sequences: *R3a* $\rightarrow$ *R2b*; *R3a* $\rightarrow$ *R3b* $\rightarrow$ *R2b*.

Fig. 4 illustrates the two solutions identified in Proposition 4. In both cases, price is positive but decreases over time; τ is the instant of time where the warehouse inventory is exhausted (cf. Proposition 2).

Fig. 4. Solution of centralized problem when Proposition 2 applies.

In the sequence $R3a \rightarrow R2b$, the warehouse inventory is not replenished at all and the manager stops replenishing the in-store inventory precisely when the warehouse inventory is empty. The central manager transfers goods from the warehouse to the store only if the shadow price of the in-store inventory exceeds that of the warehouse inventory. Appendix B shows that as of time τ , the shadow price of the warehouse inventory equals c_N . At time τ , the shadow price of the in-store inventory falls below c_N and therefore it is no longer worthwhile to replenish the in-store inventory.

In the sequence $R3a \rightarrow R3b \rightarrow R2b$, the warehouse inventory is replenished during an intermediate interval from τ to t^* . Fig. 4 shows that the in-store replenishment rate is still positive at the instant τ at which the warehouse is empty. The reason is that along $R3b$, that is, during the interval $[\tau, t^*]$, the shadow price of the in-store inventory exceeds the cost c_N and hence the in-store inventory should be replenished. According to Proposition 2 it holds that $q_N = q_D$ on the interval $[\tau, T]$ and since q_D is positive on $[\tau, t^*]$, q_N must also be positive.

4.2. The case of Proposition 1

To state our result for this case we employ the four regimes Ria , $i = 1, 2, 3, 4$, introduced in Section 4.1. Then we can establish the following proposition.

Proposition 5. Suppose that $I_{N0} \geq S(T)$. The optimal solution of the centralized problem is as follows. Cases A and C2: $R2a$; Case B: $R1a \rightarrow R2a$; Case C1: $R3a \rightarrow R2a$.

Fig. 5 illustrates Cases A and C2. (The solution in Case B looks like the one in A, and the solution in Case C1 looks like C2.) In Case A, the in-store inventory has an “intermediate” holding cost, cf. (19). The in-store replenishment rate is given by $q_D = [\lambda_D + s_N(T - t)]/c_D$, where the shadow price λ_D is negative (except at the horizon date) and increasing. Thus, if the holding cost s_N of the warehouse inventory is sufficiently large, the in-store replenishment rate will be positive, at least on an initial interval of time. The relationship between the two holding costs makes it worthwhile to transfer goods from the warehouse to the store.

Fig. 5. Solution of centralized problem when Proposition 1 applies.

In Case C2, the holding cost of the in-store inventory is lower than that of Case A. The in-store replenishment rate is now positive for all $t < T$. Owing to the fact that the shadow price λ_D is positive, but is decreasing to zero at time T , both the in-store inventory replenishment rate and price are decreasing over time. When price decreases, demand increases which reduces the inventory. This works in synergy with the decreasing replenishment rate.

4.3. Uniqueness of optimal solutions

Propositions 4 and 5 show that in Cases A, B, and C2 there is a unique solution. In Case C1, however, there are three candidates: the two solutions stated in Proposition 4 and the last one of Proposition 5. The problem of determining which one of the three candidates to apply can be solved as follows. The shadow price of the in-store inventory is given as the solution of (17). For $t \in [0, \tau]$ the replenishment rate of the in-store inventory is

$$q_D = [\lambda_D - \lambda_N]/c_D. \quad (20)$$

In (20), the shadow price λ_N of the warehouse inventory is given by

$$\lambda_N = c_N + s_N(t - \tau), \quad (21)$$

where τ is the instant at time at which the warehouse inventory is exhausted. (See Appendix A for the derivation of (21).) Substitute the costate variables from (17) and (21) into the right-hand side of (20) and denote the resulting replenishment rate by $\hat{q}_D(t)$. Inserting the latter into state Eq. (1b), one can determine τ from

$$\dot{I}_N = -\hat{q}_D, \quad I_N(0) = I_{N0}, \quad I_N(\tau) = 0$$

and the conclusion follows: If $\tau > T$, the optimal solution is the sequence $R3 \rightarrow R2$ that occurs in Case C1 of Proposition 5. If $\tau < T$, $\lambda_D(\tau) = c_N$, the optimal solution is the sequence $R3a \rightarrow R2b$ of Proposition 4. If $\tau < T$, $\lambda_D(\tau) > c_N$, the optimal solution is $R3a \rightarrow R3b \rightarrow R2b$ of Proposition 4.

5. Policy implications

This section addresses two questions. What is the impact on optimal policies of having a stock-dependent demand rate (Section 5.1) and what are the relative merits of centralized decision making in the inventory system (Section 5.2)?

5.1. Impact of stock-dependent demand

We confine our interest to the decentralized problem, although the issue could also be addressed in the centralized setup. To identify optimal policies in the absence of the stock effect, one can consider the following modification of problem D . Omitting the kI_D term in the demand function (2), and revising the assumptions in (4) accordingly, yields a model which is a special case of the one considered by Pekelman (1974), viz., a case where demand parameters α and β are stationary. (Pekelman used time-dependent demand parameters to reflect a fluctuating demand rate.)

Propositions 1 and 2 are still valid for problem *N*. Confining our interest to the case where the initial in-store inventory is insufficient to meet the accumulated demand (an analogy to Proposition 2), we can establish the following proposition.

Proposition 6. *If the demand function only depends on price, that is, $x(t) = \alpha - \beta p(t)$ and if $0 < I_{D0} < \int_0^T [\alpha - \beta p^*(t)] dt$, optimal pricing and in-store inventory replenishment policies are*

$$\begin{aligned} p^*(t) &= \max \left\{ 0, \left[\alpha + \beta s_D(t - \bar{t}) \right] / 2\beta \right\}, \\ q_D^*(t) &= 0 & \text{for } t \in [0, \bar{t}], \\ p^*(t) &= \left[\alpha + \beta (\tilde{\lambda} + s_D(t - \tau_1)) \right] / 2\beta > 0, \\ q_D^*(t) &= \left[\tilde{\lambda} + s_D(t - \tau_1) \right] / c_D > 0 & \text{for } t \in [\bar{t}, \tau_1], \\ p^*(t) &= \alpha(1 + \beta c_D) / \beta(2 + \beta c_D) > 0, \\ q_D^*(t) &= \alpha / (2 + \beta c_D) & \text{for } t \in [\tau_1, T], \end{aligned}$$

where

$$\begin{aligned} \tilde{\lambda} &= c_D \alpha / (2 + \beta c_D), \\ \tau_1 - \bar{t} &= \alpha c_D / (2 + \beta c_D) s_D, \\ I_D(t) &> 0 \quad \text{for } t \in [0, \tau_1], \\ I_D(t) &= 0 \quad \text{for } t \in [\tau_1, T]. \end{aligned} \tag{22}$$

Fig. 6 illustrates the solution stated in Proposition 6. On the initial interval $[0, \bar{t}]$ there is no replenishment and price may be zero. Price and replenishment rate are positive and increasing on an intermediate interval $(\bar{t}, \tau_1)$. As of time τ_1 , where the in-store inventory is exhausted, price and replenishment rate are constant.

Fig. 6. Solution of the problem without stock-dependent demand.

To assess the policy effects of a stock-dependent demand we proceed as follows. Since the parameters are unspecified, we can only compare qualitatively the results of Propositions 3 (stock-dependent demand) and 6 (stock-independent demand). We make the comparison for a case where the holding cost for the in-store inventory is small. Using (22) shows that as this holding cost becomes smaller, the instant $\bar{t}$ moves to the left but the instant τ_1 moves to the right. Then the replenishment rate will be positive (and increasing) most of the time. Moreover, the moment at which the stock is empty is close to the horizon date. This is intuitive. In the first two cases in Proposition 3, the replenishment rate is zero. Despite of this, the store's inventory remains positive since zero inventory implies zero demand. The only case where the replenishment rate is positive is the third one, Case C. Fig. 3 shows that the replenishment rate is positive for $t < T$.

In case of positive replenishment rates, the shape of replenishment policy differs among the demand structures. In Proposition 3, the replenishment rate decreases whereas in Proposition 6 it is increasing on the intermediate time interval, and constant at a terminal interval. These results are driven by the evolution of the shadow prices, noting that replenishment rates are proportional to these prices. In Proposition 3, the shadow price is decreasing, but in Proposition 6 it is increasing (at least most of the time). In Fig. 3, depending on the level of initial stock, the stock first increases, then decreases (cf. the trajectory marked "II"), or it always decreases (trajectory marked "I"). In contrast, Fig. 6 exhibits a stock level which decreases (most of the time). The upshot is, when the stock has no effect on demand, it clearly does not pay to let the inventory increase, due to the holding costs. However, if there is such an effect, then in order to benefit from the effect, the inventory: (i) should never be exhausted and (ii) it should be increased during an initial interval of time if the initial stock is small. Conclusion (i) is valid for all values of the in-store holding cost (cf. Fig. 3).

For the case of a stock-dependent demand, Figs. 2 and 3 show that the holding cost s_D and the efficiency parameter k are the key parameters. In Fig. 3, the steady-state values $\hat{\lambda}^L$, $\hat{I}_D$ are positive. Using the expressions in Appendix A for these values we obtain

$$\begin{aligned} \partial \hat{\lambda}^L / \partial s_D &< 0, \quad \partial \hat{I}_D / \partial s_D < 0, \quad \partial \hat{\lambda}^L / \partial k > 0, \\ \partial \hat{\lambda}_D / \partial k &\begin{pmatrix} > \\ < \end{pmatrix} 0 \quad \text{for } s_D \begin{pmatrix} > \\ < \end{pmatrix} \alpha^2 k / 16\beta. \end{aligned}$$

It is intuitive that the steady-state stock and its shadow price decrease if the holding cost increases. We expect the steady-state stock and its shadow price to increase if the efficiency parameter increases; when the stock effect is significant, it should be worthwhile to have a large long-run stock. But this property is only guaranteed globally for the shadow price. For the inventory it holds that $\partial \hat{I}_D / \partial k > 0$ in Cases A and B (see Proposition 3). In Case C we have $\partial \hat{I}_D / \partial k > 0$ if $\alpha^2 k / 16\beta < s_D < \alpha^2 k / 4\beta$. If $s_D < \alpha^2 k / 16\beta$, then $\partial \hat{I}_D / \partial k < 0$. Now, the latter is the case of the smallest holding cost or, equivalently, the highest efficiency k . This provides the interesting observation that in an optimal solution there are decreasing marginal returns to the stock effect, although this was not assumed for the demand function (2). When the stock already is very efficient in stimulating demand, a further increase in efficiency actually lowers the long-run inventory level. But for lower values of k , an increase in efficiency does raise the long-run inventory level.

5.2. Centralized vs decentralized decision making

The essential differences between the optimal policies of Sections 3 (decentralized) and 4 (centralized) are the following. In Section 4, the replenishment policy of the in-store inventory depends, apart from λ_D , also on the shadow price λ_N . In Section 3, the in-store inventory replenishment policy does not depend on λ_N . The implication is that the threshold for λ_D , which distinguishes between zero and positive replenishment, is zero in Section 3. In Section 4, however, the threshold is C_N , that is, the warehouse replenishment cost.

These differences are intuitive, since in the decentralized case the store manager is not concerned with neither the replenishment cost c_N nor the shadow price λ_N of the warehouse inventory.

Consider Proposition 4 and Fig. 4. They show that when $\lambda_D(\tau) \leq c_N$, a central decision maker stops replenishing the in-store inventory precisely at the moment τ at which the warehouse is empty. The reason is that the manager only transfers goods from the warehouse to the store (i.e., replenishes the in-store inventory) if the shadow price of the in-store inventory is above that of the warehouse inventory. As of time τ , the shadow price of the warehouse inventory is constant, equal to c_N . However, at time τ the shadow price of the in-store inventory falls below c_N and it is no longer worthwhile to replenish the in-store inventory. On the other hand, in the decentralized problem, the store manager does not take the replenishment cost c_N into account and continues to replenish the in-store inventory as long as its shadow price is positive. In fact, whenever the replenishment rate is positive, replenishment is continued till the horizon date.

In Fig. 4, the warehouse inventory is replenished during the intermediate interval from τ to t^* . The figure shows that the in-store replenishment rate is still positive at the instant τ where the warehouse is empty. The reason is that the shadow price of the in-store inventory exceeds the cost c_N during the intermediate interval $[\tau, t^*]$; hence the in-store inventory should be replenished. As opposed to the decentralized solution, there is in the centralized case a final interval of zero replenishment of the in-store inventory. The explanation is that in the centralized solution, replenishment of the in-store inventory stops already when its shadow price falls below the replenishment cost c_N . In the decentralized solution, replenishment of the in-store inventory does not stop until the shadow price becomes zero.

In the decentralized solution, the only case in which the warehouse inventory is empty is Case C. It occurs if $s_D \in [0, \alpha^2 k / 4\beta)$. In the centralized problem, the warehouse inventory is empty only in Case C1; it happens at time τ . Case C1 is defined by $s_D \in [0, k(\beta c_N - \alpha)^2 / 4\beta)$. The assumption $\beta c_N < \alpha$ implies $\beta c_N < 2\alpha$. We can conclude that $[0, k(\beta c_N - \alpha)^2 / 4\beta) \subset [0, \alpha^2 k / 4\beta)$. Thus, a scenario in which the warehouse inventory is exhausted at time τ is more likely to occur in the decentralized case. The reason again is that in a decentralized organization, the store manager does not care about the warehouse inventory replenishment cost, and orders more from the warehouse, than in a centralized setup.

Fig. 5 illustrates Case A in Proposition 5. In this case, the in-store inventory has a medium-sized holding cost, cf. (19). Appendix B shows that $q_D = [\lambda_D + s_N(T - t)] / c_D$ and the shadow price λ_D is negative (except at the horizon date) and increasing. Thus, if the holding cost s_N of the warehouse inventory is sufficiently large, the in-store replenishment rate will be positive (at least on an initial interval of time). In the decentralized problem, this replenishment rate is zero throughout. This difference arises from the following reason. In the decentralized problem, the store manager is not concerned with the size of the warehouse inventory. In the centralized problem, the manager realizes that the initial warehouse inventory is sufficient to meet all the orders from the store. Since the central manager takes into account that the warehouse inventory carries a holding cost, it is worthwhile to move goods from the former to the latter (by choosing a positive in-store replenishment rate). As the shadow price of the in-store inventory increases, price is increased to conserve a stock that has an increasing value.

6. Concluding remarks

This paper has extended the literature on EOQ models with inventory-dependent demand to a serial, two-stock setting in which one of the inventories affects demand. Moreover, optimal control models of inventory and pricing with continuous review policies have been extended with an additional determinant of demand, viz., the level of displayed inventory. Optimal solutions have been obtained for three setups: (1) decentralized and (2) centralized decision making under stock-dependent demand, and (3) decentralized decision making under stock-independent demand. Our results for the first two setups are new; the result for the third one is a straightforward modification of results in Pekelman (1974) and Feichtinger and Hartl (1986).

The paper raised two main questions. What are the policy impacts of having a stock-dependent demand, and are there disadvantages of letting the two managers make their decisions independently? The impacts of a stock-dependent demand are that replenishment rates are decreasing rather than increasing over time, and the in-store inventory is never exhausted. Indeed, it increases over time on an initial interval if the initial stock is small.

In contrast to the decentralized decision making scenario, in the centralized setup the central manager takes explicitly into account the costs that are incurred when replenishing the warehouse inventory. The period during which the warehouse inventory needs replenishment will be longer in the decentralized case. Furthermore, when the warehouse inventory is large, centralized decision making will imply larger orders of goods from the warehouse to the store, in order to reduce warehouse holding costs.

The aim of the paper was to study the impacts of a stock-dependent demand on optimal pricing and inventory policies, by establishing a series of analytical results. To keep the problem tractable, and the derived effects interpretable, functional forms were as simple as possible. One example is the choice of a demand function being linear in the inventory level, implying constant returns to the inventory level. Another research strategy would have been to design a more complex model, e.g., allowing for decreasing returns to the inventory level and introducing lead times in the ordering process, and rely on numerical analysis. It is clearly a worthwhile task for future research to extend our model in such, seemingly more realistic, directions.

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Appendix A. Proof of Propositions 1–3

It is convenient to prove the propositions in the following order: 3, 1, and 2.

A.1. Proof of Proposition 3

Define the Hamiltonians

$$H^D = \lambda_0^D \left[pf(p, I_D) - \frac{c_D}{2} q_D^2 - s_D I_D \right] + \lambda_D^D [q_D - f(p, I_D)] + \lambda_N^D [q_N - q_D],$$

$$H^N = -\lambda_0^N [c_N q_N + s_N I_N] + \lambda_D^N [q_D - f(p, I_D)] + \lambda_N^N [q_N - q_D]$$

in which λ_0^k , $k \in \{D, N\}$, are nonnegative constants and λ_j^k , $j \in \{D, N\}$, $k \in \{D, N\}$, is the costate of state variable I_j in the problem of manager k . Let v be a multiplier associated with the state constraint (5) and define the Lagrangians $L^N = H^N + v I_N$, $L^D = H^D$.

First we show that the “mixed” costates λ_j^k , $j \neq k$, are identically zero. Consider the costate equations

$$\dot{\lambda}_D^N(t) = \lambda_D^N(t) [\alpha - \beta p(t)] k,$$

$$\dot{\lambda}_N^N(t) = \lambda_N^N(t) s_N - v(t),$$

$$\dot{\lambda}_D^D(t) = [\lambda_D^D(t) - \lambda_0^D p(t)] [\alpha - \beta p(t)] k + \lambda_0^D s_D,$$

$$\dot{\lambda}_N^D(t) = 0.$$

Using the transversality conditions $\lambda_j^k(T) = 0$, $j, k \in \{D, N\}$, yields $\lambda_D^N(t) \equiv 0$, $\lambda_N^D(t) \equiv 0$. For notational convenience we set $\lambda_D^D \triangleq \lambda_D$ and $\lambda_N^N \triangleq \lambda_N$.

The following is a sufficient optimality condition (Feichtinger and Hartl, 1986). Let $(\lambda_D(t), \lambda_N(t))$ be continuous and piecewise continuously differentiable costates, $v(t)$ a piecewise continuous multiplier function, and γ a constant such that for $\lambda_0^k = 1$, $k \in \{D, N\}$, and for all t , the following conditions are satisfied.

The Hamiltonian H^D is strictly concave in q_D and (14) assures that it is maximized with respect to $q_D \geq 0$. H^D is strictly concave in p and (13) assures that it is maximized with respect to p , subject to (3). The Hamiltonian H^N is linear in q_N and is maximized with respect to $q_N \geq 0$ when

$$-c_N + \lambda_N(t) \leq 0 \quad \text{if } q_N^*(t) = 0 \quad \text{for all } t. \quad (\text{A.1})$$

If on a time interval of nonzero length it holds that $\partial H^N / \partial q_N = 0 \iff \lambda_N(t) = c_N$, the control q_N is singular. If $\partial H^N / \partial q_N > 0 \iff \lambda_N(t) > c_N$, no finite q_N maximizes the Hamiltonian.

The costate equations and their transversality conditions are satisfied:

$$\dot{\lambda}_D(t) = [\lambda_D(t) - p^*(t)][\alpha - \beta p^*(t)]k + s_D \quad \text{for v.e. } t, \quad (\text{A.2})$$

$$\dot{\lambda}_N(t) = s_N - v(t) \quad \text{for v.e. } t, \quad (\text{A.3})$$

$$\lambda_j(T) = 0, \quad j \in \{D, N\}. \quad (\text{A.4})$$

Admitting a discontinuity of the costate $\lambda_N(t)$ at the horizon date T , it must hold that

$$\lambda_N(T^-) - \lambda_N(T) = \gamma, \quad \gamma \geq 0, \quad \gamma I_N^*(T) = 0. \quad (\text{A.5})$$

It is not necessary to allow for discontinuities at other points of time because if $dI_N^*(t)/dt$ is discontinuous at a point where the constraint $I_N(t) \geq 0$ becomes (ceases to be) active, then $\lambda_N(t)$ is continuous at such a point. This holds true in our model. Since the nonnegativity constraint on state variable $I_D(t)$ is never binding, the costate $\lambda_D(t)$ is continuous.

Finally, for all t the maximized Hamiltonians are concave in I_D and I_N , respectively, and it holds that

$$v(t) \geq 0, \quad v(t)I_N^*(t) = 0 \quad \text{for all } t. \quad (\text{A.6})$$

Then the sets $(I_D^*(t), q_D^*(t), p^*(t))$, $(I_N^*(t), q_N^*(t))$ are optimal in problems D and N , respectively.

To prove concavity of the maximized Hamiltonians, note that the optimal controls in (13), (14) and (A.1) do not depend on the state variables. For D it suffices to check if the function

$$\hat{H}^D(I_D, \lambda_D) = (p^*(\lambda_D) - \lambda_D)f(p^*(\lambda_D), I_D) - s_D I_D$$

is concave in I_D for v.e. t . Noting that function f is linear in I_D provides the desired result. The concavity of $\hat{H}^N$ in I_N is obvious.

Now we look at problem D and verify the phase diagrams in Figs. 2 and 3. For this purpose, insert the optimal controls from (13) and (14) into (1a) and (A.2) to obtain the four regimes:

$$\begin{aligned} R1 : \quad & \lambda_D(t) \leq -\alpha/\beta, \quad p^*(t) = q_D^*(t) = 0, \\ & \dot{I}_D(t) = -\alpha k I_D(t), \quad \dot{\lambda}_D(t) = s_D + \alpha k \lambda_D(t), \\ R2 : \quad & -\alpha/\beta < \lambda_D(t) \leq 0, \\ & p^*(t) = [\alpha + \beta \lambda_D(t)]/2\beta, \quad q_D^*(t) = 0, \\ & \dot{I}_D(t) = -[\alpha - \beta \lambda_D(t)]k I_D(t)/2, \\ & \dot{\lambda}_D(t) = s_D - k(\alpha - \beta \lambda_D(t))^2/4\beta, \end{aligned}$$

$$\begin{aligned}
R3: \quad & 0 < \lambda_D(t) < \alpha/\beta, \\
& p^*(t) = [\alpha + \beta\lambda_D(t)]/2\beta, \quad q_D^*(t) = \lambda_D(t)/c_D, \\
& \dot{I}_D(t) = \lambda_D(t)/c_D - [\alpha - \beta\lambda_D(t)]kI_D(t)/2, \\
& \dot{\lambda}_D(t) = s_D - k(\alpha - \beta\lambda_D(t))^2/4\beta, \\
R4: \quad & \lambda_D(t) \geq \alpha/\beta, \quad p^*(t) = \alpha/\beta, \\
& q_D^*(t) = \lambda_D(t)/c_D, \\
& \dot{I}_D(t) = \lambda_D(t)/c_D, \quad \dot{\lambda}_D(t) = s_D.
\end{aligned}$$

Next we address the synthesis problem: which sequences of regimes that can be candidates for optimality? From (A.4), (13) and (14) we obtain

$$\lambda_D(T) = 0 \Rightarrow q_D^*(T) = 0, \quad p^*(T) = \alpha/2\beta > 0, \quad \dot{I}_D^*(T) = -\alpha k I_D^*(T)/2 < 0,$$

which shows that only R2 can be a terminate path, and the inventory is decreasing at the horizon date. Since the costate λ_D is continuous, only R1 and R3 can precede R2. Thus there are two candidate sequences, each consisting of two regimes: R1 $\rightarrow$ R2 and R3 $\rightarrow$ R2. To see if a regime could precede R1 or R3, consider the costate equation in (16), being valid for both R2 and R3. This Riccati differential equation has the particular (steady-state) solutions

$$\begin{pmatrix} \hat{\lambda}^U \\ \hat{\lambda}^L \end{pmatrix} = \frac{\alpha}{\beta} \pm \frac{1}{\beta} \sqrt{\frac{4s_D\beta}{k}}. \quad (\text{A.7})$$

Noting that

$$\hat{\lambda}^U > \alpha/\beta > 0, \quad \hat{\lambda}^L \begin{pmatrix} > \\ = \\ < \end{pmatrix} 0 \quad \text{for } k\alpha^2/4\beta \begin{pmatrix} > \\ = \\ < \end{pmatrix} s_D \quad (\text{A.8})$$

shows that in R3 only the positive smaller root $\hat{\lambda}^L$ is usable. In R2 only the negative smaller root is usable.

For a phase diagram analysis of R2 and R3, note that in both regimes the costate satisfies

$$\dot{\lambda}_D \begin{pmatrix} > \\ = \\ < \end{pmatrix} 0 \quad \text{for } \lambda_D \begin{pmatrix} > \\ = \\ < \end{pmatrix} \hat{\lambda}^L$$

and for the state variable we have

$$R2: \dot{I}_D < 0, \quad R3: \dot{I}_D \begin{pmatrix} > \\ = \\ < \end{pmatrix} 0 \quad \text{for } \lambda_D \begin{pmatrix} > \\ = \\ < \end{pmatrix} \frac{c_D\alpha k I_D}{2 + c_D\beta k I_D}.$$

In Regimes 2 and 3, the $\dot{\lambda}_D = 0$ isocline equals $\hat{\lambda}^L$. In R2, the $\dot{I}_D = 0$ isocline does not exist since I_D will not be zero. In R3, the $\dot{I}_D = 0$ isocline is a concave and increasing function that starts out at the origin. Each regime has a unique steady-state point

$$\begin{aligned}
R2: \quad & (I_D, \lambda_D) = (0, \hat{\lambda}^L), \quad \hat{\lambda}^L < 0, \\
R3: \quad & (I_D, \lambda_D) = \left(\frac{2\hat{\lambda}^L}{c_D k (\alpha - \beta\hat{\lambda}^L)}, \hat{\lambda}^L \right), \quad 0 < \hat{\lambda}^L < \frac{\alpha}{\beta}.
\end{aligned}$$

It is straightforward to verify that these steady-states are saddle points. In $R1$ it holds that

$$\dot{I}_D < 0 \text{ since } I_D > 0, \quad \dot{\lambda}_D \begin{pmatrix} > \\ = \\ < \end{pmatrix} 0 \quad \text{for } \lambda_D \begin{pmatrix} > \\ = \\ < \end{pmatrix} \frac{-s_D}{\alpha k}$$

and the steady-state $(I_D, \lambda_D) = (0, -s_D/\alpha k)$ is a saddle. Finally, in $R4$ both I_D and λ_D are increasing.

The results in (A.7) and (A.8), and the fact that the comparison between λ_D and the number $-\alpha/\beta$ is crucial for the distinction between $R1$ and $R2$, suggest that we must consider three cases, depending on the size of the holding cost s_D , relative to the parameter k (and α, β).

Case A :

$$k\alpha^2/4\beta < s_D < k\alpha^2/\beta \Rightarrow -\alpha/\beta < \hat{\lambda}^L < 0, \quad -s_D/\alpha k > -\alpha/\beta,$$

Case B :

$$s_D > k\alpha^2/\beta \Rightarrow \hat{\lambda}^L < -\alpha/\beta, \quad -s_D/\alpha k < -\alpha/\beta,$$

Case C :

$$s_D < k\alpha^2/4\beta \Rightarrow \hat{\lambda}^L > 0.$$

Letting a superscript identify regimes we also observe that

$$\dot{\lambda}_D^2(t) |_{\dot{z}_D^2(t)=-\alpha/\beta} = \dot{\lambda}_D^1(t) |_{\dot{z}_D^1(t)=-\alpha/\beta} = s_D - k\alpha^2/\beta$$

and

$$\dot{\lambda}_D^3(t) |_{\dot{z}_D^3(t)=0} = \dot{\lambda}_D^2(t) |_{\dot{z}_D^2(t)=0} = s_D - k\alpha^2/4\beta. \quad (\text{A.9})$$

Now we can answer the question whether sequences $R1 \rightarrow R2$ and $R3 \rightarrow R2$ can be preceded by some regime. Continuity of the costate prohibits that $R1$ is preceded by $R3$ or $R4$ and hence we only need to test the sequence $R2 \rightarrow R1 \rightarrow R2$. This sequence is infeasible: for feasibility $\dot{\lambda}_D(t)$ has to be positive when $R1$ passes into $R2$. This can only happen under the assumption of Case B. But then $R2$ cannot precede $R1$ since this would require $\dot{\lambda}_D(t)$ to be negative. Continuity of the costate prohibits that $R3$ be preceded by $R1$. Thus we only need to test the sequences $R4 \rightarrow R3 \rightarrow R2$ and $R2 \rightarrow R3 \rightarrow R2$. The first one is infeasible since in $R3$ and $R4$ it holds that $\dot{\lambda}_D(t)$, evaluated at $\lambda_D = \alpha/\beta$, equals $s_D > 0$. Hence, motion from $R4$ into $R3$ is impossible. Neither is the second sequence feasible: to pass from $R3$ into $R2$, $\dot{\lambda}_D(t)$ must be negative when $R3$ passes into $R2$. By (A.9) this can only happen under the assumption of Case C. If $R2$ were to precede $R3$ it would require $\dot{\lambda}_D(t)$ to be positive at the time when $R2$ passes into $R3$, but this violates the assumption of Case C.

The conclusion is that the only feasible sequences are the following: $R2$ throughout (occurring in Case A), $R1 \rightarrow R2$ (occurring in Case B), and $R3 \rightarrow R2$ (occurring in Case C). This completes the proof of optimality of the policies in the D problem. Proposition 3 therefore has been proved. $\square$

A.2. Proof of Proposition 1

Clearly $q_N(t) = 0 \forall t \in [0, T]$ yields $I_N(t) = I_{N0} - S(t)$ and, using the assumption of the proposition, we get $I_N(T) \geq 0$. Consider any policy having $q_N(t) > 0$ during $(t_1, t_2) \subseteq [0, T]$. Using this policy has two consequences: It gives a positive ordering cost and leads to a higher level of inventory, and hence a higher holding cost. Although the terminal stock under this policy will be higher than under the no-replenishment

policy, a larger terminal stock will not compensate for the increased costs because the value of the terminal stock is zero. This completes the proof of Proposition 1. $\square$

A.3. Proof of Proposition 2

We shall show that the solution in (11) satisfies optimality conditions (1b) and (A.1) and (A.3)–(A.6). First note that $\dot{I}_N(\tau^-) = -q_D(\tau^-) \leq 0$, $\dot{I}_N(\tau^+) = 0$. Thus, at the entry point to the boundary segment $I_N = 0$ it holds that the time-derivative of I_N is discontinuous, unless $q_D(\tau^-)$ happens to be zero (which is immaterial since we can assign any value to the control at single instant of time without changing the value of the objective function). Then the costate λ_N is continuous at $t = \tau$ (Feichtinger and Hartl, 1986, Corollary 6.3). Consider two subintervals $[0, \tau]$ and $(\tau, T]$.

$[0, \tau]$: Using (A.6) shows that $v(t) = 0$ on the interval $[0, \tau]$. Then (A.3) yields

$$\dot{\lambda}_N(t) = s_N \Rightarrow \lambda_N(t) = \lambda_N(0) + s_N t. \quad (\text{A.10})$$

Integrating in (1b) provides $I_N(t) = I_{N0} - S(t)$ and the instant τ is given by $S(\tau) = I_{N0}$.

$(\tau, T]$: Here the control is singular and in (A.1) we have $\lambda_N = c_N$, implying $\dot{\lambda}_N(t) = 0$. From (A.3) we get $\dot{\lambda}_N(t) = s_N - v(t)$. Using these results shows that $v(t) = s_N > 0$ and, by (A.6), we obtain $I_N(t) = 0$. Since the costate variable λ_N is continuous at $t = \tau$, it holds that

$$\lambda_N(0) + s_N \tau = c_N \Rightarrow \lambda_N(0) = c_N - s_N \tau$$

which inserted into (A.10) yields

$$\lambda_N(t) = c_N + s_N(t - \tau) \quad \text{for } t \in [0, \tau]. \quad (\text{A.11})$$

To satisfy the transversality condition (A.4) there must be a discontinuity in λ_N at time T (since $\lambda_N = c_N$ on the second subinterval). The jump condition (A.5) is satisfied if we set $\gamma = c_N > 0$. This completes the proof of Proposition 2. $\square$

Appendix B. Proof of Propositions 4 and 5

Given (A.4), and the fact that only b -regimes are feasible at $t = T$, the only final regime is $R2b$. Next we check which regime can precede $R2b$. It requires $\dot{I}_N(\tau^-) < 0$ for $R1a$ and $R2a$ to precede $R2b$. For $R1a$ it holds that $\lambda_D(\tau^-) < \lambda_N(\tau^-) = c_N$, implying $q_D(\tau^-) = 0$. Then (1a) yields $\dot{I}_N(\tau^-) > 0$ which violates the requirement. A similar argument applies for $R2a$. Indeed, $R1a$ and $R2a$ cannot pass into any b -regime starting at τ . Continuity of the costate λ_D , and using $\beta c_N < \alpha$, shows that $R4a$ and $R4b$ cannot precede $R2b$. Only $R1b$, $R3a$, and $R3b$ are candidates to precede $R2b$. We test these three candidate sequences to see if they are feasible.

(I) $R1b \rightarrow R2b$: When $R1b$ passes into $R2b$ it must hold that $\dot{\lambda}_D = s_D - k\alpha^2/\beta \geq 0$. The sequence is only feasible in Case B.

(II) $R3a \rightarrow R2b$: When $R3a$ meets $R2b$ it must hold that $\dot{\lambda}_D = s_D - k(\alpha - \beta c_N)^2/4\beta \leq 0$. The sequence is only feasible in Case C1.

(III) $R3b \rightarrow R2b$: The sequence is only feasible in Case C1 for the same reason as in (II).

The next step is to see which regimes can precede the three sequences in I–III, that is, to test for feasibility sequences that consist of three regimes.

(I) $R1b \rightarrow R2b$ cannot be preceded by $R3a$, $R3b$, $R4a$, and $R4b$ since any of these couplings would require the costate λ_D to be discontinuous. Moreover, we have already shown that neither $R1a$ nor $R2a$ can precede a b -regime; hence they cannot precede the sequence $R1b \rightarrow R2b$. The only three-regime sequence

left is $R2b \rightarrow R1b \rightarrow R2b$. When $R2b$ passes into $R1b$ it must hold that $\dot{\lambda}_D = s_D - k\alpha^2/\beta \leq 0$, but this violates the defining condition of Case B under which $R1b \rightarrow R2b$ was the only feasible sequence. The conclusion is that no regime can precede $R1b \rightarrow R2b$. Recalling that we consider the case of Proposition 2, where an optimal trajectory must start with an a -regime, we see that $R1b \rightarrow R2b$ cannot be optimal.

(II) By definition, $R3a \rightarrow R2b$ can only be preceded by an a -regime. $R1a$ would violate continuity of the costate and only $R2a$ and $R4a$ are left. Consider the sequence $R2a \rightarrow R3a \rightarrow R2b$ and note that when $R2a$ meets $R3a$, the inequality $\dot{\lambda}_D = s_D - k(\alpha - \beta c_N)^2/4\beta \geq 0$ must be satisfied. This is, however, incompatible with Case C1. Next consider the sequence $R4a \rightarrow R3a \rightarrow R2b$. Along $R4a$ it holds that $\dot{\lambda}_D = s_D > 0$ and hence the costate cannot decrease. Then $R4a$ cannot precede $R3a$. In conclusion, no regime can precede the sequence $R3a \rightarrow R2b$ which makes this sequence optimal in Case C1.

(III) By continuity of the costate, the sequence $R3b \rightarrow R2b$ cannot be preceded by neither $R1a$ nor $R1b$. On $R4a$ and $R4b$, the costate increases which makes it impossible that $R4a$ and $R4b$ can precede $R3b$. Neither can the sequence be preceded by $R2a$ since we know that this regime cannot precede any b -regime. Thus, we are left with $R2b$ and $R3a$. Hence, consider the sequence $R2b \rightarrow R3b \rightarrow R2b$. When $R2b$ passes into $R3b$ it must hold that $\dot{\lambda}_D = s_D - k(\alpha - \beta c_N)^2/4\beta \geq 0$ which is incompatible with Case C1.

The sequence $R3a \rightarrow R3b \rightarrow R2b$ is feasible and it remains to be checked if there are regimes that can precede this sequence. By the same arguments that proved that no regimes could precede $R3a \rightarrow R2b$ one can show that no regime can precede $R3a \rightarrow R3b \rightarrow R2b$. The conclusion is that $R3a \rightarrow R3b \rightarrow R2b$ is optimal in Case C1. This ends the proof of optimality in problem D for the case in which Proposition 2 applies.

Next we prove optimality in the case that Proposition 1 applies. The only feasible final regime still is $R2$. We check which regimes can precede $R2$.

$R1 \rightarrow R2$: When $R1$ passes into $R2$ it must hold that $\dot{\lambda}_D = s_D - k\alpha^2/\beta \geq 0$. This is only possible in Case B.

$R3 \rightarrow R2$: When $R3$ meets $R2$ it must hold that $\dot{\lambda}_D = s_D - k(\alpha - \beta c_N)^2/4\beta \leq 0$. This is only possible in Case C1.

$R4 \rightarrow R2$: The sequence violates continuity of the costate and is infeasible.

In Cases A and C2 there are no feasible sequences. The conclusion is that $R2$ is optimal for all $t \in [0, T]$. It remains to be checked if a regime can precede $R1 \rightarrow R2$ and $R3 \rightarrow R2$. For the sequence $R2 \rightarrow R1 \rightarrow R2$ it holds that $\dot{\lambda}_D = s_D - k\alpha^2/\beta \leq 0$ at the time when $R2$ passes into $R1$. This is incompatible with Case B. Moreover, neither $R3$ nor $R4$ can precede the sequence $R1 \rightarrow R2$ due to continuity of the costate. Thus, the optimal solution in Case B is $R1 \rightarrow R2$. Next consider $R3 \rightarrow R2$. Due to continuity of the costate, $R1$ cannot precede this sequence. Neither can $R4$ since the costate increases along $R4$. This leaves $R2 \rightarrow R3 \rightarrow R2$ but at the time where $R2$ passes into $R3$ it holds that $\dot{\lambda}_D = s_D - k(\alpha - \beta c_N)^2/4\beta \geq 0$ which is incompatible with Case C1. The conclusion is that the optimal solution in Case C1 is the sequence $R3 \rightarrow R2$. This ends the proof. $\square$

Appendix C. Proof of Proposition 6

As in Feichtinger and Hartl (1986, Lemma 9.1), the following lemma is true for the model where demand does not depend on the stock level. It is intuitive and needs no proof.

Lemma A. If $I_{D0} < \int_0^T [\alpha - \beta p(t)] dt$ then $I_D(T) = 0$.

Using the Lagrangian

$$L = p(\alpha - \beta p) - \frac{c_D}{2} q_D^2 - s_D I_D + \lambda(q_D - (\alpha - \beta p)) + \mu_1 q_D + \mu_2 p + \mu_3 \left(\frac{\alpha}{\beta} - p \right) + v I_D$$