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e Investigación Operativa***

**On a Relative Measure of Skill for
Games with Chance Elements**

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Abstract

In various countries, including the Netherlands and Austria, legislation is such that the question whether a specific game should be considered as a game of chance or as a game of skill is predominant in the exploitation decision of private casinos. This paper aims for an objective and operational criterium to quantify the relative level of skill with respect to chance of games in order to provide a juridical tool for classification. The various concepts are illustrated by means of variations of Poker.

Key Words: Games, skill, chance elements, poker.

AMS subject classification: 91A10, 91A60.

1 Introduction

Does one really need skills to play the dice game backgammon or is it basically just a game of chance? Perhaps the nice thing about this question is the fact that almost everybody will arrive at a distinct opinion rather quickly. For those who play backgammon frequently it will definitely be a game of skill, in this way implicitly justifying the amount of time spent to this game. For other people, for example for fanatic chess players, the fact that a player at every move is dependent of the throwing of dices will be unbearable and for this reason alone these people will be inclined to qualify backgammon as a game of chance. This, in our opinion, is typical for many games: opposite distinct opinions about the qualification of the game. One person will judge the level of skill of a casino game high if already one individual can beat the casino in the long run. Another person will only

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be convinced of the presence of skill if a vast majority of players actually displays skill. Still another person will perhaps say that by definition every casino game is a game of chance.

It is not the aim of this paper to present a general treatment of the skill concept. We restrict ourselves to a juridical point of view as reflected by gaming acts in several European countries, such as in the Dutch Gaming Act:

... it is not allowed to: exploit games with monetary prizes if the participants in general do not have a dominant influence on the probability to win, unless in compliance to this act, a license is granted ...

In practice the Dutch state only grants such a license to its own Holland Casino's foundation. The main reason behind this monopoly is that the exploitation of games of chance is a lucrative affair for the proprietor. The government wants to have both the control and the profits of this market. The formulation of the Gaming Act clearly implies that skill should be considered relatively with respect to chance. Therefore we will discuss skill only in this restricted context. In the remainder we call a game that is covered by the Gaming Act a *game of chance*. A game which is not a game of chance will be called a *game of skill*. So, in particular, to commercialize games of skill no license within the meaning of the Gaming Act is needed. This opens the way for private casinos to compete with the games of chance that are exploited in the Holland Casino's. Of course the proprietor of a game and the legislator can have different views on how to qualify a game since such a judgement about the role of chance is rather subjective and the financial stakes are high. If it would be possible to rank a broad class of games with chance elements by means of an operational and objective criterium which quantifies the level of skill relatively to chance, e.g. on a scale from zero to one, the legislator would be able to decide on a certain bound on the level of skill, below which a game should be considered as a game of chance. In this article we propose a specific definition to measure skill in this way and provide approximations for various casino games. The analysis is partly based on the results in Van der Genugten and Borm (1991, 1994). The various concepts will be illustrated in some detail by means of simple variations of the game of Poker. Some general references on specific (computational) aspects on Poker are Karlin (1959a, b), Epstein

(1977), Packel (1981), Sakaguchi (1993), Scarne (1990), Tamburin (1993) and Sakaguchi and Mazalov (1996). A recent paper on the topic of skill in games is Larkey et al. (1997). This paper provides an interesting discussion on the interpretation and relevance of the concept of skill in analyzing and solving games. Contrary to the current exposition however the authors do not focus on a possible answer the juridical question underlying the current paper, but mainly discuss and illustrate the difference in skill incorporated in various strategies in a simplification in Stud Poker using the technique of simulation tournaments.

The paper is organized in the following way.

Section 2 discusses the qualitative requirements we think should be satisfied by a suitable relative measure of skill. The important concepts under consideration will be the learning effect and the random effect. Then, in Section 3, we propose a definition of the level of skill for one-person games like Roulette, Blackjack and Golden Ten. This definition is extended to two-person zero-sum games like Chess, Schnapsen and two-person Poker in section 4, and, finally, for general n -person zero-sum games in section 5. At this stage some general remarks about the differences in complexity between the various classes of games should be made. In a one-person game like Roulette each player's betting strategy is all-decisive upon the possible expected gains: these gains do not depend on the behaviour of other players. In a more-person game like Poker it is typically the case that a player's gains also depend on the opponents' behaviour. So, in particular, strategic aspects come in. For three- or more-person games also the feature of possible cooperation between subgroups of players should be taken into account. These aspects will be considered in detail in the various sections. A numerical two-person variation of Poker will be used to illustrate the underlying concepts in section 4, a three-person variation in section 5. Section 6 concludes with the possible implications of our measure of skill.

2 Some qualitative criteria for skill

In our analysis we will restrict attention to games which, in principle, can be repeated under the same conditions. This assumption guarantees an objective quantification of uncertainty in terms of probability. This does not mean that such a quantification is impossible beforehand for games of a different kind but that it necessarily will be of a more subjective nature.

We formulate three requirements which in our opinion summarize the basic ideas underlying the Dutch legislation concerning the exploitation of games with chance elements.

- R1. Legislation applies exclusively to situations which involve the exploitation of games with monetary prizes. For this reason we only consider games in which the "game-result" of a player can be expressed in some way in terms of a certain gain (or loss) of money.
- R2. The skill of a player should be measured as his result in the long run, i.e. in terms of probabilistic expectation. Necessary for a game of skill is that these expected results vary among players.
- R3. The fact that there is a difference between players with respect to their expected result, does not immediately imply that the underlying game is a game of skill. Sufficient for a game of skill is that the chance elements involved do not prohibit these differences to be substantial.

In Van der Genugten and Borm (1994, 1996) each of these requirements is discussed in detail in the light of the Dutch jurisprudence on specific games as Saturne, Blackjack, Roulette and Golden Ten. Implicitly these cases make clear that one should distinguish between three types of players:

- (1) a *beginner* who plays the game in the naive way of somebody who has just mastered the rules of the game.
- (2) a *real average player* who can be thought to represent the vast majority of players.
- (3) a *virtual average player* whom we tell in advance (i.e. before he has to decide) the outcome of the chance elements.

One may conclude that legislation is concerned with the difference between a beginner and a real average player. Therefore, we call the difference in expected result between those two types of players the *learning* effect. This effect should be judged substantial in relation with the restrictive possibilities within the game set by the chance elements. This judgement we base on the so-called *random effect*: the difference in expected result between a virtual average player and a real average player. So in our view skill,

as a juridical concept, reflects the relative judgement between the notions of learning effect and random effect as defined above. An objective judgement is possible if the value of the following type parameters have been determined:

- result of beginner
- result of real average player
- result of virtual average player.

The terminology above is not restricted to one-person games (e.g. Black-jack), but can be applied to more-person games (e.g. Poker as well). Given the three type parameters, the following definition gives an appropriate measure for skill as a relative judgement between the learning and the random effect:

$$\begin{aligned} \text{skill} &= \frac{\text{learning effect}}{\text{learning effect} + \text{random effect}} \\ &= \frac{\text{result real average player} - \text{result beginner}}{\text{result virtual average player} - \text{result beginner}} \end{aligned}$$

Clearly, this implies that

$$\begin{array}{ccc} 0 & \leq \text{skill} \leq & 1. \\ \text{(pure chance)} & & \text{(pure skill)} \end{array}$$

So, games in which the random effect dominates the learning effect will have a small skill and games in which the learning effect dominates will have a high skill.

The above definition of skill makes the way in which games can be judged upon objective. It forces to think about the learning effect and the random effect in terms of the three player types above. In order to give precise and meaningful definitions of the various concepts we have to restrict the class of games under consideration. We briefly summarize these restrictions:

- games can be repeated

* with objective probabilities determined by the rules of the game

* with independency between chance elements during repetitions

- players can be identified with their strategies
- the result of a player is based on the mean of the result in the long run (i.e. expectation) of the corresponding strategy

for more-person games:

- the type parameters are defined with respect to the behaviour of a collective of average players
- the result of the types is the mean over the different playing roles of a player.

Under these restrictions results of strategies of players can be described numerically. However, the type parameters reflect judgements of experts and therefore can differ. It is our experience that agreement can be obtained as far as it concerns the type beginner. However, commitment about the type average player seems to be difficult. For this we are forced to find other ways.

A very pragmatic solution is to replace the concept of the rather subjective average player with the more objective *advanced* (= almost optimal) *player*. The effect of this substitution will be an increase in the result of the real average player to the result of the real advanced player, but also an increase of the result of the virtual average player to the result of the virtual advanced player. Therefore, it may be expected that the relative change through both increases will almost remain the same. And that is just the way we defined skill: in a relative way. More formally,

$$\begin{aligned} \text{skill} &\approx \frac{\text{result real advanced player} - \text{result beginner}}{\text{result virtual advanced player} - \text{result beginner}} \\ &= \frac{\text{potential leaning effect}}{\text{potential learning effect} + \text{potential random effect}} \end{aligned}$$

In the last expression we call the difference of the real advanced player with the beginner the *potential learning effect* because it does not measure what an average player will learn but what a game allows potentially to be learned. The same interpretation holds for potential random effect as

the difference between the virtual and real advanced player. This quantitative way of measuring skill is quite effective as long as the strategies of advanced players can be determined. This is especially the case for one-person games. Advanced strategies of more-person games are more difficult to obtain. However, analysis of simplified versions of complex games can provide additional information.

Before we proceed on the quantitative way in some detail, we want to discuss a more qualitative way to get some impression about skill via the learning effect and random effect by considering indicators for those effects (cf. Van der Genugten et al. (1997)).

An important tool is to look at the structure of the game as it evolves in time. This will generate a game tree of all possible random moves (outcomes of chance elements) and at each stage of the game the possible moves of each player together with the history of play up to this stage and the information the player has acquired at that stage. In game theory this game tree is a standard concept and is called the representation of the game in its extensive form. Basically, the game tree is the structure to look at for all aspects of the game including skill. For games in practice these game trees are very complex. Roughly speaking, the more random moves the higher the random effect and the more players moves and information the higher the learning effect. These two indicators have an opposite effect on skill.

In one-person games the historical information at a certain stage of the player can only help him with the choice of his moves (e.g. remembering cards played in Blackjack). It helps him to choose a good move. We call the ability to exploit the historical information *observational skill*.

In more-person games observational skill is an important ability too. However, in such games the other players not necessarily have the same historical information a particular player has when having to choose his move. A player can profit from the uncertainty of the others by varying his moves in a reasonable chosen way. This is often far more better than making each time the same move in a particular situation. We call this *strategic skill*. In fact, this way of playing can result in moves which are very bad if the other players would have the same information he has (e.g. raising in Poker with low cards). We call the ability of playing in such a way *strategic bluffing*. There is nothing mysterious about this bluffing skill: its effectiveness can be proved in a mathematical precise way. It is part of strategic skill.

There is a lot of misunderstanding about bluffing in card games, especially in Poker. Besides strategic bluffing one can observe also psychological bluffing. By this we mean misleading opponents with tools outside the formal rules of the game (e.g. playing attitude as is always strongly suggested in Western movies). We want to emphasize that in our analysis we totally disregard psychological bluffing in this sense.

Summarizing, important indicators for a learning effect are observational skill and, for more-person games, strategic skill (including strategic bluffing).

3 One-person games

In this section we formally introduce a definition of the level of skill for one-person games like Roulette and Blackjack. In practice, even for relatively simple casino games, a player has to choose between various actions, possibly all with different implications. For this reason the analysis of one-person games cannot be based on the probabilities of winning for a beginner and an optimal player and on the gains in case of a loss and win, simply because these concepts cannot be defined in an unambiguous way.

Instead we will start out directly from the following three notions for the different types of players:

- W_0 : result of a beginner
- W_m : result of an optimal player
- W_u : result of a virtual optimal player.

Here, the following question arises. What is the appropriate definition of the *result* of a player type? To answer this question one has to take into account not only the fact that our aim is to compare different games in a consistent way (a global argument) but also the fact that we want to have a "fair" comparison between the different types of players within a specific game with its own typical characteristics (a local argument). The underlying (local) problem mainly has to do with the fact that variation in stakes can be an important part of the strategy of a player. For example using high stakes in Blackjack with good cards and low stakes with bad cards.

Recall that we supposed (R1) that the stakes and the game-result can

be both expressed in terms of money. The first possibility to define the result of a player is to disregard the stakes entirely and to define the result of a player-type as the (expected) *direct payoffs* arising from the game. Let P_0, P_m and P_u denote these payoffs for the three different types of players distinguished before. A second possibility is to disregard only the variation in stakes between players in the evaluation of skill and to define the result of a player type as the expected *net gains* per instance of the game. Denoting these expected net gains by G_0, G_m and G_u and the expected stakes per instance by I_0, I_m and I_u , it is clear that $G_0 = P_0 - I_0, G_m = P_m - I_m$ and $G_u = P_u - I_u$. A third possibility would be to look at the expected *returns* which takes the variation of stakes into account, per unit of stake-money. Denoting these returns by R_0, R_m and R_u we have

$$R_0 = \frac{P_0}{I_0} = \left(\frac{G_0}{I_0} + 1 \right),$$

$$R_m = \frac{P_m}{I_m} = \left(\frac{G_m}{I_m} + 1 \right),$$

$$R_u = \frac{P_u}{I_u} = \left(\frac{G_u}{I_u} + 1 \right).$$

The choice of a particular notion of result is not as innocent as it may perhaps seem: it implicitly implies an assumption on the goals of the players in a game. It will be typically the case that the strategy choice of a player (and in particular of an advanced player) will be different if he focusses on net gains or on returns and use the corresponding notations G_0, G_m and G_u for result. In this paper we will focus on result measured in terms of net gains. An analysis based on return is very well possible but the games under consideration become more complex. The main reason is that in a game with payoffs representing returns one loses linearity of the payoffs functions.

Note that G_0 and G_m are fully determined by a description of the strategy of a beginner and an optimal player, respectively and that $G_0 \leq G_m \leq G_u$.

For one-person games with

$$G_0 < G_u$$

we define the level of skill by

$$S := \frac{G_m - G_0}{(G_u - G_m) + (G_m - G_0)} = \frac{G_m - G_0}{G_u - G_0}.$$

Example 1 (American Roulette)

In American Roulette a player can choose between a great variety of actions. The main choice however is between "simple" (e.g. rouge/noir, pair/impair) or not (e.g. plein, cheval, carré). For precise details also on the calculations below, we refer to Van der Genugten and Borm (1994).

Suppose one chooses not to play "simple". Then, no matter what the precise action is, one's expected gains are $-\frac{1}{37}$ if we normalize the stake-money to 1.

For a fair comparison between the three player types in Roulette it seems reasonable to assume that each player-type can only use in each instance of the game (there is no real difference between instances) stakes equal to one.

Now suppose one chooses to play "simple". Then the winning outcomes never include "0", but if the outcome of a game is "0", one gets half of the state-money back. This leads to the expected gains of $-\frac{1}{74}$.

Consequently, an optimal player will choose "simple". Hence, $G_u = 35$ (knowing the outcome a virtual optimal player will choose plein on this outcome) and $G_m = -\frac{1}{74}$.

What about G_0 ? One can discuss on the behaviour of a beginner in American Roulette. If one assumes that a beginner chooses to play simple, then $G_0 = -\frac{1}{74}$ and $S = 0$.

If one assumes that a beginner chooses not to play simple, then $G_0 = -\frac{1}{37}$.

By substitution it then follows that

$$S = \frac{-\frac{1}{74} + \frac{1}{37}}{35 + \frac{1}{37}} \approx 0.0004.$$

This is a very low value and we may conclude that American Roulette approximately is a pure game of chance.

Example 2 (Blackjack)

Blackjack as played in the Holland Casino's is extensively studied in Van der Genugten (1993,1997). Using the technique of simulation the following numbers were derived for the case where the maximum stake is normalized to 50 and the minimum stake is 1:

$$\begin{aligned}G_0 &= -0.057 \\G_m &= 0.11 \\G_u &= 27.\end{aligned}$$

The value of G_0 corresponds to the strategy "mimic the dealer" with minimum stakes. The value of G_u reflects the fact that the rules of the game are such that even if one knows the outcome of the chance mechanism, i.e. one knows the precise order of the cards, one can not always succeed in winning.

By substitution we find that $S = 0.006$.

The Supreme Court in the Netherlands has explicitly qualified Blackjack as a game of chance. Hence, if one uses the above measure for the notion of skill, a consistency argument implies that each game with a level of skill below that of blackjack should be considered as a game of chance w.r.t. Dutch legislation.

Example 3 (Golden Ten)

The skill of this game was analyzed in Van der Genugten and Borm (1991) on the base of statistical data concerning the results of actual players. It was the starting point for the very thorough study of De Vos (1997). Here the main part is a statistical analysis of experimental data of trajectories using the underlying laws of mechanics. Combined with results of actual play in different locations and settings the conclusion is reached that the skill of Golden Ten is somewhere between 0.02 and 0.10 (see Ch. 2, section 2.3.3).

4 Two-person zero-sum games

In a two-person game typically the net gains of a player do not only depend on his own strategy choice but also on the strategy choice of the other player. In this section we will consider strictly competitive games in the sense that the gains of one player equal the losses of the other in each instance of the game. Put differently, the players exchange an amount

of money specified by the outcome corresponding to one specific strategy-combination. This type of game is called a zero-sum game because the total gains of the players equal zero. For example, a Poker game as Seven Card Stud typically can be classified as a zero-sum game: apart from an admission fee, money is reallocated among the participants.

To study skill in this type of games we will use an objective analysis of strategic behaviour (such as strategic bluffing) based on the assumption of rational players. This approach follows the lines set out by *game theory*: a mathematical theory of conflict situations which started out from the pioneering work of von Neumann and Morgenstern (1944).

Consider an arbitrary two-person zero-sum game where the payoffs are assumed to represent net gains in which both player 1 and player 2 can choose from a finite number of actions. In this game we will allow a player to choose a mixed strategy, which provides a probability distribution on the set of actions.

Then this game has a uniquely determined (minimax) *value* V in the sense that player 1 can choose a mixed strategy which guarantees him an expected payoff of at least V (independent of player 2's strategy choice), and player 2 has a mixed strategy which keeps his losses down to at most V . The strategies concerned are called minimax strategies and can be interpreted as describing optimal play. Due to these facts we are able to extend the definition of the level of skill to an arbitrary (finite) zero-sum game with two players 1 and 2.

Assume we are in the role of player 1. If we fix a specific minimax strategy of player 2 in the sense that we assume that player 2 will act accordingly, the original two-person game can be reduced to a one-person game because player 1 faces a fixed but probabilistic environment, and the three basic notions of result (w.r.t. player 1) based on net gains are well-defined: $G_0(1) \leq G_m(1) \leq G_u(1)$. Note that, by definition, $G_m(1)$ equals the value of the game.

Analogously, in the role of player 2, we assume a specific minimax strategy of player 1 to be given, which reduces the situation to a one-person game. This leads to the following notions of expected gains w.r.t. player 2: $G_0(2) \leq G_m(2) \leq G_u(2)$, where $G_m(2)$ equals minus the value of the game.

To define the overall notion of skill in the game it is reasonable to average between the role of player 1 and player 2, implicitly assuming a player of

the game to take both the role of player 1 and player 2 every two instances of the game. This leads to

$$\begin{aligned} G_0 &= \frac{1}{2}(G_0(1) + G_0(2)), \\ G_m &= \frac{1}{2}(G_m(1) + G_m(2)), \\ G_u &= \frac{1}{2}(G_u(1) + G_u(2)), \end{aligned}$$

and, clearly, $G_0 \leq G_m \leq G_u$.

Further note that $G_m = 0$ since $G_m(1) = -G_m(2)$. As before, we will assume that

$$G_0 < G_u.$$

Subsequently, we define

$$S = \frac{G_m - G_0}{G_u - G_0}$$

as the level of skill.

It is readily verified that the level of skill for two-person games generalizes the definition for one-person games. Consider a one-person game with player 1 as its only participant who receives or pays money from or to a bankholder. This game can be viewed as a two-person zero-sum game in which the role of player 2 is taken by the bankholder. By definition of a one-person game, player 2 has no freedom in his choice of action and follows a prescribed strategy.

Hence, $G_0(1)$, $G_m(1)$ and $G_u(1)$ equal the corresponding one-person notions introduced in section 3 and the level of skill S^1 in terms of (net) gains in the one-person framework is given by

$$S^1 = \frac{G_m(1) - G_0(1)}{G_u(1) - G_0(1)}.$$

In the two-person framework we also have to analyze the role of player 2. However, for this analysis we assumed optimal behaviour of player 1 and since player 2 has no freedom of strategy choice it follows that

$$G_0(2) = G_m(2) = G_u(2) = -G_m(1).$$

Hence,

$$G_0 = \frac{1}{2}(G_0(1) - G_m(1)), G_m = 0 \text{ and } G_u = \frac{1}{2}(G_u(1) - G_m(1))$$

	<u>1</u>	<u>2</u>	<u>3</u>	<u>4</u>	<u>5</u>	<u>6</u>	<u>7</u>	<u>8</u>
10:	C	C	C	C	R	R	R	R
20:	C	C	R	R	C	C	R	R
30:	C	R	C	R	C	R	C	R

Table 1: Pure strategies of player 1. (C = Check, R = Raise)

and by substitution it follows that the level of skill S for the two-person game equals S^1 .

Example 4 (Two-person Mini Poker)

Two-person Mini Poker is a game of cards played by two players, named player 1 and player 2, and with three cards of which only the numeric value is important. These values are 10, 20 and 30, respectively.

Before playing both players donate \$ 1 to the stakes. After (re)shuffling the deck of cards each player is dealt one card. Each player knows (the value of) his own card but not the card of this opponent. Thus, the one card which remains in the deck is not shown to either of the players.

Player 1 starts the play and has to decide between “checking” or “raising”. If he decides to check, player 2 has to check too and “showdown” follows. If player 1 decides to raise, he has to add one extra dollar to the stakes. Subsequently, player 2 has to decide between “folding” or “calling”. If he decides to fold, player 1 gets the stakes. If player 2 decides to call, he also has to add one extra dollar to the stakes and “showdown” follows. If the players have decided upon “showdown” both cards are compared and the player with the highest card value gets the stakes.

In this simple game of Poker both chance (dealing cards) and skill (w.r.t. a good betting strategy) play a role. We will reach the conclusion that the level of skill S equals 0.2.

First we will describe the possible strategies of both players. For each of the three possible cards, both player 1 and player 2 have to choose between two possible actions, leading to a total of 8 combinations. The so-called pure strategies are represented in table 1 and table 2.

For example, the strategy 4 of player 1 should be interpreted as

“Check with 10, Raise with 20, Raise with 30”,

	<u>1</u>	<u>2</u>	<u>3</u>	<u>4</u>	<u>5</u>	<u>6</u>	<u>7</u>	<u>8</u>
10:	F	F	F	F	C	C	C	C
20:	F	F	C	C	F	F	C	C
30:	F	C	F	C	F	C	F	C

Table 2: Pure strategies of player 2. (F = Fold, C = Call)

and strategy 6 of player 2 as

“Call with 10, Fold with 20, Call with 30”,

if you are called upon to act.

Allowing for mixed strategies, each player may put a probability measure on his set of pure strategies. Mixed strategies will be denoted in the following way: the strategy $(\frac{1}{2}2, \frac{1}{2}4)$ of player 1 will indicate that player 1 will choose the pure strategy 2 with probability $\frac{1}{2}$ and the pure strategy 4 with probability $\frac{1}{2}$. In the equivalent behavioral interpretation this boils down to:

“Check with 10, Raise with 30, and with 20:

“Check with probability $\frac{1}{2}$ and Raise with probability $\frac{1}{2}$ ”.

In analysing the level of skill of this game of Poker we first need the strategy of a beginner in both the role of player 1 and player 2.

W.r.t. player 1 we find it reasonable for a naive player to check with 10 and to raise with 30. How to act with 20? Probably it is wise to vary between checking and raising, and using symmetry arguments, to check and raise with equal probability. In fact, the same reasoning applies w.r.t. player 2. If he is called upon to act, we assume he folds with 10, calls with 30 and folds and calls with equal probability with 20. Using the notation introduced above, this boils down to the beginner's strategy $(\frac{1}{2}2, \frac{1}{2}4)$ for both roles.

The next step is to determine minimax strategies for both player 1 and player 2. To this aim we first calculate the 8×8 matrix which describes the expected payoff to player 1 for each possible pure strategy combination. The result is presented in table 3.

More specifically, in table 3 the result of a specific strategy combination is a fair average over the outcomes w.r.t. the six possible card combinations.

pl. 2 pl. 1	<u>1</u>	<u>2</u>	<u>3</u>	<u>4</u>	<u>5</u>	<u>6</u>	<u>7</u>	<u>8</u>
<u>1</u>	0	0	0	0	0	0	0	0
<u>2</u>	0	0	1/6	1/6	1/6	1/6	1/3	1/3
<u>3</u>	1/3	-1/6	1/3	-1/6	1/2	0	1/2	0
<u>4</u>	1/3	-1/6	1/2	0	2/3	1/6	5/6	1/3
<u>5</u>	2/3	1/6	1/6	-1/3	2/3	1/6	1/6	-1/3
<u>6</u>	2/3	1/6	1/3	-1/6	5/6	1/3	1/2	0
<u>7</u>	1	0	1/2	-1/2	7/6	1/6	2/3	-1/3
<u>8</u>	1	0	2/3	-1/3	4/3	1/3	1	0

Table 3: Two-person Mini Poker as a matrix game

pl. 2 pl. 1	<u>2</u>	<u>4</u>
<u>2</u>	0	1/6
<u>6</u>	1/6	-1/6

Table 4: The reduced game

By iteratively deleting dominated strategies (e.g. strategy 1 of player 1 is dominated by strategy 2), the 8×8 matrix game of table 4.3 can be reduced to the simple 2×2 matrix game represented in table 4. From table 4 it is readily derived that the value of the game is $\frac{1}{18}$ and the minimax strategies of the players are uniquely determined:

$(\frac{2}{3}\underline{2}, \frac{1}{3}\underline{6})$ for player 1 and $(\frac{2}{3}\underline{2}, \frac{1}{3}\underline{4})$ for player 2.

This boils down to the following (behavioral) strategy-schemes:

For player 1:

With 10: C with probability $\frac{2}{3}$, R with probability $\frac{1}{3}$

With 20: C

With 30: R

For player 2 (if called upon to act):

With 10: F

	player 2: $(\frac{2}{3}\underline{2}, \frac{1}{3}\underline{4})$					
card comb.	(10, 20)	(10, 30)	(20, 10)	(20, 30)	(30, 10)	(30, 20)
Check	-1	-1	1	-1	1	1
Raise	0	-2	1	-2	1	$\frac{4}{3}$
Max	0	-1	1	-1	1	$\frac{4}{3}$

Table 5: The calculation of $G_u(1) = \frac{2}{9}$.

With 20: F with probability $\frac{2}{3}$, C with probability $\frac{1}{3}$

With 30: C .

For determining $G_0(1)$, $G_m(1)$ and $G_u(1)$ we assume that player 2 behaves according to his unique optimal strategy $(\frac{2}{3}\underline{2}, \frac{1}{3}\underline{4})$.

Since we have chosen $(\frac{1}{2}\underline{2}, \frac{1}{2}\underline{4})$ as a beginner's strategy for player 1, it follows from table 3 that

$$G_0(1) = \frac{1}{2} \cdot \frac{2}{3} \cdot (0) + \frac{1}{2} \cdot \frac{1}{3} \cdot \left(\frac{1}{6}\right) + \frac{1}{2} \cdot \frac{2}{3} \cdot \left(-\frac{1}{6}\right) + \frac{1}{2} \cdot \frac{1}{3} \cdot (0) = -\frac{1}{36}.$$

Further, since $G_m(1)$ equals the value of the game, we have that

$$G_m(1) = \frac{1}{18}.$$

To determine $G_u(1)$, we have to consider a virtual optimal player in the role of player 1. By assumption, a virtual optimal player will know the precise card combination (C_1, C_2) , with $C_i \in \{10, 20, 30\}$, $C_1 \neq C_2$, denoting the value of the card of player i , before having to decide upon checking or raising.

Moreover, since we fixed the strategy of player 2 to $(\frac{2}{3}\underline{2}, \frac{1}{3}\underline{4})$ one can calculate for each card combination what the expected gains of the two possible actions will be. For each card combination a virtual optimal player will subsequently choose the action with minimal (maximal) gains. The number $G_u(1)$ then will be the fair average of the six corresponding gains. This computation, which is illustrated in table 5, leads to $G_u(1) = \frac{2}{9}$.

In an analogous way one derives

$$G_0(2) = -\frac{1}{18}, G_m(2) = -\frac{1}{18} \text{ and } G_u(2) = \frac{1}{9}.$$

Hence,

$$G_0 = -\frac{1}{24}, G_m = 0, G_u = \frac{1}{6}$$

and, by substitution, the level of skill equals

$$S = 0.2.$$

Example 4 is rather special in the sense that the minimax strategies of both player 1 and player 2 are uniquely determined. If this is not the case, the analysis requires a specific choice between all minimax strategies of a player. A possibility to do this would be to select what we would like to call a *learning minimax* strategy by following a specific (modification of) learning procedure as initially proposed by Brown (1949) and Robinson (1950), and elaborated on by Shapiro (1958) and Karlin (1959a, 1959b).

5 *n*-Person Zero-Sum Games

For a two-person zero-sum game the players are direct opposites in the sense that the gains of one player equal the losses of the other in each instance of the game.

Although in each instance of an *n*-person zero-sum game with $n \geq 3$ money is reallocated among the players, two particular participants cannot be viewed as direct opposites in the sense that they should act in a way to oppose each other in the strongest sense. The mutual competition is of a more indirect and complex nature. Moreover, it is conceivable that subgroups of players will (possibly implicitly) cooperate during certain stages of the game to oppose the other participants in a more effective way. Put in game theoretic terms this means that coalitions (alliances) between players are formed. The situation is even more complex since coalitions might change during one instance of the game.

Since our aim is to measure skill relatively to the impact of chance elements in a game, we will not analyze an *n*-person game in all its strategic and cooperative details. Just to determine the level of skill for this type of games in a consistent way, we make the assumption that in each instance of the game player *i* evaluates the situation as if he faces only one opponent: the coalition of all other participants which coordinates actions in such a way to oppose player *i* in the strongest sense.

These considerations boil down to the following definitions w.r.t. the level of skill of an arbitrary zero-sum game with player set $N = \{1, 2, \dots, n\}$ with $n \geq 3$.

Assume one is in the role of player $i \in N$. By assumption we consider the two-person zero-sum game between player i and the coalition $N \setminus \{i\}$ in our analysis of skill. If we fix a specific minimax strategy of "player" $N \setminus \{i\}$ in this two-person game, a one-person situation is obtained and the following basic notions of expected gains of player i are well-defined:

$$G_0(i) \leq G_m(i) \leq G_u(i).$$

To define the general notion of skill, we average between the possible roles a participant can have:

$$G_0 = \frac{1}{n} \sum_{i \in N} G_0(i), G_m = \frac{1}{n} \sum_{i \in N} G_m(i), G_u = \frac{1}{n} \sum_{i \in N} G_u(i)$$

and, clearly, $G_0 \leq G_m \leq G_u$. As before, we will assume that

$$G_0 < G_u.$$

Subsequently, we define

$$S(G) = \frac{G_m - G_0}{G_u - G_0}$$

as the level of skill.

Note that this definition of the level of skill also can be adopted if $n = 2$. Obviously, this renders the same definition as provided in section 4.

Example 5 (Three-person Mini Poker)

Three-person Mini Poker is a game of cards played by three players 1, 2 and 3 and four cards with numerical values 10, 20, 30 and 40.

The rules are similar to the rules of two-person Mini Poker. After donating \$ 1 to the stakes each player is dealt one card. Player 1 starts the play and has to decide between checking or raising. If he checks, player 2 and 3 have to check too and showdown follows. If he raises, then first player 2 has to decide between folding or calling, after which player 3 has to make the same choice. If a player raises or calls, he has to add one extra dollar to the stakes. If both player 2 and 3 fold the stakes go to player

Card of 1	10	20	30	40
Action	C	C	R	R

Table 6: Player 1's strategy as a beginner (C = Check, R = Raise)

Card of 2	10	20	30	40
Action	F	F	C	C

Table 7: Player 2's strategy as a beginner (F = Fold, C = Call)

1. Otherwise, showdown between the two or three players left follows. If showdown occurs, the player with the highest (value of his) card gets the stakes.

We will not analyze this game in all its details but only highlight some important aspects. In principle player 1 can choose between $16 = 2^4$ possible (pure) strategies: for each of his four possible cards he can choose between checking or raising. For the strategy of a beginner in the role of player 1 we take the pure strategy of Table 6.

Player 2 is only called to play if player 1 raises. In that case he has to decide between folding or calling. So, in principle, also player 2 can choose between $16 = 2^4$ possible pure strategies because his decision might depend on his card.

For a beginner in the role of player 2 we take the choices of table 7.

For player 3 the situation is a bit more complicated in the sense that he could let his choice between folding or calling depend not only on his card but also on the action taken by player 2. This implies a total of $256 = 2^8$ pure strategies. For a beginner in the role of player 3 we take the strategy represented in table 8.

Card of 3	10	10	20	20	30	30	40	40
Observed action of 2	F	C	F	C	F	C	F	C
Action of 3	F	F	F	F	C	C	C	C

Table 8: Player 3's strategy as a beginner (F = Fold, C = Call)

Card of 2	10	20	30	40
Action of 2	F	F	F	C

Card of 3	10	10	20	20	30	30	40	40
Observed action of 2	F	C	F	C	F	C	F	C
Action of 3	F	F	F	C	$(\frac{1}{2} F, \frac{1}{2} C)$	C	C	C or F

Table 9: The minimax strategy of coalition $\{2, 3\}$

Card of 1	10	20	30	40
Action of 1	$(\frac{3}{4} C, \frac{1}{4} R)$	$(\frac{3}{4} C, \frac{1}{4} R)$	C	R

Table 10: The minimax strategy of player 1

Note that in the game of player 1 against the coalition $\{2, 3\}$ which we use in our analysis of skill player 1 can choose between the same 16 pure strategies, while the coalition $\{2, 3\}$ can choose between 4096 ($= 16 \times 256$) pure strategies. For the corresponding two-person game the (coordinated) minimax strategy of "player" $\{2, 3\}$ is given in Table 9 (with obvious notation).

For this two-person game the minimax strategy for player 1 is given in table 10.

The value of this game is $\frac{1}{24}$. Following a similar analysis as in example 4 we find

$$G_0(1) = -0.125, G_m(1) = 0.041 \text{ and } G_u(1) = 0.375.$$

Moreover, proceeding along the same lines and using learn minimax strategies (for details we refer to van der Genugten and Borm (1994)):

$$\begin{aligned} G_0(2) &= -0.028, & G_m(2) &= -0.028, & G_u(2) &= 0.140 \\ G_0(3) &= -0.041, & G_m(3) &= -0.021, & G_u(3) &= 0.063 \end{aligned}$$

and, consequently

$$G_0 = -0.064, G_m = -0.002 \text{ and } G_u = 0.193.$$

Hence, for the level of skill we find that

$$S = 0.24.$$

	Observational skill	Strategic skill
Draw Poker	-	+
Texas Hold'Em	0	+
Seven Card Stud.	+	+

Table 11: A qualitative judgement of skill for Poker

6 A classification of games

In this section we present a global ordering of some well-known round games w.r.t. the level of skill. The corresponding order is partly based on the (quantitative) analysis introduced in this paper and partly on a (qualitative) comparison between the various possibilities of exercising skill in the underlying games. For a more elaborate discussion and motivation we refer to Van der Genugten and Borm (1994) and Van der Genugten et al. (1997).

The main results are summarized in the following overview.

Pure games of chance

Roulette, Craps, Trente et Quarante
 Blackjack
 Golden Ten
 Schnapsen
 Draw Poker
 Texas Hold'Em
 Seven Card Stud
 Bridge
 Chess, Checkers

Pure games of skill

The ordering between the three variants of Poker is based on the following considerations summarized in table 11 with respect to observational skill and strategic skill.

Interestingly, Dutch jurisprudence has classified Golden Ten as a game of chance while according to Austrian jurisprudence Schnapsen is a game of skill (cf. Grossmann (1997)). If we combine these judgements, the decision

bound between games of chance and games of skill should be laid between Golden Ten and Schnapsen.

Of course, the classification given above should not be interpreted as the final order in which no shifts are possible. A further analysis, in particular w.r.t. more sophisticated variants of Poker than the ones discussed in section 4 and 5, is needed to further sharpen and quantify the classification.

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