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An infinite family of quasi-symmetric designs

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Dedicated to S.S. Shrikhande

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Abstract

Presented is a construction of quasi-symmetric $2-(q^3, q^2(q-1)/2, q(q^3 - q^2 - 2)/4)$ designs with block intersection numbers $q^2(q-2)/4$ and $q^2(q-1)/4$, where q is a power of 2. The framework is given by the 3-dimensional affine space over $\mathbf{F}_q$.

In 1963, S.S. Shrikhande and D. Raghavarao [4] presented a construction method for block designs by which a $2-(q^3, q^2(q-1)/2, q(q^3 - q^2 - 2)/4)$ design can be made from a resolvable $2-(q^3, q, 1)$ design and a symmetric $2-(q^2, q(q-1)/2, q(q-2)/4)$ design. If q is a power of 2 the smaller designs both exist and thus the method applies. In this note it is shown that this construction can be organised such that the obtained design has just two intersection numbers $q^2(q-2)/4$ and $q^2(q-1)/4$.

The reader is assumed to be familiar with the basic notions and ideas from design theory and finite geometry (see for example [1] and [5]). Nevertheless we recall some relevant definitions. A 2-design $\mathcal{D}$ is α -*resolvable* if the blocks of $\mathcal{D}$ can be partitioned into classes (called resolution classes), such that for each point p of $\mathcal{D}$ there are exactly α blocks containing p in each resolution class. If $\alpha = 1$ the design is usually just called *resolvable*, and resolution classes are *parallel* classes. An α -resolvable design is *strongly resolvable* if there exist integers x and y , such that two distinct blocks from one class meet in x points and two blocks from different classes meet in y points. The numbers that occur as the size of the intersection of two distinct blocks are the *intersection numbers* of the design. A design with just two intersection numbers is called *quasi-symmetric*. Clearly strongly resolvable designs are quasi-symmetric, and so

are all Steiner 2-designs and their complements. For a survey of these designs see M.S. Shrikhande and S.S. Sane [2] and M.S. Shrikhande [3]. The designs presented here are new if $q \geq 4$ ($q = 2$ gives the trivial $2-(8, 2, 1)$ design). If $q = 4$ one gets parameter set number 67 in the table given in [3].

The mentioned construction of S.S. Shrikhande and D. Raghavarao can be described in terms of matrices as follows.

Lemma 1 *Let $A = [A_1 \ A_2 \ \dots \ A_r]$ be the $v \times b$ incidence matrix of a resolvable $2-(v, k, \lambda)$ design $\mathcal{D}$, wherein $A_1, \dots, A_r$ are the $v \times m$ ($m = b/r = v/k$) incidence matrices of the resolution classes. Let $B_1, \dots, B_r$ be the $m \times b'$ incidence matrices of r $2-(m, k', \lambda')$ designs $\mathcal{D}'_1, \dots, \mathcal{D}'_r$ with r' blocks through a point. Let $\tilde{B}$ be the block diagonal matrix $\text{diag}(B_1, \dots, B_r)$. Then $Q = A\tilde{B}$ is the incidence matrix of a $2-(v, kk', r\lambda' + r'\lambda - \lambda\lambda')$ design $\mathcal{Q}$.*

Proof. Define $K = I_r \otimes J_m$ (J_m is the $m \times m$ all-one matrix). Then $\tilde{B}\tilde{B}^\top = \lambda'K + (r' - \lambda')I_b$ and $AK = J_b$. So

$$\begin{aligned} QQ^\top &= A\tilde{B}\tilde{B}^\top A^\top = \lambda'AKA^\top + (r' - \lambda')AA^\top \\ &= (r\lambda' + r'\lambda - \lambda\lambda')J_v + (r' - \lambda')(r - \lambda)I_v, \end{aligned}$$

and $Q^\top \mathbf{1} = \tilde{B}^\top A^\top \mathbf{1} = k'k\mathbf{1}$ ($\mathbf{1}$ is the all-one vector). □

The intersection numbers of $\mathcal{Q}$ are given by the off-diagonal entries of

$$Q^\top Q = \tilde{B}^\top \begin{bmatrix} A_{11} & \dots & A_{1r} \\ \vdots & & \vdots \\ A_{r1} & \dots & A_{rr} \end{bmatrix} \tilde{B},$$

where $A_{ij} = A_i^\top A_j$, and in particular $A_{ii} = kI_m$. Suppose that each $\mathcal{D}'_i$ is a symmetric design. Then the diagonal blocks of $Q^\top Q$ are equal to $B_i^\top A_{ii} B_i = kB_i^\top B_i = k\lambda'J_m + k(k' - \lambda')I_m$, so $\mathcal{Q}$ is quasi-symmetric whenever there exist an integer y say, such that each entry of $B_i^\top A_{ij} B_j$ equals y or $k\lambda'$ ($i \neq j$). For example if $\mathcal{D}$ is strongly resolvable with intersection numbers $x = 0$ and y , then $A_{ij} = yJ_m$ ($i \neq j$), and it follows that $\mathcal{Q}$ is also strongly resolvable with intersection numbers $k\lambda'$ and k^2y .

We shall apply Lemma 1 with the $2-(q^3, q, 1)$ design $\mathcal{D}$ of points and lines in $AG(3, q)$ (the 3-dimensional affine space over $\mathbf{F}_q$) and symmetric $2-(q^2, q(q-1)/2, q(q-2)/4)$ designs $\mathcal{D}'_i$, where q is a power of 2. Then, for each parallel class $\mathcal{C}_i$ of $\mathcal{D}$, the points of $\mathcal{D}'_i$ have to correspond to the lines of $\mathcal{C}_i$. This correspondence will be realised by defining the required symmetric designs $\mathcal{D}'_i$ on the lines of $\mathcal{C}_i$, using the fact that the lines of $\mathcal{C}_i$ and the planes spanned by these lines form the affine plane $AG(2, q)$. Each block of $\mathcal{D}'_i$ will be a $(q(q-1)/2, q/2)$ -arc in $AG(2, q)$. An arc Ω with these parameters is maximal and consists of $q(q-1)/2$ points, such that each line meets Ω in 0 or $q/2$ points. The lines not

meeting Ω (including the line at infinity) form a dual hyperoval in $PG(2, q)$, which is a set of $q + 2$ lines no three concurrent. So in $AG(2, q)$ this gives a set of $q + 1$ lines, no three concurrent, all having different slopes. And vice versa, the points not on a line of a dual hyperoval in $PG(2, q)$ containing the line at infinity give a maximal arc in $AG(2, q)$ with the above parameters.

Lemma 2 *For q even, there exist a collection of q^2 maximal arcs in $AG(2, q)$ that forms a symmetric 2 -($q^2, q(q-1)/2, q(q-2)/4$) design $\mathcal{D}'$.*

Proof. Let Ω be any $(q(q-1)/2, q/2)$ -arc. The blocks of $\mathcal{D}'$ are the q^2 translates of Ω . For any two distinct blocks Ω_1 and Ω_2 there is exactly one line (the one of the direction of the translation) that misses both. Counting flags (p, ℓ) with $p \in \Omega_1 \setminus \Omega_2$ and ℓ missing Ω_2 , yields that $2 \cdot |\Omega_1 \setminus \Omega_2| = q \cdot q/2$. So Ω_1 and Ω_2 intersect in $q(q-2)/4$ points and therefore $\mathcal{D}'$ is a symmetric design with the required parameters. $\square$

Now it is clear how $\mathcal{Q}$ is constructed. The points of $\mathcal{Q}$ are the points of $AG(3, q)$; the blocks are the unions of all sets of $q(q-1)/2$ parallel lines that correspond to a maximal arc from Lemma 2 in the affine plane formed by the parallel class.

Lemma 3 *With the notation above, if $i \neq j$ then each entry of $B_i^\top A_{ij} B_j$ equals $q^2(q-2)/4$ or $q^2(q-1)/4$.*

Proof. Fix i and j ($i \neq j$). The matrix A_{ij} gives the intersection of the lines of the two parallel classes $\mathcal{C}_i$ and $\mathcal{C}_j$ of $AG(3, q)$. These lines are lying in q parallel planes. For this set of planes, let S_i and S_j be the $q^2 \times q$ line-plane incidence matrix of class $\mathcal{C}_i$ and $\mathcal{C}_j$, respectively. Then $A_{ij} = S_i S_j^\top$.

In $AG(2, q)$ there is exactly one line in each direction that does not meet a $(q(q-1)/2, q/2)$ -arc Ω . This implies that in each row of $B_i^\top S_i$, one entry equals 0 and $q-1$ entries are equal to $q/2$. Hence, for $i \neq j$, each entry of $B_i^\top A_{ij} B_j = B_i^\top S_i S_j^\top B_j$ equals $q^2(q-2)/4$ or $q^2(q-1)/4$. $\square$

The off-diagonal entries of $Q^\top Q$ are equal to $k\lambda' = q^2(q-2)/4$ on the diagonal blocks and, by Lemma 3, equal to $q^2(q-2)/4$ or $q^2(q-1)/4$ on the off-diagonal blocks. This proves the main result:

Theorem 1 *$\mathcal{Q}$ is a quasi-symmetric 2 -($q^3, q^2(q-1)/2, q(q^3-q^2-2)/4$) design with intersection numbers $q^2(q-2)/4$ and $q^2(q-1)/4$.*

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