Bridging the gap between a stationary point process and its palm distribution
Nieuwenhuis, G.

Publication date: 1991

Document Version
Publisher's PDF, also known as Version of record

Link to publication in Tilburg University Research Portal

Citation for published version (APA):

General rights
Copyright and moral rights for the publications made accessible in the public portal are retained by the authors and/or other copyright owners and it is a condition of accessing publications that users recognise and abide by the legal requirements associated with these rights.

• Users may download and print one copy of any publication from the public portal for the purpose of private study or research.
• You may not further distribute the material or use it for any profit-making activity or commercial gain.
• You may freely distribute the URL identifying the publication in the public portal.

Take down policy
If you believe that this document breaches copyright please contact us providing details, and we will remove access to the work immediately and investigate your claim.
BRIDGING THE GAP BETWEEN A STATIONARY POINT PROCESS AND ITS PALM DISTRIBUTION

Gert Nieuwenhuis

FEW 502
BRIDGING THE GAP BETWEEN A STATIONARY POINT PROCESS AND ITS PALM DISTRIBUTION

Gert Nieuwenhuis
Tilburg University
Department of Econometrics
P.O. Box 90153
NL-5000 LE Tilburg
The Netherlands

Summary. Let $P$ be the distribution of a stationary point process on the real line and let $P^0$ be its Palm distribution. In this paper we consider probability measures which are equivalent to $P^0$, having simple relations with $P$. Relations between $P$ and $P^0$ are derived with these intermediate measures as bridges. With the resulting Radon-Nikodym derivatives several well-known results can be proved easily. New results are derived. As a corollary of cross ergodic theorems a conditional version of the well-known inversion formula is proved. Several approximations of $P^0$ are considered, for instance the local characterization of $P^0$ as a limit of conditional probability measures $P_{1,n}, n \in \mathbb{N}$. The total variation distance between $P^0$ and $P_{1,n}$ can be expressed in terms of the $P$-distribution function of the forward recurrence time.

AMS 1980 subject classifications. Primary 60G55; secondary 60G10. Key words and phrases. Palm distribution, local characterization, inversion formula, ergodicity.
1 Introduction

In queueing theory it is often wanted to express expectations of time-stationary processes in terms of expectations of customer-stationary sequences. It turns out that the underlying theory for many problems of this type concerns the relationship between two probability measures, the distribution $P$ of a stationary (marked) point process and the Palm distribution $P^0$ (intuitively arising from $P$ by conditioning on the occurrence of a point (with some mark) in the origin). See e.g. Franken et al. (1982) and Baccelli & Brémaud (1987). As an example we mention Little's law (cf. page 41 of the second reference), linking quantities as the mean number of customers in a queueing system and the mean waiting time. The first mean is considered under $P$ and the second under $P^0$. For this reason it is important to obtain a good understanding of the relationship between $P$ and $P^0$.

In this paper we will try to bridge the gap between $P$ and $P^0$. We will confine ourselves to unmarked point processes, although in the final section a generalization to marked point processes is briefly indicated.

The approach in this paper could be called the Radon-Nikodym approach. Several probability measures are considered which are equivalent to $P^0$ (in the sense of mutual domination), having simple relations with $P$. The resulting Radon-Nikodym derivatives are used to express $P^0$-expectations in terms of $P$-expectations (and vice versa).

Some of the results in this paper are also obtained elsewhere by more conventional methods. Usually, however, our approach is faster and more natural, adding some special elements.

The formal definition of the Palm distribution (see (1.3) below) is one possibility to go from $P$ to $P^0$. With the classical inversion formula (see (1.5) below) we can go the other way. We will, however, consider probability measures which are intermediate between $P$ and $P^0$, having simple relations with both.

Examples of such intermediate probability measures are considered in Section 2. They are equivalent to $P^0$ or to $P$ with simple Radon-Nikodym derivatives. The advantages of using these measures as a bridge is illustrated. As a result of this approach some cross
ergodic theorems are proved in Section 3. No ergodicity conditions are assumed here.
As a corollary a conditional version of the well-known inversion formula (1.5) is derived.
Starting from $P$, some strong or pointwise approximations of $P^0$ are considered in Section 4. For these approximations to hold necessary and sufficient conditions are formulated.
For this purpose a notion weaker than ergodicity of the point process is introduced. Some other intermediate probability measures, all equivalent to $P^0$, are considered. The well-known (and intuitively clear) uniform approximation of $P^0$ by conditional probability measures $P_{1,n}$, usually referred to as local characterization of the Palm distribution (cf., e.g., Franken et al. (1982; Th. 1.3.7)), is also considered. We derive a very simple expression for the total variation distance of $P^0$ and $P_{1,n}$. Conditions are given such that the rate of the resulting uniform convergence is of order $1/n$. In Section 5 a generalization to marked point processes is briefly indicated.

At the end of this section we formalize some of the notions mentioned above and give some other definitions and notations.
A point process on $\mathbb{R}$ is a random element $\Phi$ in the class $M$ of all integer-valued measures $\varphi$ on the $\sigma$-field $\text{Bor} \ \mathbb{R}$ of Borel sets on $\mathbb{R}$ for which

$$\varphi(B) < \infty \text{ for all bounded } B \in \text{Bor} \ \mathbb{R}.$$ 

Let $\mathcal{M}$ be the $\sigma$-field generated by the sets $[\varphi(B) = k] := \{\varphi \in M : \varphi(B) = k\}, k \in \mathbb{N}_0$ and $B \in \text{Bor} \ \mathbb{R}$. See Matthes, Kerstan & Mecke (1978), Kallenberg (1983/86) or Daley & Vere-Jones (1988) for more information. Set

$$M^\infty := \{\varphi \in M : \varphi(-\infty, 0) = \varphi(0, \infty) = \infty; \ \varphi\{x\} \leq 1 \text{ for all } x \in \mathbb{R}\},$$

$$\mathcal{M}^\infty := M^\infty \cap \mathcal{M}.$$ 

We will always assume that $\Phi$ (or rather its distribution $P$) is stationary (i.e., $\Phi(t+\cdot) =_d \Phi$ for all $t \in \mathbb{R}$). We also assume that $\Phi \neq 0$ w.p.1, that $\Phi$ is simple and that the intensity $\lambda$ is finite; or, equivalently,
\[ P(M^\infty) = 1 \quad \text{and} \quad \lambda := \mathbb{E}\Phi(0,1) < \infty. \] (1.1)

The atoms of \( \varphi \in M^\infty \) are denoted by \( X_i(\varphi), \ i \in \mathbb{Z}, \) with the convention that

\[ \ldots X_{-1}(\varphi) < X_0(\varphi) \leq 0 < X_1(\varphi) < X_2(\varphi) < \ldots . \]

We interpret \( X_i(\varphi) \) as the time of the \( i \)th arrival (or point) and \( \alpha_i(\varphi) := X_{i+1}(\varphi) - X_i(\varphi) \) as the \( i \)th interarrival time (or interval length). We have \( \Phi(B) := \# \{ i \in \mathbb{Z} : X_i \in B \} \) and \( [\alpha_i \in B] := [\alpha_i(\varphi) \in B] := \{ \varphi \in M^{\infty} : \alpha_i(\varphi) \in B \}, \ B \in \text{Bor} \mathbb{R}. \)

For \( t \in \mathbb{R} \) the time shift \( T_t : M \rightarrow M \) is defined by \( T_t \varphi := \varphi(t + \cdot), \ \varphi \in M. \) By stationarity it is obvious that these mappings are measure preserving under \( P. \) The atoms of \( T_t \varphi \) are \( X_i(\varphi) - t, \ i \in \mathbb{Z}. \) For \( n \in \mathbb{Z} \) the point shift \( \theta_n : M^\infty \rightarrow M^\infty \) is defined by \( \theta_n \varphi := \varphi(X_n(\varphi) + \cdot), \ \varphi \in M^\infty. \) Note that \( \theta_n(\theta_1 \varphi) = \theta_{n+1} \varphi. \)

A random sequence \( (\xi_i) := (\xi_i)_{i \in \mathbb{Z}} \) with \( \xi_i : M^\infty \rightarrow \mathbb{R} \) is generated by the point shift \( \theta_1 \) if \( \xi_n(\theta_1 \varphi) = \xi_{n+1} \varphi \) for all \( \varphi \in M^\infty \) and \( n \in \mathbb{Z}. \) See also Nieuwenhuis (1989; p. 600). Examples of such sequences are \( (\alpha_i) \) and \( (1_A \circ \theta_i), \ A \in \mathcal{M}^\infty. \) The general form is \( (f \circ \theta_i), \ f : M^\infty \rightarrow M^\infty \) measurable.

The distribution \( P_n \) of \( \theta_n \Phi \) plays an important role in this paper. It arises from \( P \) by shifting the origin to the \( n \)th arrival.

\[ P_n := P \theta_n^{-1}, \ n \in \mathbb{Z}. \] (1.2)

We now consider the Palm distribution \( P^0 \) of \( \Phi. \) An intuitive definition of \( P^0 \) was stated before. The formal definition of the Palm distribution \( P^0 \) is

\[ P^0(A) := \frac{1}{\lambda} \mathbb{E} \left[ \sum_{i=1}^{\Phi(0,1)} 1_A(\theta_i \Phi) \right], \ A \in \mathcal{M}^\infty. \] (1.3)

Set \( M^0 := \{ \varphi \in M^\infty : \varphi(0) = 1 \} \) and \( \mathcal{M}^0 := M^0 \cap \mathcal{M}. \) It is obvious that \( P^0 \) is a probability measure on \( (M^\infty, \mathcal{M}^\infty) \) with \( P^0(M^0) = 1. \) Note also that \( P^0[\alpha_0 = 0] = 0 \) by
According to Franken et al. (1982; Th. 1.2.7) $P^0$ has the following important property:

$$P^0 = P^0\theta_n^{-1} \quad \text{for all } n \in \mathbb{Z}. \quad (1.4)$$

Consequently, any sequence $(\xi_i)$ generated by $\theta_1$ is $P^0$-stationary, i.e., $(\xi_1, \ldots, \xi_n)$ and $(\xi_{k+1}, \ldots, \xi_{k+n})$ have the same distribution under $P^0$, all $n \in \mathbb{N}$ and $k \in \mathbb{Z}$. Particularly, $(\alpha_i)$ is $P^0$-stationary.

Definition (1.3) allows us to express $P^0$ in terms of $P$. The following inversion formula expresses $P$ in terms of $P^0$ (cf. Franken et al. (1982; p. 27)).

$$P(A) = \lambda \int_0^\infty P^0[X_1(\varphi) > u; \varphi(u + \cdot) \in A] du, \quad A \in \mathcal{M}. \quad (1.5)$$

Substituting $A = M$ yields

$$E^0\alpha_0 = \frac{1}{\lambda}. \quad (1.6)$$

For $\varphi \in M$ we define

$$N(t, \varphi) := N_t(\varphi) := \begin{cases} 
\varphi(0, t] & \text{if } t \geq 0 \\
-\varphi(t, 0] & \text{if } t < 0.
\end{cases} \quad (1.7)$$

We will sometimes write $N(t)$ instead of $N_t$.

The total variation distance $d$ between two probability measures $Q_1$ and $Q_2$ on a common probability space, both dominated by a $\sigma$-finite measure $\mu$ and having densities $h_1$ and $h_2$ respectively, is defined by

$$d(Q_1, Q_2) := \int |h_1 - h_2| d\mu. \quad (1.8)$$

It is well-known that
\[ d(Q_1, Q_2) = 2 \sup_{A} |Q_1(A) - Q_2(A)| = 2(Q_1[h_1 \geq h_2] - Q_2[h_1 \geq h_2]). \tag{1.9} \]

Expectations with respect to the probability measures \( P, P_n \) and \( P^0 \), all considered on \((M^\infty, M^\infty)\), are denoted by \( E, E_n \) and \( E^0 \), respectively. In particular the distinction between \( P_0 \) and \( P^0 \) and between \( E_0 \) and \( E^0 \) should be noted. Expectation with respect to an universal probability space \((\Omega, \mathcal{F}, \mathbf{P})\) is denoted by \( \mathbf{E} \).

Let \( Q_1 \) and \( Q_2 \) be probability measures on a common probability space. We say that \( Q_1 \) is dominated by \( Q_2 \) (notation \( Q_1 \ll Q_2 \)) if the \( Q_2 \)-null-sets are also \( Q_1 \)-null-sets. A Radon-Nikodym derivative of \( Q_1 \) with respect to \( Q_2 \) will be denoted by \( \frac{dQ_1}{dQ_2} \). The supplement \( Q_2 \)-almost surely will usually be suppressed. \( Q_1 \) and \( Q_2 \) are equivalent (notation \( Q_1 \sim Q_2 \)) if an event \( A \) exists such that \( Q_1(A) = 0 \) and \( Q_2(A) = 1 \).

Independence is denoted by \( \perp \) and Lebesgue measure on \( \mathbb{R} \) by \( \text{Leb} \). Random variable is abbreviated to \( \text{rv} \) and almost surely to \( \text{as} \).

## 2 Intermediate probability measures

Although \( P \) and \( P^0 \) are mutually singular, shifts of \( P \) are equivalent to \( P^0 \) and have simple Radon-Nikodym derivatives. We collect formulas and conclusions that follow from this observation.

In (1.2) the probability measures \( P_n, n \in \mathbb{Z} \), were introduced. They can be considered as intermediate between \( P \) and \( P^0 \) since they have simple relations with both. \( P_n \) is related to \( P \) in a simple way because of its definition. The relationship to \( P^0 \) follows from the following theorem (see Nieuwenhuis (1989; Th. 2.1)).

**Theorem 2.1.** Let \( n \in \mathbb{Z} \). Then

(i) \( P_n \sim P^0 \),

(ii) \( \rho_n(\varphi) := \lambda_{\alpha_n}(\varphi), \varphi \in M^0 \), defines a Radon-Nikodym derivative of \( P_n \) with respect to \( P^0 \).
Since $P_n$ and $P^0$ apparently have the same null-sets, it is clear that convergence wp1 (just as convergence in probability) holds equivalently under both probability measures. This observation leads immediately to some cross ergodic theorems. See Section 3. In Nieuwenhuis (1989) Theorem 2.1 was applied to prove (under some mixing condition) the equivalence of a special type of functional central limit theorems under $P$ and $P^0$. The relation in (ii) can serve as a tool for transforming formulas involving $P$ into formulas involving $P^0$ and vice versa. We will give here some examples.

Suppose that $f : M^0 \to \mathbb{R}$ is $P^0$-integrable. Since

$$E^0 f = \frac{1}{\lambda} E_n \left( \frac{1}{\alpha_{-n}} f \right)$$

and $\alpha_{-n} \circ \theta_n = \alpha_0$,

we have

$$E^0 f = \frac{1}{\lambda} E \left( \frac{1}{\alpha_0} f \circ \theta_n \right), \quad n \in \mathbb{Z}. \quad (2.1)$$

This relation expresses $P^0$-expectations in terms of $P$-expectations and may be an alternative to (1.3). For a $P$-integrable function $g : M^\infty \to \mathbb{R}$ with $Eg = Eg \circ \theta_0$ it follows immediately from Theorem 2.1(ii) that

$$Eg = \lambda E^0(\alpha_0 g). \quad (2.2)$$

This relation is the counterpart of (2.1). If $Eg = Eg \circ \theta_0$ it is just a reformulation of (1.5), since by (1.5) and Fubini's theorem

$$Eg \circ \theta_0 = \lambda \int_0^\infty E^0 \left( 1_{[\alpha_0 > u]} g \circ \theta_0 \circ T_u \right) du \quad (2.3)$$

$$= \lambda E^0 \left( \int_{\alpha_0}^{\infty} g \circ \theta_0 \circ T_u du \right) = \lambda E^0(\alpha_0 g).$$

(In the last equality it was used that $\theta_0(T_u \varphi) = \varphi$ for all $\varphi \in M^0$ and $u \in (0, \alpha_0(\varphi))$.)
The formulation in (2.2) is of special interest when $g$ is a function of some sequence generated by $\theta_1$.

To illustrate the simplicity of this Radon-Nikodym approach we will derive some short results here. The well-known relation (1.6) can be obtained by (2.1) by choosing $n = 0$ and $f = \alpha_0$. Other formulas on $(\alpha_i)$ can be obtained by making simple choices for $f$, $g$ and $n$, or (probably even faster) by applying Theorem 2.1 directly.

\[
E \frac{1}{\alpha_0} = \lambda, \\
E \alpha_k = \lambda \mathcal{E}^0(\alpha_0 \alpha_k) = E^0 \alpha_0 + \operatorname{cov}_{\mathcal{P}_0}(\alpha_0, \alpha_k)/E^0 \alpha_0, \quad k \in \mathbb{Z},
\]

(2.4)

(cf. Cox & Lewis (1966; (4.28)) and McFadden (1962; (3.12)),

\[
E \frac{\alpha_k}{\alpha_0} = 1, \quad E \alpha_k = E \alpha_{-k}, \quad E \alpha_k \alpha_n = E \alpha_{-k} \alpha_{n-k}, \quad k, n \in \mathbb{Z}.
\]

Let $n \in \mathbb{N}_0$. If the $P^0$-distribution of $(\alpha_0, \ldots, \alpha_n)$ is dominated by Lebesgue measure with density $f_n$, then the $P$-distribution of $(\alpha_0, \ldots, \alpha_n)$ is also dominated by Lebesgue measure, with density $g_n$ defined by

\[
g_n(x_0, \ldots, x_n) = \lambda x_0 f_n(x_0, \ldots, x_n), \quad x_0, \ldots, x_n \in (0, \infty).
\]

(2.5)

This relation holds since for $y_0, \ldots, y_n \in (0, \infty)$, $A := [\alpha_0 \leq y_0, \ldots, \alpha_n \leq y_n]$, and $B := X^n_{i=0}(0, y_i)$ we have

\[
P(A) = P_0(A) = \lambda \mathcal{E}^0(\alpha_0 1_A) = \lambda \int_B x_0 f_n(x_0, \ldots, x_n) dx_n \ldots dx_0.
\]

In Cox & Lewis (1966; p. 61) Relation (2.5) is proved by heuristic arguments.

Since $X_1(T_0 \varphi) = \alpha_0(\varphi) - u$ for all $\varphi \in M^0$ and $u \in (0, \alpha_0(\varphi))$, we have by (1.5) and (2.1) that

\[
E \left[1_{[\alpha_0 \in B]}E(g(X_1)|\alpha_0)\right] = E \left(1_{[\alpha_0 \in B]}(g(X_1))\right)
\]
for all $B \in \text{Bor } \mathbb{R}^+$ and $g : \mathbb{R}^+ \to \mathbb{R}$ such that $E|g(X_1)| < \infty$. Consequently,

$$
\text{the conditional } P\text{-distribution of } X_1 \text{ given } \alpha_0 \text{ is } U(0, \alpha_0). \quad (2.6)
$$

This well-known result will be applied next. By (2.6), Fubini’s theorem and Theorem 2.1 we obtain

$$
P[X_1 \leq x] = E P[X_1 \leq x | \alpha_0] = E \left[ \frac{1}{\alpha_0} \int_0^x 1_{(0, \alpha_0)}(s) ds \right]$$

$$= \int_0^x E \left( \frac{1}{\alpha_0} 1_{[\alpha_0 > s]} \right) ds = \lambda \int_0^x P^0[\alpha_0 > s] ds.
$$

It follows immediately that

$$
P_{X_1} \ll \text{Leb and } \frac{dP_{X_1}}{d\text{Leb}}(s) = \lambda P^0[\alpha_0 > s] \text{ Leb ae.} \quad (2.7)
$$

Relation (2.7) can also be derived from (1.2.21) in Franken et al. (1982).

The following result will be applied in Section 4. By (2.7) and Fubini’s theorem we have

$$
P[X_1 > t] = \lambda \int_t^\infty P^0[\alpha_0 > s] ds = \lambda E^0 \left[(\alpha_0 - t) 1_{[\alpha_0 > t]} \right], \quad t \in [0, \infty).
$$

By Theorem 2.1 we obtain for $t \in [0, \infty)$:

$$
P[X_1 > t] = P[\alpha_0 > t] - \lambda t P^0[\alpha_0 > t], \quad (2.8)
$$

$$
P[X_1 \leq t] = P[\alpha_0 \leq t] + \lambda t P^0[\alpha_0 > t]. \quad (2.9)
$$
We will prove another corollary of Theorem 2.1 which will be useful in Section 3. Let $\mathcal{I}$ be the invariant $\sigma$-field under the point shift $\theta_1$, i.e.,

$$\mathcal{I} := \{ A \in \mathcal{M}^\infty : \theta_1^{-1} A = A \}. \quad (2.10)$$

Note that $P(A) = P_1(A)$ for all $A \in \mathcal{I}$. Hence,

$$P \big|_\mathcal{I} \sim P^0 \big|_\mathcal{I}. \quad (2.11)$$

In Baccelli & Brémaud (1987; p. 28) Relation (2.11) is proved directly from the definition of $P^0$. We need, however, expressions for the Radon-Nikodym derivatives. For $A \in \mathcal{I}$ we have

$$P^0(A) = \frac{1}{\lambda} E_1 \left( \frac{1}{\alpha_1} 1_A \right) = \frac{1}{\lambda} E \left( \frac{1}{\alpha_0} 1_{\theta_1^{-1} A} \right) = \frac{1}{\lambda} E \left[ 1_A E \left( \frac{1}{\alpha_0} I \right) \right],$$

$$P(A) = P_1(A) = \lambda E^0(\alpha_1 1_A) = \lambda E^0(\alpha_0 1_A) = \lambda E^0[1_A E^0(\alpha_0 I)].$$

Hence,

$$\frac{dP \big|_\mathcal{I}}{dP^0 \big|_\mathcal{I}} = \lambda E^0(\alpha_0 I) \quad \text{and} \quad \frac{dP^0 \big|_\mathcal{I}}{dP \big|_\mathcal{I}} = \frac{1}{\lambda} E(1_\mathcal{I}). \quad (2.12)$$

Another probability measure on $(\mathcal{M}^\infty, \mathcal{M}^\infty)$ which is in some sense intermediate between $P$ and $P^0$ is the measure $P'$ defined by

$$P'(A) := \frac{1}{\lambda} E \left( \frac{1}{\alpha_0} 1_A \right), \quad A \in \mathcal{M}^\infty. \quad (2.13)$$

Note that $P'$ is indeed a probability measure (see (2.4)), that $P' \perp P^0$, and that

$$P' \sim P \quad \text{with} \quad \frac{dP}{dP'} = \lambda \alpha_0. \quad (2.14)$$
By (2.13) and Theorem 2.1 we obtain for $A \in \mathcal{M}^\infty$ and $n \in \mathbb{Z}$ that

$$P'(\theta_n^{-1} A) = \frac{1}{\lambda} E \left( \frac{1}{\alpha_0} 1_A(\theta_n \cdot) \right) = \frac{1}{\lambda} E_0 \left( \frac{1}{\alpha_0} 1_A(\theta_n \cdot) \right) = E_0(1_A(\theta_n \cdot)) = P^0(A).$$

Consequently,

$$P'\theta_n^{-1} = P^0, \quad n \in \mathbb{Z}. \quad (2.15)$$

This relation implies that random sequences on $M^\infty$ generated by $\theta_1$ are not only $P^0$-stationary but also $P'$-stationary. If $\Phi$ is a renewal process, then the sequence $(\alpha_i)$ is both iid under $P^0$ and under $P'$ (note that the $P'$- and the $P'\theta_0^{-1}$-distribution of $(\alpha_1, \ldots, \alpha_n)$ are the same).

The following diagram comprises some of the above results.

```
P ~ P'  
\theta_0 \downarrow \quad \downarrow \quad \theta_0  
P_0 ~ P^0
```

The Radon-Nikodym derivative of $P$ with respect to $P'$ is not affected by applying $\theta_0$ (cf. Theorem 2.1 and (2.14)). By (2.15) and the above diagram it is obvious that the position of $P'$ as intermediate probability measure between $P$ and $P^0$ is similar to the position of $P_0$. Relations (2.1) and (2.2) can also be derived with $P'$. In Nieuwenhuis (1989; Th. 7.4) the measure $P'$ has been used to prove that a functional central limit theorem holds equivalently under $P$ and $P^0$.

In Section 4 some other intermediate probability measures will be considered.

## 3 Cross ergodic theorems

Birkhoff's ergodic theorem holds for stationary sequences. Although sequences $(\xi_i)$ (generated by $\theta_1$) and $(\Phi(i-1, i])$ are usually not stationary under $P$ and $P^0$ respectively, we
can derive strong laws also under these probability measures. In literature these so-called cross ergodic theorems are usually formulated under ergodicity conditions, see Franken et al. (1982; Th. 1.3.12), Baccelli & Brémaud (1987; p. 29/30), Rolski (1981; § 3.3).

By applying Theorem 2.1 we can give simple proofs for more general results without assuming ergodicity.

We need some preliminaries first. Set

\[ I' := \{ A \in M^\infty : T_i^{-1}A = A \text{ for all } t \in \mathbb{R} \} \]  

(3.1)

and recall the definition of \( I \) in (2.10). For \( A \in I' \) we have \( \varphi \in A \) iff \( T_i \varphi \in A \) for all \( t \in \mathbb{R} \). Consequently, \( \varphi \in A \) iff \( \theta_i \varphi \in A \). So, \( A = \theta_i^{-1}A \) and \( I' \subset I \).

A stationary point process \( \Phi \) (or its distribution \( P \)) with \( P(M^\infty) = 1 \) is called \emph{ergodic} if \( P(A) \in \{0, 1\} \) for all \( A \in I' \) or, equivalently, if \( E(f|I') = Ef P \)-as for all \( f : M^\infty \to \mathbb{R} \) with \( E|f| < \infty \). \( P^0 \) is called \emph{ergodic} if \( P^0(A) \in \{0, 1\} \) for all \( A \in I \) or, equivalently, if \( E^0(g|I) = E^0g \) \( P^0 \)-as for all \( g : M^0 \to \mathbb{R} \) with \( E^0|g| < \infty \).

Recall the definition of \( N_t \) in (1.7). Let \( g : M^\infty \to \mathbb{R} \) be \( P \)-integrable. By ergodic type theorems we have

\[ P \left[ \frac{1}{t} N_t \to E(N_1|I') \right] = 1, \]  

(3.2)

\[ P \left[ \frac{1}{t} \int_0^t g \circ T_s ds \to E(g|I') \right] = 1, \]  

(3.3)

and, if \((\xi_i)\) is \( P^0 \)-stationary (in particular if \((\xi_i)\) is generated by \( \theta_1 \)) and \( E^0|\xi_0| < \infty \),

\[ P^0 \left[ \frac{1}{n} \sum_{i=1}^n \xi_i \to E^0(\xi_0|I) \right] = 1. \]  

(3.4)

Set \( U := E^0(\alpha_0|I) \) and \( V' := E(N_1|I') \). In the proof of the next theorem it will be used repeatedly that any \( I \) (or \( I' \)) measurable function \( h : M^\infty \to \mathbb{R} \) satisfies \( h \circ \theta_i = h \) for all \( i \in \mathbb{Z} \).
Theorem 3.1.

(a) If \((\xi_i)\) is generated by \(\theta_1\) and \(E^0|\xi_0| < \infty\), then (3.4) holds as well with \(P\) instead of \(P^0\).

(b) Relations (3.2) and (3.3) hold as well with \(P^0\) instead of \(P\).

Proof. Since \(P_0 \sim P^0\), Relation (3.4) holds with \(P_0\) as well. Part (a) follows immediately.

For (b), consider

\[
P\left[ \frac{1}{t} N_t \to V' \right] = P\left[ \frac{1}{t} \varphi(X_1(\varphi), X_1(\varphi) + t] \to V'(\varphi) \right]
= P\left[ \frac{1}{t} \varphi(0, X_1(\varphi) + t] \to V'(\varphi) \right]
= P\left[ \frac{\varphi(0, X_1(\varphi) + t]}{X_1(\varphi) + t} \cdot \frac{X_1(\varphi) + t}{t} \to V'(\varphi) \right] = 1.
\]

Since \(P_1 \sim P^0\), the first part of (b) follows. For \(\varphi \in M^\infty\) we have

\[
\frac{1}{t} \int_0^t g(T_s(\theta_0\varphi)) ds = \frac{1}{t} \int_{X_0(\varphi)}^{t + X_0(\varphi)} g(T_s\varphi) ds.
\]

By this observation it is obvious that (3.3) is also valid with \(P_0\) and thus with \(P^0\). \(\Box\)

Remarks. It is easy to prove that the events in (3.2)-(3.4) are elements of \(\mathcal{I}\). This observation, combined with (2.11), leads to another proof of Theorem 3.1 (see Baccelli & Brémaud (1987; p. 29/30) for the ergodic case).

Application of Theorem 3.1(a) with \(\xi_i = g \circ \theta_i\) for \(P^0\)-integrable functions \(g : M^0 \to \mathbb{R}\) yields:

\[
\frac{1}{n} \sum_{i=1}^{n} g \circ \theta_i \to E^0(g|\mathcal{I}) \quad P^0-\text{and} \quad P-\text{as}.
\]

See also Franken et al. (1982; (1.3.18)) for the ergodic case.
By conditioning on $\mathcal{I}$ we obtain

$$E^0 \left[ \alpha_0 1_{U=0} \right] = E^0 \left[ U 1_{U=0} \right] = 0.$$

Since $P^0[\alpha_0 = 0] = 0$, we have (apply Theorem 2.1)

$$U > 0 \quad P^0-\text{and } P\text{-as.} \quad (3.5)$$

Application of Theorem 3.1(a) with $\xi_i = \alpha_i$ yields

$$1 = P \left[ \frac{1}{n} X_n \rightarrow U \right] = P \left[ \frac{1}{N_t} X_{N_t} \rightarrow U \right] = P \left[ \frac{1}{t} N_t \rightarrow \frac{1}{U} \right]. \quad (3.6)$$

(The last equality holds since

$$\frac{1}{N_t(\varphi)} X_{N_t(\varphi)}(\varphi) \leq \frac{t}{N_t(\varphi)} \leq \frac{1}{N_t(\varphi)} X_{N_t(\varphi)+1}(\varphi)$$

for all $\varphi \in M^\infty$ with $N_t(\varphi) > 0$. Use (3.5).) By (3.5), (2.12), (3.2) and Theorem 2.1(i) we have

$$E \left( \frac{1}{\alpha_0} | \mathcal{I} \right) = \frac{1}{E^0(\alpha_0 | \mathcal{I})} = E(N_1 | \mathcal{I}') \quad P^0-\text{and } P\text{-as.} \quad (3.7)$$

Note the resemblance between (3.7) and Relations (1.6) and (2.4).

A similar almost sure limit result for

$$I(t) := \frac{1}{t} \int_0^t g \circ T_s ds$$

can be derived directly from (a) and the first part of (b). $I(t)$ can be decomposed as follows:

$$\frac{N(t)}{t} \frac{1}{N(t)} \sum_{i=1}^{N(t)} \int_{X_{i-1}}^{X_i} g \circ T_s ds + \frac{1}{t} \int_{X_N(t)}^{t} g \circ T_s ds - \frac{1}{t} \int_0^0 g \circ T_s ds. \quad (3.8)$$
Note that the sequences \((f_{X_{i-1}} g \circ T_s ds)\) and \((f_{X_{i-1}} |g \circ T_s| ds)\) are both generated by \(\theta_1\).

By Theorem 3.1(a), the first part of (b) and (3.7) we have

\[
\left| \frac{1}{t} \int_{X_{N(t)}}^{t} g \circ T_s ds \right| \leq \frac{N(t) + 1}{t} \frac{1}{N(t) + 1} \int_{X_{N(t)}}^{X_{N(t)+1}} |g \circ T_s| ds \to 0 \quad \text{as} \ t \to \infty,
\]

\(P^0\)- and \(P\)-as, and

\[
I(t) \to \frac{1}{E^0(\alpha_0|I)} E^0\left( \int_{0}^{\alpha_0} g \circ T_s ds |I \right) \quad P^0\text{- and } P\text{-as.} \tag{3.9}
\]

Combining the limit results in (3.9) and the second part of Theorem 3.1(b) yields

\[
E(g|I') = \frac{1}{E^0(\alpha_0|I)} E^0\left( \int_{0}^{\alpha_0} g \circ T_s ds |I \right) \quad P^0\text{- and } P\text{-as.} \tag{3.10}
\]

This relation is a conditional version of the inversion formula (1.5) (replace \(1_A\) in (1.5) by \(g\) and apply Fubini's theorem). Conditional versions of (2.1) and (2.2) can be derived from (3.10). For \(f : M^0 \to \mathbb{R}\) with \(E^0[f] < \infty\) we have

\[
E^0(f|I) = E^0\left( \frac{1}{\alpha_0} \int_{0}^{\alpha_0} f \circ \theta_0 \circ T_s ds |I \right) \quad P^0, \text{ P-as.}
\]

By (3.10) and (3.7) we obtain (take \(g = f \circ \theta_0/\alpha_0\))

\[
E^0(f|I) = \frac{1}{E(N_1|I')} \left( \frac{1}{\alpha_0} f \circ \theta_0 |I' \right) \quad P^0, \text{ P-as.} \tag{3.11}
\]

If \(g : M^\infty \to \mathbb{R}\) is such that \(E|g| < \infty\) and \(E(g|I') = E(g \circ \theta_0|I')\) \(P\)-as, then (3.10) implies

\[
E(g|I') = \frac{E^0(\alpha_0 g|I)}{E^0(\alpha_0|I)} \quad P^0\text{- and } P\text{-as.} \tag{3.12}
\]

By Relation (3.11) it can easily be proved that \(P\)-ergodicity implies \(P^0\)-ergodicity, since
\[ E^0(f|\mathcal{I}) = \frac{1}{\lambda} E\left( \frac{1}{\alpha_0} f \circ \theta_0 \right) = E^0 f \quad \text{Po.- and P.-as} \]

for any \( P^0 \)-integrable \( f : M^0 \to \mathbb{R} \), provided that \( P \) is ergodic. Since \( P |_I \sim P^0 |_I \) (see (2.11)) and \( I' \subset I \), this implication may also be reversed. With this uncommon proof we have established the following well-known result (cf. e.g. Franken et al. (1982; Th. 1.3.9) or Baccelli & Brémaud (1987; p. 28/29)):

\[ P \text{ is ergodic} \iff P^0 \text{ is ergodic.} \tag{3.13} \]

The choice \( g = 1/\alpha_0 \) in (3.9) yields

\[ \frac{1}{t} \int_0^t \frac{1}{\alpha_0 \circ T_s} ds \to \lambda \quad \text{Po.- and P.-as,} \tag{3.14} \]

provided that \( E^0(\alpha_0|\mathcal{I}) = 1/\lambda \ P^0 \text{-as.} \) This condition is weaker than ergodicity of \( \Phi \); see also Section 4.

4 Approximations of \( P^0 \)

In this section we will consider several expressions tending in some sense to \( P^0 \) as \( n \to \infty \). For this purpose a notion is introduced which is weaker than ergodicity of \( \Phi \). Several new intermediate probability measures are defined, all equivalent to \( P^0 \). The corresponding Radon-Nikodym derivatives are used to approximate \( P^0 \) starting from \( P \).

The following theorem is a generalization of Franken et al. (1982; (1.3.20)). See also Matthes, Kerstan & Mecke (1978; Th. 9.4.5) and Miyazawa (1977; Th. 3.2').

**Theorem 4.1.** The following statements are equivalent:

(i) \( P^0[\alpha_0(\varphi) \in B \land \theta_n \varphi \in A] \to P^0[\alpha_0 \in B]\ P^0(A) \) for all \( B \in \text{BorR}^+ \) and \( A \in \mathcal{M}^0 \),

(ii) \( P[\alpha_0(\varphi) \in B \land \theta_n \varphi \in A] \to P[\alpha_0 \in B]\ P^0(A) \) for all \( B \in \text{BorR}^+ \) and \( A \in \mathcal{M}^0 \).

**Proof.** Assume (i). For all \( B \in \text{BorR}^+ \) and \( A \in \mathcal{M}^0 \) we have (cf. Theorem 2.1)
\[
P[\alpha_0 \in B] \cap [\theta_n \varphi \in A] = \lambda E^0 \left[ \alpha_0 1_{[\alpha_0 \in B]} 1_{[\theta_n \varphi \in A]} \right]
\]

\[
= \lambda \int_0^\infty P^0[\alpha_0 > x] \cap [\alpha_0 \in B] \cap [\theta_n \varphi \in A] dx
\]

\[
\to \lambda \int_0^\infty P^0[\alpha_0 > x \text{ and } \alpha_0 \in B] dx P^0(A) \text{ as } n \to \infty
\]

because of (i) and dominated convergence. This limit is equal to

\[
\lambda E^0 \left[ \alpha_0 1_{[\alpha_0 \in B]} \right] P^0(A) = P[\alpha_0 \in B] P^0(A),
\]

which proves (ii). The implication (ii) \implies (i) can be proved the same way. \qed

Hypothesis (i) is weaker than the mixing (ergodic-sense) property for \( P^0 \) (cf. e.g. Franken et al. (1982; p. 37)); hypothesis (ii) could equivalently be formulated as (cf. Nieuwenhuis (1989; Section 5))

\[
P_n = P[\theta_n \varphi \in \cdot] \to P^0 \text{ pointwise, independently of } \sigma(\alpha_0).
\]

Next we consider strong approximation of \( P^0 \). For \( n \in \mathbb{N} \) the empirical distribution \( \hat{P}_n \) is defined by

\[
\hat{P}_n(A, \varphi) := \frac{1}{n} \sum_{i=1}^n 1_A(\theta_i \varphi), \ A \in \mathcal{M}^\infty \text{ and } \varphi \in M^\infty.
\]

Since the sequence \( (1_A \circ \theta_i) \) is generated by \( \theta_1 \), we obtain by (3.4) and Theorem 3.1(a) that

\[
\hat{P}_n(A) \to E^0(1_A|T) \text{ P^0- and P-as.}
\]

Note that for each \( \varphi \in M^\infty \) \( \hat{P}_n(\cdot, \varphi) \) is a probability measure on \((M^\infty, \mathcal{M}^\infty)\) and that \( \hat{P}_n(A) \) is a \( P^0 \)-unbiased estimator of \( P^0(A) \). The next statement follows immediately from (3.13) and (4.3). It characterizes strong approximation of \( P^0 \) by \( \hat{P}_n \) under \( P \).
\( \Phi \) is ergodic iff \( \hat{P}_n(A) \to P^0(A) \) \( P \)-as for all \( A \in \mathcal{M}^\infty \). \tag{4.4}

Starting with (4.3) under \( P \) we obtain:

\[
E\hat{P}_n(A) = \frac{1}{n} \sum_{i=1}^{n} P_i(A) \to E[E^0(1_A[I])] = Q^0(A), \ A \in \mathcal{M}^\infty.
\] \tag{4.5}

\( Q^0 \) is a probability measure on \((\mathcal{M}^\infty, \mathcal{M}^\infty)\) having \( Q^0(M^0) = 1 \), since \( E^0(1_{M^0}|I) = 1 \) \( P^0 \)- and \( P \)-as (cf. Th. 2.1(i)).

**Lemma 4.2.** \( Q^0 \) and \( P^0 \) are equivalent. The Radon-Nikodym derivative of \( Q^0 \) with respect to \( P^0 \) is:

\[
\frac{dQ^0}{dP^0} = \lambda E^0(\alpha_0|I).
\] \tag{4.6}

**Proof.** By Theorem 2.1 we have:

\[
Q^0(A) = \lambda E^0(\alpha_0 E^0(1_A[I])) = \lambda E^0[E^0(\alpha_0|I)E^0(1_A[I])]
\]

\[
= \lambda E^0[E^0(1_A E^0(\alpha_0|I)|I)] = \lambda E^0[1_A E^0(\alpha_0|I)].
\]

In the second equality we conditioned on \( I \). Since \( P^0[E^0(\alpha_0|I) = 0] = 0 \) by (3.5), the conclusions of the lemma follow immediately. \( \square \)

By (4.6) we obtain

\[
Q^0 = P^0 \iff E^0(\alpha_0|I) = \frac{1}{\lambda} P^0 \text{-as.}
\] \tag{4.7}

If \( \Phi \) is ergodic, then \( E^0(\alpha_0|I) = E^0\alpha_0 = \lambda^{-1} P^0 \text{-as.} \) Relation (4.5) could then be taken as a definition of \( P^0 \). If, however, \( \Phi \) is not ergodic, then it is possible that \( Q^0 \neq P^0 \).
Example 4.3. Set \( \varphi_k \):= \#(\cdot \cap k\mathbb{Z})\), \( k \in \{1, 2\} \). Let \( \Phi^0 \) be a random element in \( M^0 \) such that \( \mathbb{P}[\Phi^0 = \varphi_1] = p \) and \( \mathbb{P}[\Phi^0 = \varphi_2] = 1 - p \), \( p \in (0, 1) \). Then \( \mathbb{E}(\alpha_i(\Phi^0)) = 2 - p \) for all \( i \in \mathbb{Z} \) and \( (\alpha_i(\Phi^0)) \) is stationary. According to Franken et al. (1982; Th. 1.3.4) there exists exactly one distribution \( P \) of a stationary point process \( \Phi \) such that its Palm distribution \( P^0 \) equals the distribution of \( \Phi^0 \). For \( B_1 := [\alpha_i(\varphi) = 1 \text{ for all } i \in \mathbb{Z}] \) and \( B_2 := [\alpha_i(\varphi) = 2 \text{ for all } i \in \mathbb{Z}] \) it can easily be proved that \( P^0(B_1) = p \), \( P^0(B_2) = 1 - p \), that \( B_1, B_2 \in \mathcal{I} \), and that \( \mathbb{E}^0(\alpha_0|\mathcal{I}) = 1_{B_1} + 21_{B_2} \) \( P^0 \)-as. Consequently, \( \Phi \) is not ergodic and \( Q^0 \neq P^0 \).

Definition 4.4. A stationary point process \( \Phi \) with \( \mathbb{P}[\Phi \in M^\infty] = 1 \) and \( \lambda \in (0, \infty) \) is called pseudo-ergodic if \( \mathbb{E}^0(\alpha_0|\mathcal{I}) = \lambda^{-1} \) \( P^0 \)-as.

An ergodic point process is pseudo-ergodic. A pseudo-ergodic point process need not be ergodic.

Example 4.5. Let \( \varphi_1 \) be as in Example 4.3, \( A_1 := [\alpha_i = 1 \text{ for all } i \in \mathbb{Z}] \), and \( A_2 := [\alpha_i \in \{1/2, 3/2\} \text{ for all } i \in \mathbb{Z}] \). Consider the following experiment. A fair coin is tossed. If head appears, then \( \varphi_1 \) is taken as outcome of \( \Phi^0 \). If, however, tail appears, then we let for each \( i \in \mathbb{Z} \) the coin decide whether \( \alpha_i \) equals 1/2 or 3/2, and take the resulting \( \varphi \in A_2 \cap M^0 \) as outcome for \( \Phi^0 \). Note that \( (\alpha_i(\Phi^0)) \) is stationary and that \( \mathbb{E}(\alpha_i(\Phi^0)) = 1 \). Let \( \Phi \) (with distribution \( P \)) be the stationary point process for which the corresponding \( P^0 \) equals the distribution of \( \Phi^0 \). Then \( \Phi \) is not ergodic, since \( P^0(A_1) = P^0(A_2) = \frac{1}{2} \) and \( A_1, A_2 \in \mathcal{I} \). Since \( P^0[\mathbb{E}^0(\alpha_0|\mathcal{I}) = 1] = 1 \), \( \Phi \) is pseudo-ergodic.

Since \( E\hat{P}_n \ll P^0 \) with Radon-Nikodym derivative \( \lambda n^{-1} \sum_{i=1}^{n} \alpha_{-i} \) (see Theorem 2.1), we obtain by (4.6) that \( d(E\hat{P}_n, Q^0) = \lambda E^0|n^{-1} \sum_{i=1}^{n} \alpha_{-i} - \mathbb{E}^0(\alpha_0|\mathcal{I})| \) (recall the definition of \( d \) in (1.8)). We want to prove that this last expression tends to 0 as \( n \to \infty \).

A sequence \( (Y_n)_{n \in \mathbb{N}} \) of integrable rv's is uniformly integrable if
\[
\lim_{a \to \infty} \sup_{n \in \mathbb{N}} \mathbb{E}[Y_n | 1 \{Y_n > a\}] = 0, \tag{4.8}
\]

or, equivalently,

\[
\sup_{n \in \mathbb{N}} \mathbb{E}[Y_n] = M < \infty \text{ and for every } \varepsilon > 0 \text{ there exists } \delta > 0 \text{ such that for all events } A \text{ with } \mathbb{P}(A) < \delta \text{ we have:}
\]

\[
\sup_{n \in \mathbb{N}} \mathbb{E}[Y_n | 1_A] < \varepsilon. \tag{4.9}
\]

If \((Y_n)_{n \in \mathbb{N}}\) is uniformly integrable, then so is \((n^{-1} \sum_{i=1}^{n} Y_i)_{n \in \mathbb{N}}\) as is obvious by (4.9). A random sequence with identically distributed elements is uniformly integrable. Consequently, \((n^{-1} \sum_{i=1}^{n} \alpha_{-i})_{n \in \mathbb{N}}\) is uniformly \(P^0\)-integrable. Since \(n^{-1} \sum_{i=1}^{n} \alpha_{-i} \to E^{0}(\alpha_0|\mathcal{I})\) \(P^0\)-as, we obtain that \(d(E^n, Q^0) \to 0\) as \(n \to \infty\) (cf. e.g. Th. T26 in Brémaud (1981)). We conclude that the convergence in (4.5) is uniform in \(A\):

\[
\sup_{A \in \mathcal{M}^{\infty}} \left| \frac{1}{n} \sum_{i=1}^{n} P_i(A) - Q^0(A) \right| \to 0. \tag{4.10}
\]

The consequences of this observation for the Palm distribution are explained in the next theorem.

**Theorem 4.6.** For stationary point processes with \(P(M^{\infty}) = 1\) and \(\lambda \in (0, \infty)\) the following statements are equivalent:

(i) \(\frac{1}{n} \sum_{i=1}^{n} P_i(A) \to P^0(A)\) for all \(A \in \mathcal{M}^{\infty}\),

(ii) \(\sup_{A \in \mathcal{M}^{\infty}} \left| \frac{1}{n} \sum_{i=1}^{n} P_i(A) - P^0(A) \right| \to 0,\)

(iii) \(\Phi\) is pseudo-ergodic,

(iv) \(P^0[\frac{1}{n} \sum_{i=1}^{n} \alpha_{-i} \to \frac{1}{\lambda}] = 1,\)

(v) \(P[\frac{1}{\lambda} N_i \to \lambda] = 1,\)

(vi) \(P^0 = P\) on \(\mathcal{I}\).
Proof. Relations (4.5), (4.10), (4.7), and (2.12) imply (i) $\iff$ (ii), (i) $\iff$ (iii), and (iii) $\iff$ (vi). The equivalence of (iii) and (iv) is an immediate consequence of Birkhoff's ergodic theorem. The implication (iv) $\implies$ (v) is a corollary of Theorem 3.1(a) and observations as in (3.6), with $U$ replaced by $\lambda^{-1}$. Theorem 3.1(b) and

$$P^0 \left[ \frac{1}{t} N_t \to \lambda \right] \leq P^0 \left[ \frac{1}{X_n} N_{X_n} \to \lambda \right] = P^0 \left[ \frac{1}{n} X_n \to \frac{1}{\lambda} \right]$$

yield the implication (v) $\implies$ (iv).

The main conclusion of Theorem 4.6 is that it is not always correct to define $P^0$ as the limit of $n^{-1} \sum_{i=1}^n P_i$, attractive as it may be. It is, however, possible to obtain $P^0(A)$ as another limit without any restraint, uniformly in $A \in \mathcal{M}$. Note that

$$P^0(A) = E^0[ E^0(1_A | \mathcal{I}) ] = \lambda^{-1} E[ \alpha_0^{-1} E^0(1_A | \mathcal{I}) ]$$

$$= \lambda^{-1} E[ E(\alpha_0^{-1} | \mathcal{I}) E^0(1_A | \mathcal{I}) ] = \lambda^{-1} E[ E^0(1_A E(\alpha_0^{-1} | \mathcal{I}) | \mathcal{I}) ] \quad (4.11)$$

Since the sequence $(\lambda^{-1} 1_A(\theta_i) E(\alpha_0^{-1} | \mathcal{I}))_{i \in \mathcal{I}}$ is generated by $\theta_1$, we obtain by Theorem 3.1(a) that

$$\frac{1}{\lambda} E(\frac{1}{\alpha_0} | \mathcal{I}) \hat{P}_n(A) \to \frac{1}{\lambda} E^0(1_A E(\frac{1}{\alpha_0} | \mathcal{I}) | \mathcal{I}) \quad P\text{-as.} \quad (4.12)$$

So (cf. (4.11)),

$$Q_n(A) := \frac{1}{\lambda} E[ E(\frac{1}{\alpha_0} | \mathcal{I}) \hat{P}_n(A) ] \to P^0(A), \quad A \in \mathcal{M} \quad (4.13)$$

By Relation (2.4) $Q_n$ is a probability measure. By Theorem 2.1, (3.7), (1.4) and the observation preceeding Theorem 3.1 we have

$$Q_n(A) = E^0 \left[ \frac{\alpha_0}{E^0(\alpha_0 | \mathcal{I})} \hat{P}_n(A) \right] = \frac{1}{n} \sum_{i=1}^n E^0 \left[ \frac{\alpha^{-i}}{E^0(\alpha_0 | \mathcal{I})} 1_A \right] .$$
Hence, $Q_n \sim P_0$ and

$$
\frac{dQ_n}{dP_0} = \frac{1}{n} \sum_{i=1}^{n} \frac{\alpha_i}{E^0(\alpha_0|I)} \to 1 \quad P_0\text{-as.} \quad (4.14)
$$

For $B \in \text{Bor}\mathbb{R}^+$ we have for $k \in \mathbb{Z}$

$$
P_0 \left[ \frac{\alpha_{-k}}{E^0(\alpha_0|I)} \in B \right] = P_0 \left[ \frac{\alpha_0}{E^0(\alpha_0|I)} \in B \right].
$$

So, the random sequence $(\alpha_{-i}/E^0(\alpha_0|I))$ is identically $P_0$-distributed and hence $(n^{-1} \sum_{i=1}^{n} \alpha_{-i}/E^0(\alpha_0|I))_{n \in \mathbb{N}}$ is uniformly $P_0$-integrable (cf. the arguments preceding (4.10)). By (4.14) it is obvious that the convergence in (4.13) is uniform in $A \in \mathcal{M}^\infty$.

Note that $Q_n = n^{-1} \sum_{i=1}^{n} P_i = E\hat{P}_n$ iff $\Phi$ is pseudo-ergodic.

According to (4.4) the sequence $(\hat{P}_n)$, considered as a sequence of estimators of $P_0$, is strongly $P$-consistent iff $\Phi$ is ergodic. By Theorem 4.6 it is asymptotically $P$-unbiased iff $\Phi$ is pseudo-ergodic. It is an easy exercise to prove that $E(\hat{P}_n(A) - P_0(A))^2$, the mean squared error under $P$, tends to 0 iff $\Phi$ is ergodic.

In the next theorem we examine for sequences $(\xi_i)$ generated by $\theta_1$ the asymptotic $P$-unbiasedness of the estimator $n^{-1} \sum_{i=1}^{n} \xi_i$ of $E^0\xi_0$.

**Theorem 4.7.** Suppose that $(\xi_i)$ is generated by $\theta_1$ and that $E^0\alpha_0^3 \vee E^0\xi_0^3 < \infty$. Then

$$
\frac{1}{n} \sum_{i=1}^{n} E\xi_i \to \lambda E^0[\alpha_0 E^0(\xi_0|I)] \quad \text{as} \quad n \to \infty.
$$

(4.15)

If $\Phi$ is pseudo-ergodic, then $n^{-1} \sum_{i=1}^{n} \xi_i$ is asymptotically $P$-unbiased for $E^0\xi_0$.

**Proof.** By Theorem 2.1 we have

$$
\frac{1}{n} \sum_{i=1}^{n} E\xi_i = \lambda E^0 \left[ \frac{1}{n} \sum_{i=1}^{n} \alpha_0 \xi_i \right].
$$

(4.16)
Since

\[ E^0[\alpha_0 \xi_n | 1[\alpha_0 \xi_n > a]] \leq E^0[\alpha_0 \xi_n | 1[\alpha_0 \xi_n > a]] + E^0[\alpha_0 \xi_n | 1[\xi_n > a]] \leq \left( E^0[\alpha_0^2 1[\alpha_0 > a]] \right)^{1/2} \left( E^0[\xi_n^2 1[\xi_n > a]] \right)^{1/2} \]

and since this upper bound tends to zero, it is obvious that \((\alpha_0 \xi_n)_{n \in \mathbb{N}}\) and \((n^{-1} \sum_{i=1}^{n} \alpha_0 \xi_i)_{n \in \mathbb{N}}\) are uniformly \(P^0\)-integrable (see (4.8) and the arguments following (4.9)). Note also that by (3.4)

\[ \frac{1}{n} \sum_{i=1}^{n} \alpha_0 \xi_i \to \alpha_0 E^0(\xi_0 | I) \quad P^0\text{-as.} \]

By (4.16) and Brémaud (1981; T26) Relation (4.15) follows immediately. The limit in (4.15) is equal to

\[ \lambda E^0[E^0(\alpha_0 | I)E^0(\xi_0 || I)] = E^0(\xi_0), \]

provided that \(\Phi\) is pseudo-ergodic.

\[ \square \]

**Corollary 4.8.** Suppose that \(E^0\alpha_0^2 < \infty\). The estimator \(n^{-1} \sum_{i=1}^{n} \alpha_i\) of \(E^0\alpha_0 = \lambda^{-1}\) is asymptotically \(P\)-unbiased iff \(\Phi\) is pseudo-ergodic.

**Proof.** The if-part is a consequence of Theorem 4.7. If \(n^{-1} \sum_{i=1}^{n} \alpha_i\) is asymptotically \(P\)-unbiased, then we obtain by (4.15) that \(E^0[\alpha_0 E^0(\alpha_0 | I)] = (E^0\alpha_0)^2\). Consequently, \(\text{Var}_{P^0} E^0(\alpha_0 | I) = 0\) and \(E^0(\alpha_0 | I) = 1/\lambda\) \(P^0\text{-as.} \)

\[ \square \]

The point process in Example 4.3 satisfies \(\lambda^{-1} = E^0(1_{B_1} + 21_{B_2}) = 2 - p\) and (cf. (4.15))

\[ \frac{1}{n} \sum_{i=1}^{n} E\alpha_i \to \frac{E^0(E^0(\alpha_0 | I))^2}{2 - p} = \frac{4 - 3p}{2 - p}. \]
This limit is indeed not equal to $\lambda^{-1} = 2 - p$.

It is well known that $P^0$ can also be approximated by the probability measures $P_{1,n}$, $n \in \mathbb{N}$, defined by

$$P_{1,n}(A) := P[\theta_1 \varphi \in A|X_1(\varphi) \leq \frac{1}{n}], \ A \in \mathcal{M}^\infty.$$  

(4.17)

Franken et al. (1982; Th. 1.3.7) prove that $d(P^0, P_{1,n}) \to 0$ as $n \to \infty$. We will, however, express $d(P^0, P_{1,n})$ in terms of $F$, the distribution function of $X_1$ under $P$.

**Theorem 4.9.** Let $\Phi$ be a stationary point process with $P(M^\infty) = 1$ and $\lambda \in (0, \infty)$. Then

(i) $P_{1,n} \sim P^0$ and $\frac{dP_{1,n}}{dP^0} = \frac{\lambda}{F(1/n)}(1/n)^\wedge \alpha_{-1} =: \sigma_n$,

(ii) $\sup_{A \in \mathcal{M}^\infty} |P[\theta_1 \varphi \in A|X_1(\varphi) \leq \frac{1}{n}] - P^0(A)| = \frac{1}{2} E^0|\sigma_n - 1| = 1 - \frac{F(F(1/\lambda))/F(1/\lambda)}{F(1/\lambda)} \to 0$.

**Proof.** By (1.5) we obtain

$$P[\theta_1 \varphi \in A; X_1(\varphi) \leq \frac{1}{n}] = \lambda \int_{M^\infty} \int_0^{\alpha_0(\varphi)} 1_{[\theta_1 \varphi \in A; \alpha_0(\varphi) - u \leq \frac{1}{n}]}dudP^0(\varphi)$$

$$= \lambda E^0 \left[1_{[\theta_1 \varphi \in A]} \left(\frac{1}{n} \wedge \alpha_0\right)\right] = \lambda E^0[1_A \left(\frac{1}{n} \wedge \alpha_{-1}\right)].$$

Hence $P_{1,n}(A) = \lambda E^0[1_A \left(\frac{1}{n} \wedge \alpha_{-1}\right)]/F(1/\lambda)$, which proves (i). By (1.8) it is obvious that $d(P_{1,n}, P^0) = E^0|\sigma_n - 1|$. We will express this $P^0$-expectation in terms of $F$.

First we note that (cf. Theorem 2.1)

$$P[\alpha_0 \leq x] = \lambda E^0[\alpha_0 1_{[\alpha_0 \leq x]}] \leq \lambda x P^0[\alpha_0 \leq x], \ x \in [0, \infty),$$

and (cf. (2.9))

$$F(\frac{1}{n}) = P[\alpha_0 \leq \frac{1}{n}] + \frac{\lambda}{n} - \frac{\lambda}{n} P^0[\alpha_0 \leq \frac{1}{n}] \leq \frac{\lambda}{n}. \quad \quad (4.18)$$
Set $h(n) := F(1/n)/\lambda$. By (4.18), Theorem 2.1 and (2.9) we obtain

$$E^0|\sigma_n - 1| = \frac{1}{h(n)}E^0_0|\frac{1}{n} \wedge \alpha_0 - h(n)|$$

$$= \left( E^0(h(n) - \alpha_0)1_{[\alpha_0 \leq h(n)]} + E^0(\alpha_0 - h(n))1_{[h(n) < \alpha_0 \leq \frac{1}{n}]} \right) / h(n)$$

$$= 2P^0[\alpha_0 \leq h(n)] - \frac{1/n}{h(n)}P^0[\alpha_0 \leq \frac{1}{n}] + \frac{1/n - h(n)}{h(n)}$$

$$- \frac{2}{\lambda h(n)}P[\alpha_0 \leq h(n)] + \frac{1}{\lambda h(n)}P[\alpha_0 \leq \frac{1}{n}]$$

$$= 2 - 2F(h(n)) + \frac{\lambda h(n) - \lambda/n}{\lambda h(n)} + \frac{1/n - h(n)}{h(n)}$$

$$= 2 - 2F(h(n)) + \frac{F(h(n))}{\lambda h(n)} = 2 - 2\frac{F(F(1/n)/\lambda)}{F(1/n)}.$$

The convergence to 0 follows immediately since $F(x) = \lambda x + o(x)$ as $x \to 0$, cf. e.g. Franken et al. (1982; Th. 1.2.12). \qed

Because of (ii) it is possible to determine in many situations the rate at which $P_{1,n}$ tends to $P^0$. If $\Phi$ is a Poisson process, then it is an easy exercise to prove that $d(P_{1,n}, P^0) = \frac{1}{2}F(\frac{1}{n}) + o(F(\frac{1}{n})) = \frac{1}{2}\lambda/n + o(\frac{1}{n})$ as $n \to \infty$. This rate $1/n$ is not universal; it turns out that the renewal process with $P^0[\alpha_0 \leq x] = x^{1-p}$ for $0 < x < 1$, $p \in (0, 1)$, satisfies $d(P_{1,n}, P^0) = cn^{-(1-p)} + o(n^{-(1-p)})$ as $n \to \infty$. (Here $c \in (0, \infty)$ is some constant, not depending on n.) We can, however, give conditions such that the rate $1/n$ is satisfied.

**Corollary 4.10.** Suppose that $G$ is differentiable on $(0, \varepsilon)$ for some $\varepsilon > 0$ with bounded derivative $g := G'$. Then

$$\sup_{A \in \mathcal{M}_\infty} |P[\theta_1 \varphi \in A | X_1(\varphi) \leq \frac{1}{n}] - P^0(A)| = O(\frac{1}{n}) \quad \text{as} \quad n \to \infty.$$

**Proof.** Because of the continuity of $G$ it is obvious (see (2.7)) that $F$ is differentiable on
(0, ε) with \( F' = \lambda(1 - G) \). By the mean value theorem we have for \( n \) sufficiently large:

\[
F(F(\frac{1}{n})/\lambda) = F(0) + F(\frac{1}{n})(1 - G(\eta_n))
\]

for some \( \eta_n \in (0, F(1/n)/\lambda) \). Since \( F(0) = 0 \) and \( F(1/n) \leq \lambda/n \), see (4.18), we obtain by Theorem 4.9:

\[
E_0^0|\sigma_n - 1| = 2G(\eta_n) \leq 2G(\frac{1}{n}).
\]

Another application of the mean value theorem yields for \( n \) sufficiently large:

\[
E_0^0|\sigma_n - 1| \leq 2G(\delta_n) \leq \frac{2c}{n}
\]

for some \( \delta_n \in (0, \frac{1}{n}) \). Here \( c := \sup\{g(x) : x \in (0, \epsilon)\} \), not depending on \( n \). \( \Box \)

**Remark.** The condition in Corollary 4.10 may equivalently be replaced by

\[ F \text{ is twice differentiable on } (0, \epsilon) \text{ for some } \epsilon > 0 \tag{4.19} \]

with bounded second derivative \( F'' \).

### 5 Generalization to marked point processes

The results of Sections 1 to 4 can be generalized to marked point processes. We briefly consider this extension.

Let \( K \) be a complete and separable metric space. A *marked point process on \( \mathbb{R} \) with mark space \( K \) is a random element \( \Phi \) in the class of all integer-valued measures \( \varphi \) on the \( \sigma \)-field \( \text{Bor} \mathbb{R} \times \text{Bor} K \) such that:

\[ \varphi(A \times K) < \infty \text{ for all bounded } A \in \text{Bor} \mathbb{R}. \]
Let $M_K$ be this class and endow it with the $\sigma$-field $\mathcal{M}_K$ generated by the sets $[\varphi(A \times L) = k] := \{ \varphi \in M_K : \varphi(A \times L) = k \}$, $k \in \mathbb{N}_0$, $L \in \text{Bor } K$ and $A \in \text{Bor } \mathbb{R}$.

Here are some further notations and definitions. For $\varphi \in M_K$ and $L \in \text{Bor } K$ we define $\bar{\varphi}_L \in M_K$ and $\varphi_L \in M$ by $\bar{\varphi}_L(B) := \varphi(B \cap (\mathbb{R} \times L))$ and $\varphi_L(A) := \varphi(A \times L)$, $B \in \text{Bor } \mathbb{R} \times \text{Bor } K$ and $A \in \text{Bor } \mathbb{R}$. Note that $\bar{\varphi}_L(\mathbb{R} \times L^c) = 0$ and $\bar{\varphi}_K = \varphi$. Furthermore, set

\[
\begin{align*}
\mathcal{M}^\infty_L &:= \{ \varphi \in M_K : \varphi(-\infty, 0) = \varphi_L(0, \infty) = \infty; \varphi_K(\{s\}) \leq 1 \text{ for all } s \in \mathbb{R} \}, \\
\mathcal{M}^0_L &:= \{ \varphi \in M^\infty_L : \varphi_L(\{0\}) = 1 \}, \\
\mathcal{M}^\infty_L &:= M^\infty_L \cap M_K \text{ and } M^0_L := M^0_L \cap M_K,
\end{align*}
\]

$L \in \text{Bor } K$. Let $T_t : M_K \rightarrow M_K$, $t \in \mathbb{R}$, be the time shifts determined by $T_t \varphi(A \times L) = \varphi((t + A) \times L)$. We will assume that $\Phi$ (or its distribution $P$) is stationary with respect to these time shifts (cf. Section 1). We also assume that $\lambda := \mathbb{E}\Phi((0, 1] \times K) < \infty$, so that $\lambda(L) := \mathbb{E}\Phi((0, 1] \times L) < \infty$ for all $L \in \text{Bor } K$. We will confine our attention to $L$ with $P(M^\infty_L) = 1$.

The atoms of $\varphi \in M^\infty_K$ are denoted by $(X_i(\varphi), k_i(\varphi))$, $i \in \mathbb{Z}$, enumerated such that $(X_i(\varphi))_{i \in \mathbb{Z}}$ represents $\varphi_K$ as indicated in Section 1. For $\varphi \in M^\infty_L$ we write $X^L_i(\varphi) := X_i(\bar{\varphi}_L)$, the ‘ith $L$-point of $\varphi$', and $k^L_i(\varphi) := k_i(\bar{\varphi}_L)$, the ‘mark of the ith $L$-point of $\varphi$’. Note that $T_t \varphi =: \varphi(t + \cdot)$ can be represented by $\{(X_i(\varphi) - t, k_i(\varphi))\}$. Some other notations:

\[
\begin{align*}
\alpha_i &:= X_{i+1} - X_i, \\
\alpha^L_i &:= X^L_{i+1} - X^L_i, \\
\theta_{n,L} &:= M^\infty_L \rightarrow M^0_L \text{ with } \theta_{n,L} \varphi := \varphi(X^L_n(\varphi) + \cdot), \\
P_{n,L} &:= P_{\theta_{n,L}^{-1}}, \\
I' &:= \{ A \in \mathcal{M}^\infty_K : T_t^{-1}A = A \text{ for all } t \in \mathbb{R} \}, \\
I'^L &:= \{ A \in \mathcal{M}^\infty_L : T_t^{-1}A = A \text{ for all } t \in \mathbb{R} \}, \\
I_L &:= \{ A \in \mathcal{M}^\infty_L : \theta^L_{1,A}A = A \},
\end{align*}
\]
where $i, n \in \mathbb{Z}$ and $L \in \text{Bor } K$. $P_{n,L}$ is obtained from $P$ by shifting the origin to the $n$th $L$-point. Note that $T'_L \subset T$, $T'_L \subset T_L$ and $T' \cap M^\infty_L = T'_L$.

The Palm distribution $P^0_L$ of $P$ with respect to $L$ is defined by:

$$P^0_L(A) := \frac{1}{\lambda(L)} \mathbb{E} \left[ \sum_{i=1}^{\Phi((0,1] \times L)} 1_A(\theta_i L \Phi) \right], \quad A \in \mathcal{M}_K,$$

which intuitively arises from $P$ by shifting the origin to an arbitrary $L$-point. Now $P^0_L$ is a probability measure on $(M_K, \mathcal{M}_K)$ with $P^0_L(M^\infty_L) = 1$ and having the following properties (cf. (1.4) and (1.5)):

(i) $P^0_{n,L} \rightarrow P^0_L$ for all $n \in \mathbb{Z},$

(ii) $P(A) = \lambda(L) \int_0^\infty P^0_L[X^1(t)(\varphi) > u; \varphi(u + \cdot) \in A] du, \quad A \in \mathcal{M}_K,$

see e.g. Franken et al. (1982).

We now generalize the results of Sections 1 to 4. Our emphasis is on conditioning on $L$-points in the origin with $L \in \text{Bor } K$ such that $P(M^\infty_L) = 1$. Hence, we must replace $M, \mathcal{M}, M^\infty, \mathcal{M}^\infty, M^0, \mathcal{M}^0, \lambda, \alpha_i, X_i, P^0, \theta_n, I, I', U, V', N(t)$ by $M_K, \mathcal{M}_K, M^\infty_L, \mathcal{M}^\infty_L, M^0_L, \mathcal{M}^0_L, \lambda(L), \alpha^L, X^L, P^0_L, P_{n,L}, P_{\theta_n,L}, I_L, I'_L, U_L, V_L, N_L(t)$ respectively. (The definitions of $P^L_U, U_L$ and $V'_L$ are clear by (2.13) and the definitions following (3.4); $N_L(t, \varphi) := \varphi_L(0, t]$ if $t \geq 0$ and $N_L(t, \varphi) := -\varphi_L(t, 0]$ if $t < 0$, see (1.7).) We must replace 'pseudo-ergodic' by 'pseudo-$L$-ergodic'.

With these modifications all results remain true. In fact only some of the proofs need an argument. Since $P(M^\infty_L) = 1$ and $I' \cap M^\infty_L = I'_L$, it is obvious that ergodicity of $\Phi$ can (indeed) equivalently be defined with $I'_L$ instead of $I'$ and that $E(g|I') = E(g|I'_L)$ $P$-as for all $P$-integrable functions $g : M^\infty_L \rightarrow \mathbb{R}$. With this in mind the generalized results of Section 3 follow immediately.

Examples of sequences $(\xi_i)$ generated by $\theta_{1,L}$ (see Section 1) are given by
\[ \alpha_t^L(\varphi), \]
\[ \varphi_{L'}(X_{t_1}^L(\varphi) + t_1, X_{t_1}^L(\varphi) + t_2), \]
\[ \varphi_{L'}(X_{t-1}^L(\varphi), X_t^L(\varphi)), \]
\[ k_t^L(\varphi). \]

Here \( L, L' \in \text{Bor} \ K \) with \( P(M_L^\infty) = P(M_{L'}^\infty) = 1 \) and \( t_1 < t_2 \). The third sequence is interesting. If \( N_{L'}(t_1, t_2) := N_{L'}(t_2) - N_{L'}(t_1) \), it can be defined as \( (N_{L'}(X_{t-1}^L, X_t^L)) \). By (the generalization of) Theorem 3.1 we obtain

\[ \frac{1}{n} N_{L'}(0, X_n^L) \to E^0_L(N_{L'}(0, X_1^L) \| I_L) \quad \text{as} \quad n \to \infty \quad P_L^0 \text{- and P-as.} \quad (5.1) \]

Since

\[ \frac{N_L(t)}{t} \frac{N_{L'}(0, X_{N_L(t)})}{N_{L'}(0, t)} \leq \frac{1}{t} N_{L'}(0, t) \leq \frac{N_{L'}(0, X_{N_L(t)+1})}{N_{L'}(t) + 1} N_{L'}(t) + 1 \]

it follows from (5.1) and Theorem 3.1(b) that

\[ \frac{1}{t} N_{L'}(t) \to E(N_{L'}(1) \| I_L) E^0_L(N_{L'}(0, X_1^L) \| I_L) \quad \text{as} \quad t \to \infty \quad P_L^0 \text{- and P-as.} \quad (5.2) \]

Set \( M_{L,L'}^\infty := M_{L}^\infty \cap M_{L'}^\infty \), \( \mathcal{M}_{L,L'}^\infty := \mathcal{M}_{L,L'} \cap \mathcal{M}_K \) and \( I_{L,L'}' := \{ A \in \mathcal{M}_{L,L'}^\infty : T^{-1}A = A \text{ for all } t \in \mathbb{R} \} \). Note that \( I_{L,L'} \subset I_L, \; I_{L,L'}' = T' \cap \mathcal{M}_{L,L'}^\infty \) and \( P(M_{L,L'}^\infty) = 1 \). By arguments as in the proof of the first part of Theorem 3.1(b) we have

\[ \frac{1}{t} N_{L'}(t) \to E(N_{L'}(1) \| I_{L,L'}') \quad P_L^0 \text{- and P-as.} \quad (5.3) \]

Combining (5.2) and (5.3) yields

\[ E^0_L(N_{L'}(0, X_1^L) \| I_L) = \frac{E(N_{L'}(1) \| I_{L,L'}')}{E(N_{L'}(1) \| I_L)} \quad P_L^0 \text{- and P-as.} \quad (5.4) \]
which is a generalization of Relation (3.4.2) in Baccelli & Brémaud (1987).

Parts of this paper are also valid for special classes of non-stationary point processes. We are preparing a publication on these matters.
References


IN 1990 REEDS VERSCHENEN

419 Bertrand Melenberg, Rob Alessie
A method to construct moments in the multi-good life cycle consumption model

420 J. Kriens
On the differentiability of the set of efficient \((\mu, \sigma^2)\) combinations in the Markowitz portfolio selection method

421 Steffen Jørgensen, Peter M. Kort
Optimal dynamic investment policies under concave-convex adjustment costs

422 J.P.C. Blanc
Cyclic polling systems: limited service versus Bernoulli schedules

423 M.H.C. Paardekooper
Parallel normreducing transformations for the algebraic eigenvalue problem

424 Hans Gremmen
On the political (ir)relevance of classical customs union theory

425 Ed Nijssen
Marketingstrategie in Machtsperspectief

426 Jack P.C. Kleijnne
Regression Metamodels for Simulation with Common Random Numbers: Comparison of Techniques

427 Harry H. Tigelaar
The correlation structure of stationary bilinear processes

428 Drs. C.H. Veld en Drs. A.H.F. Verboven
De waardering van aandelenwarrants en langlopende call-opties

429 Theo van de Klundert en Anton B. van Schaik
Liquidity Constraints and the Keynesian Corridor

430 Gert Nieuwenhuis
Central limit theorems for sequences with \(m(n)\)-dependent main part

431 Hans J. Gremmen
Macro-Economic Implications of Profit Optimizing Investment Behaviour

432 J.M. Schumacher
System-Theoretic Trends in Econometrics

433 Peter M. Kort, Paul M.J.J. van Loon, Mikulás Luptacik
Optimal Dynamic Environmental Policies of a Profit Maximizing Firm

434 Raymond Gradus
Optimal Dynamic Profit Taxation: The Derivation of Feedback Stackelberg Equilibria
Jack P.C. Kleijnen
Statistics and Deterministic Simulation Models: Why Not?

M.J.G. van Eijs, R.J.M. Heuts, J.P.C. Kleijnen
Analysis and comparison of two strategies for multi-item inventory systems with joint replenishment costs

Jan A. Weststrate
Waiting times in a two-queue model with exhaustive and Bernoulli service

Alfons Daems
Typologie van non-profit organisaties

Drs. C.H. Veld en Drs. J. Grazell
Motieven voor de uitgifte van converteerbare obligatieleningen en warrantobligatieleningen

Jack P.C. Kleijnen
Sensitivity analysis of simulation experiments: regression analysis and statistical design

C.H. Veld en A.H.F. Verboven
De waardering van conversierechten van Nederlandse converteerbare obligaties

Drs. C.H. Veld en Drs. P.J.W. Duffhues
Verslaggevingsaspecten van aandelenwarrants

Jack P.C. Kleijnen and Ben Annink
Vector computers, Monte Carlo simulation, and regression analysis: an introduction

Alfons Daems
"Non-market failures": Imperfecties in de budgetsector

J.P.C. Blanc
The power-series algorithm applied to cyclic polling systems

L.W.G. Strijbosch and R.M.J. Heuts
Modelling (s,Q) inventory systems: parametric versus non-parametric approximations for the lead time demand distribution

Jack P.C. Kleijnen
Supercomputers for Monte Carlo simulation: cross-validation versus Rao's test in multivariate regression

Jack P.C. Kleijnen, Greet van Ham and Jan Rotmans
Techniques for sensitivity analysis of simulation models: a case study of the CO₂ greenhouse effect

Harrie A.A. Verbon and Marijn J.M. Verhoeven
Decision-making on pension schemes: expectation-formation under demographic change
450 Drs. W. Reijnders en Drs. P. Verstappen
Logistiek management marketinginstrument van de jaren negentig

451 Alfons J. Daems
Budgeting the non-profit organization
An agency theoretic approach

452 W.H. Haemers, D.G. Higman, S.A. Hobart
Strongly regular graphs induced by polarities of symmetric designs

453 M.J.G. van Eijs
Two notes on the joint replenishment problem under constant demand

454 B.B. van der Genugten
Iterated WLS using residuals for improved efficiency in the linear model with completely unknown heteroskedasticity

455 P.A. van der Duyn Schouten and S.G. Vanneste
Two Simple Control Policies for a Multicomponent Maintenance System

456 Geert J. Almekinders and Sylvester C.W. Eijffinger
Objectives and effectiveness of foreign exchange market intervention
A survey of the empirical literature

457 Saskia Oortwijn, Peter Borm, Hans Keiding and Stef Tijs
Extensions of the $\tau$-value to NTU-games

458 Willem H. Haemers, Christopher Parker, Vera Pless and Vladimir D. Tonchev
A design and a code invariant under the simple group $\text{Co}_3$

459 J.P.C. Blanc
Performance evaluation of polling systems by means of the power-series algorithm

460 Leo W.G. Strijbosch, Arno G.M. van Doorne, Willem J. Selen
A simplified MOLP algorithm: The MOLP-S procedure

461 Arie Kapteyn and Aart de Zeeuw
Changing incentives for economic research in The Netherlands

462 W. Spanjers
Equilibrium with co-ordination and exchange institutions: A comment

463 Sylvester Eijffinger and Adrian van Rixtel
The Japanese financial system and monetary policy: A descriptive review

464 Hans Kremers and Dolf Talman
A new algorithm for the linear complementarity problem allowing for an arbitrary starting point

465 René van den Brink, Robert P. Gilles
A social power index for hierarchically structured populations of economic agents
IN 1991 REEDS VERSCHENEN

466 Prof. Dr. Th. C. M. J. van de Klundert - Prof. Dr. A. B. T. M. van Schaik
Economische groei in Nederland in een internationaal perspectief

467 Dr. Sylvester C. W. Eijffinger
The convergence of monetary policy - Germany and France as an example

468 E. Nijssen
Strategisch gedrag, planning en prestatie. Een inductieve studie binnen de computerbranche

469 Anne van den Nouweland, Peter Borm, Guillermo Owen and Stef Tijs
Cost allocation and communication

470 Drs. J. Grazell en Drs. C. H. Veld
Motieven voor de uitgifte van converteerbare obligatieleningen en warrant-obligatieleningen: een agency-theoretische benadering

471 P. C. van Batenburg, J. Kriens, W. M. Lammerts van Bueren and R. H. Veenstra
Audit Assurance Model and Bayesian Discovery Sampling

472 Marcel Kerkhofs
Identification and Estimation of Household Production Models

473 Robert P. Gilles, Guillermo Owen, René van den Brink
Games with Permission Structures: The Conjunctive Approach

474 Jack P. C. Kleijnen
Sensitivity Analysis of Simulation Experiments: Tutorial on Regression Analysis and Statistical Design

475 C. P. M. van Hoesel
An O(nlogn) algorithm for the two-machine flow shop problem with controllable machine speeds

476 Stephan G. Vanneste
A Markov Model for Opportunity Maintenance

477 F. A. van der Duyn Schouten, M. J. G. van Eijs, R. M. J. Heuts
Coordinated replenishment systems with discount opportunities

478 A. van den Nouweland, J. Potters, S. Tijs and J. Zarzuelo
Cores and related solution concepts for multi-choice games

479 Drs. C. H. Veld
Warrant pricing: a review of theoretical and empirical research

480 E. Nijssen
De Miles and Snow-typologie: Een exploratieve studie in de meubelbranche

481 Harry G. Barkema
Are managers indeed motivated by their bonuses?
Jacob C. Engwerda, André C.M. Ran, Arie L. Rijkeboer
Necessary and sufficient conditions for the existence of a positive
definite solution of the matrix equation $X + A^T X^{-1} A = I$

Peter M. Kort
A dynamic model of the firm with uncertain earnings and adjustment
costs

Raymond H.J.M. Gradus, Peter M. Kort
Optimal taxation on profit and pollution within a macroeconomic
framework

René van den Brink, Robert P. Gilles
Axiomatizations of the Conjunctive Permission Value for Games with
Permission Structures

A.E. Brouwer & W.H. Haemers
The Gewirtz graph - an exercise in the theory of graph spectra

Pim Adang, Bertrand Melenberg
Intratemporal uncertainty in the multi-good life cycle consumption
model: motivation and application

J.H.J. Roemen
The long term elasticity of the milk supply with respect to the milk
price in the Netherlands in the period 1969-1984

Herbert Hamers
The Shapley-Entrance Game

Rezaul Kabir and Theo Vermaelen
Insider trading restrictions and the stock market

Piet A. Verheyen
The economic explanation of the jump of the co-state variable

Drs. P.L.J.W. Manders en Dr. J.A.C. de Haan
De organisatorische aspecten bij systeemontwikkeling
een beschouwing op besturing en verandering

Paul C. van Batenburg and J. Kriens
Applications of statistical methods and techniques to auditing and
accounting

Ruud T. Frambach
The diffusion of innovations: the influence of supply-side factors

J.H.J. Roemen
A decision rule for the (des)investments in the dairy cow stock

Hans Kremers and Dolf Talman
An SLSP-algorithm to compute an equilibrium in an economy with
linear production technologies
L.W.G. Strijbosch and R.M.J. Heuts
Investigating several alternatives for estimating the compound lead time demand in an (s,Q) inventory model

Bert Bettonvil and Jack P.C. Kleijnen
Identifying the important factors in simulation models with many factors

Drs. H.C.A. Roest, Drs. F.L. Tijssen
Beheersing van het kwaliteitsperceptieproces bij diensten door middel van keurmerken

B.B. van der Genugten
Density of the F-statistic in the linear model with arbitrarily normal distributed errors

Harry Barkema and Sytse Douma
The direction, mode and location of corporate expansions