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ANTIDUALITY IN EXACT PARTITION GAMES

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Antiduality in exact partition games

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Highlights

- We show that the egalitarian Dutta and Ray (1989) solution for transferable utility games is self-antidual on the class of exact partition games.
- We derive several new axiomatic characterizations of the Dutta and Ray (1989) solution on the class of exact partition games.
- We point out an error in earlier work on antiduality in convex games.
- We repair and strengthen several related characterizations on the class of convex games.

Abstract

This note shows that the egalitarian Dutta and Ray (1989) solution for transferable utility games is self-antidual on the class of exact partition games. By applying a careful antiduality analysis, we derive several new axiomatic characterizations. Moreover, we point out an error in earlier work on antiduality and repair and strengthen several related characterizations on the class of convex games.

Keywords: transferable utility games; egalitarianism; antiduality; exact partition games; convex games

JEL classification: C71

1 Introduction

Allocation problems where agents are able to generate joint revenues in coalitions can be modeled as cooperative games with transferable utility. In this model, the worth of each coalition represents the monetary opportunities when the subgroup of players acts on its own. These worths serve as benchmarks for a fair allocation of the worth of the grand coalition. One of the main issues in cooperative game theory is to design solutions which prescribe for each such game a payoff allocation. The desirable features of these allocation rules are formulated as axioms.

This note focuses on the structure and implications of a specific set of axioms related to the egalitarian Dutta and Ray (1989) solution. Based on a framework where individuals believe in equality as a desirable social goal, although private preferences dictate selfish behavior, this solution assigns to each game the Lorenz undominated allocation in the Lorenz

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core. Although such allocation is unique whenever it exists, only sufficient conditions for existence are known, the central one being convexity of the underlying game. For convex games, the Dutta and Ray (1989) solution prescribes the Lorenz dominating core element. The Dutta and Ray (1989) solution is mainly studied on the class of convex games and several axiomatic characterizations are provided. Already since the work of Dutta (1990), these characterizations are generally based on consistency properties. Whereas Dutta (1990) combined a fixed solution for two-player games with the consistency properties of Davis and Maschler (1965) and Hart and Mas-Colell (1989), these results were turned into full axiomatic characterizations by Klijn et al. (2000). Moreover, Klijn et al. (2000) provided a third characterization on the class of convex games based on an alternative consistency property.

Recently, Llerena and Mauri (2017) introduced the larger class of exact partition games and showed that the Dutta and Ray (1989) solution for these games behaves as for convex games, i.e. it assigns to each such game the Lorenz dominating core element. Dietzenbacher and Yanovskaya (2020) showed that some axiomatic characterizations of Klijn et al. (2000) for convex games can be extended to the class of exact partition games. Moreover, they provided another characterization based on the consistency property of Moulin (1985).

Oishi and Nakayama (2009) introduced an antiduality notion within the context of transferable utility games to structure games and solutions. Two games are antidual if the worth of each coalition in one game equals the worth of its complement minus the worth of the grand coalition in the other game. Two solutions are antidual if one solution assigns to each game the negated set of payoff allocations assigned by the other solution to the antidual game. A solution is self-antidual if it coincides with its antidual. Oishi and Nakayama (2009) showed that some well-known classes of games are antidual to each other and several solutions are self-antidual. Oishi et al. (2016) further developed the antiduality notion and showed that the classes of all games, balanced games, and convex games are each closed under antiduality. Moreover, they introduced antiduality for axioms and characterizations to uncover new structures. Two axioms are antidual if for each two antidual solutions, one property is satisfied by one solution if and only if the other property is satisfied by the other solution. An axiom is self-antidual if for each two antidual solutions, the property is satisfied by one solution if and only if it is satisfied by the other solution. In particular, Oishi et al. (2016) showed that the Dutta and Ray (1989) solution is self-antidual on the class of convex games and revealed new axiomatic characterizations antidual to the results of Klijn et al. (2000). However, these antidual characterizations are invalid due to an incorrect claim about self-antiduality of the axiom equal division stability.

The purpose of this note is twofold. On the one hand, we show that the class of exact partition games is closed under antiduality and that the Dutta and Ray (1989) solution is self-antidual on this class. By applying a careful antiduality analysis, we derive several new axiomatic characterizations of the Dutta and Ray (1989) solution on the class of exact partition games from the results of Dietzenbacher and Yanovskaya (2020). On the other hand, we repair and strengthen the results of Oishi et al. (2016) related to the Dutta and Ray (1989) solution on the class of convex games by providing the correct antidual of equal division stability and weaken the corresponding consistency properties.

This note is organized in the following way. Section 2 provides the preliminary notions and notations for transferable utility games and the Dutta and Ray (1989) solution. Section 3 performs an antiduality analysis on the class of exact partition games. Section 4 presents concluding remarks on antiduality in larger classes of games.

2 Preliminaries

Let N be a nonempty and finite set. Denote $2^N = \{S \mid S \subseteq N\}$. For each $x \in \mathbb{R}^N$, we define

$$\begin{aligned} R_0^x &= \emptyset \text{ and } R_k^x = \{i \in N \mid \forall j \in N \setminus R_{k-1}^x : x_j \leq x_i\} \text{ for each } k \in \{1, \dots, |N|\}; \\ P_0^x &= \emptyset \text{ and } P_k^x = \{i \in N \mid \forall j \in N \setminus P_{k-1}^x : x_j \geq x_i\} \text{ for each } k \in \{1, \dots, |N|\}. \end{aligned}$$

For each $x \in \mathbb{R}^N$ and each $S \in 2^N \setminus \{\emptyset\}$, $x_S \in \mathbb{R}^S$ denotes $x_S = (x_i)_{i \in S}$, and $x(S) \in \mathbb{R}$ denotes $x(S) = \sum_{i \in S} x_i$.

A *transferable utility game* is a pair (N, v) where N is a nonempty and finite set of *players* and $v : 2^N \rightarrow \mathbb{R}$ assigns to each *coalition* $S \in 2^N$ its *worth* $v(S) \in \mathbb{R}$ such that $v(\emptyset) = 0$. Let Γ_{all} denote the class of all games. The *core* of a game (N, v) is given by

$$C(N, v) = \{x \in \mathbb{R}^N \mid x(N) = v(N), \forall S \in 2^N : x(S) \geq v(S)\}.$$

A game (N, v) is a *balanced game* if $C(v) \neq \emptyset$. A game (N, v) is an *exact partition game* (cf. Llerena and Mauri (2017)) if there is $x \in C(N, v)$ such that $x(R_k^x) = v(R_k^x)$ for each $k \in \{1, \dots, |N|\}$. A game (N, v) is a *convex game* (cf. Shapley (1971)) if $v(S) + v(T) \leq v(S \cup T) + v(S \cap T)$ for each pair $S, T \in 2^N$. Let Γ_{bal} , Γ_{exp} , and Γ_{conv} denote the class of all balanced games, exact partition games, and convex games, respectively. We have $\Gamma_{conv} \subset \Gamma_{exp} \subset \Gamma_{bal} \subset \Gamma_{all}$. Throughout this note, (N, v) is the generic notation for a game and Γ is the generic notation for a class of games.

A *solution* σ on Γ assigns to each game $(N, v) \in \Gamma$ a nonempty set of payoff allocations $\sigma(N, v) \subseteq \mathbb{R}^N$ such that $x(N) \leq v(N)$ for each $x \in \sigma(N, v)$. The Dutta and Ray (1989) solution assigns to each exact partition game $(N, v) \in \Gamma_{exp}$

$$DR(N, v) = \{x \in C(N, v) \mid \forall k \in \{1, \dots, |N|\} : x(R_k^x) = v(R_k^x)\}.^1$$

This means that for each $x \in DR(N, v)$,

$$\begin{aligned} x_i &= \frac{v(R_k^x) - v(R_{k-1}^x)}{|R_k^x \setminus R_{k-1}^x|} && \text{for each } k \in \{1, \dots, |N|\} \text{ and each } i \in R_k^x; \\ x_i &= \frac{v(N \setminus P_{k-1}^x) - v(N \setminus P_k^x)}{|P_k^x \setminus P_{k-1}^x|} && \text{for each } k \in \{1, \dots, |N|\} \text{ and each } i \in P_k^x. \end{aligned}$$

We have $|DR(N, v)| = 1$ for each $(N, v) \in \Gamma_{exp}$. Throughout this note, σ is the generic notation for a solution.

3 Antiduality

This section performs an antiduality analysis on the class of exact partition games. The antidual of a game is the negated dual game, i.e. the game in which the worth of each coalition equals the worth of its complement minus the worth of the grand coalition in the original game. Two solutions are antidual if one solution assigns to each game the negated set of payoff allocations assigned by the other solution to the antidual game. A solution is self-antidual if it coincides with its antidual.

¹Llerena and Mauri (2017) showed that this definition is equivalent to the original definition of Dutta and Ray (1989).

Definition 1 (cf. Oishi and Nakayama (2009))

The *antidual* of a game (N, v) is the game (N, v^*) defined by

$$v^*(S) = v(N \setminus S) - v(N) \text{ for each } S \in 2^N.$$

Definition 2 (cf. Oishi and Nakayama (2009))

Two solutions σ and σ^* on Γ are *antidual* if $(N, v^*) \in \Gamma$ and

$$\sigma(v) = -\sigma^*(v^*) \text{ for each } (N, v) \in \Gamma.$$

A solution σ on Γ is *self-antidual* if $(N, v^*) \in \Gamma$ and

$$\sigma(v) = -\sigma(v^*) \text{ for each } (N, v) \in \Gamma.$$

Note that the antidual of the antidual game equals the original game. Moreover, note that each solution on a domain closed under antiduality has a unique antidual. Oishi et al. (2016) showed that class of all games, the class of balanced games, and the class of convex games are each closed under antiduality. Moreover, they showed that the Dutta and Ray (1989) solution is self-antidual on the class of convex games. We show that the class of exact partition games is also closed under antiduality and that the Dutta and Ray (1989) solution is also self-antidual on this class.

Theorem 1

The Dutta and Ray (1989) solution is self-antidual on the class of exact partition games.

Proof. Let $(N, v) \in \Gamma_{exp}$. Let $x \in C(N, v)$ be such that $x(R_k^x) = v(R_k^x)$ for each $k \in \{1, \dots, |N|\}$. Then

$$-x(N) = -v(N) = v(N \setminus N) - v(N) = v^*(N)$$

and for each $S \in 2^N$,

$$-x(S) = x(N \setminus S) - x(N) \geq v(N \setminus S) - v(N) = v^*(S).$$

This means that $-x \in C(N, v^*)$. Moreover, for each $k \in \{1, \dots, |N|\}$,

$$-x(R_k^{-x}) = -x(P_k^x) = x(N \setminus P_k^x) - x(N) = v(N \setminus P_k^x) - v(N) = v^*(P_k^x) = v^*(R_k^{-x}).$$

This means that $(N, v^*) \in \Gamma_{exp}$ and $DR(N, v^*) = -DR(N, v)$. Hence, the Dutta and Ray (1989) solution is self-antidual on the class of exact partition games. $\square$

The following punctual properties are satisfied by the Dutta and Ray (1989) on the class of exact partition games.

Efficiency

for each $x \in \sigma(N, v)$, $x(N) = v(N)$.

Feasible richness

for each $x \in \sigma(N, v)$, $x(R_1^x) \leq v(R_1^x)$.²

Equal division stability

for each $x \in \sigma(N, v)$ and each $S \in 2^N \setminus \{\emptyset\}$, there is $i \in S$ such that $x_i \geq \frac{v(S)}{|S|}$.

²Klijn et al. (2000) and Oishi et al. (2016) called this the *bounded maximum payoff property* and Llerena and Mauri (2017) called this property *rich player feasibility*.

Efficiency requires that a solution fully allocates the worth of the grand coalition. Feasible richness requires that the richest players are able to obtain their payoffs by themselves. Equal division stability requires that no coalition is better off by equally dividing the worth among its members.

Lemma 1

If a solution is self-antidual, then it satisfies efficiency.

Proof. Let σ be a self-antidual solution on Γ . Let $(N, v) \in \Gamma$ and let $x \in \sigma(N, v)$. Then $x(N) \leq v(N)$ and

$$-x(N) \leq v^*(N) = v(N \setminus N) - v(N) = -v(N).$$

This means that $x(N) = v(N)$. Hence, σ satisfies efficiency. $\square$

Antiduality can also be defined for axioms and characterizations. Two axioms are antidual if for each solution satisfying one axiom, its antidual satisfies the other axiom, and conversely. An axiom is self-antidual if for each solution satisfying the axiom, its antidual satisfies the axiom as well.

Definition 3 (cf. Oishi et al. (2016))

Two properties for solutions are *antidual* if for each pair of antidual solutions σ and σ^* , one property is satisfied by σ if and only if the other property is satisfied by σ^* . A property for solutions is *self-antidual* if for each pair of antidual solutions σ and σ^* , the property is satisfied by σ if and only if it is satisfied by σ^* .

Clearly, self-antiduality is a self-antidual property. Oishi et al. (2016) claimed that efficiency is self-antidual, showed that feasible richness is antidual to limited poorness, and claimed that equal division stability is self-antidual as well. The following example shows that the latter claim is incorrect.

Example 1

Let $(N, v) \in \Gamma_{conv}$ with $N = \{1, 2, 3\}$ be the convex game given by

$$v(S) = \begin{cases} 2 & \text{if } S \in \{\{1, 2\}, \{1, 2, 3\}\}; \\ 0 & \text{otherwise.} \end{cases}$$

The antidual game $(N, v^*) \in \Gamma_{conv}$ is given by

$$v^*(S) = \begin{cases} 0 & \text{if } S \in \{\emptyset, \{3\}\}; \\ -2 & \text{otherwise.} \end{cases}$$

Let σ be a solution on Γ_{conv} satisfying equal division stability such that $(1, 0, 1) \in \sigma(N, v)$. Let σ^* be the antidual solution on Γ_{conv} . Then $(-1, 0, -1) \in \sigma^*(N, v^*)$. Since $v^*(\{3\}) > -1$, this means that σ^* does not satisfy equal division stability. Hence, equal division stability is not self-antidual. $\triangle$

We show that efficiency is indeed self-antidual, feasible richness is indeed antidual to limited poorness, and equal division stability is in fact dual to the average contribution property.

Limited poorness

for each $x \in \sigma(N, v)$, $x(P_1^x) \geq v(N) - v(N \setminus P_1^x)$.³

Average contribution property

for each $x \in \sigma(N, v)$ and each $S \in 2^N \setminus \{\emptyset\}$, there is $i \in S$ such that $x_i \leq \frac{v(N) - v(N \setminus S)}{|S|}$.

³Oishi et al. (2016) called this the *bounded minimum payoff property*.

Limited poorness requires that the poorest players get at least their joint marginal contribution to the grand coalition. The average contribution property requires that for each coalition, at least one member gets at most the average marginal contribution of this coalition to the grand coalition.

Lemma 2

(i) *Efficiency is self-antidual.*

(ii) *Feasible richness and limited poorness are antidual.*

(iii) *Equal division stability and the average contribution property are antidual.*

Proof. (i) Let σ and σ^* be two antidual solutions on Γ . Assume that σ^* satisfies efficiency. Let $(N, v) \in \Gamma$ and let $x \in \sigma(N, v)$. Then

$$-x(N) = v^*(N) = v(N \setminus N) - v(N) = -v(N).$$

This means that $x(N) = v(N)$. Hence, σ satisfies efficiency.

(ii) Let σ and σ^* be two antidual solutions on Γ . Assume that σ^* satisfies feasible richness. Let $(N, v) \in \Gamma$ and let $x \in \sigma(N, v)$. Then

$$-x(P_1^x) = -x(R_1^{-x}) \leq v^*(R_1^{-x}) = v^*(P_1^x) = v(N \setminus P_1^x) - v(N).$$

This means that $x(P_1^x) \geq v(N) - v(N \setminus P_1^x)$. Hence, σ satisfies limited poorness.

Conversely, assume that σ^* satisfies limited poorness. Let $(N, v) \in \Gamma$ and let $x \in \sigma(N, v)$. Then

$$\begin{aligned} -x(R_1^x) &= -x(P_1^{-x}) \\ &\geq v^*(N) - v^*(N \setminus P_1^{-x}) \\ &= v(N \setminus N) - v(N) - v(P_1^{-x}) + v(N) \\ &= -v(P_1^{-x}) \\ &= -v(R_1^x). \end{aligned}$$

This means that $x(R_1^x) \leq v(R_1^x)$. Hence, σ satisfies feasible richness.

(iii) Let σ and σ^* be two antidual solutions on Γ . Assume that σ^* satisfies equal division stability. Let $(N, v) \in \Gamma$, let $x \in \sigma(N, v)$, and let $S \in 2^N \setminus \{\emptyset\}$. Then there is $i \in S$ such that

$$-x_i \geq \frac{v^*(S)}{|S|} = \frac{v(N \setminus S) - v(N)}{|S|}.$$

This means that $x_i \leq \frac{v(N) - v(N \setminus S)}{|S|}$. Hence, σ satisfies the average contribution property.

Conversely, assume that σ^* satisfies the average contribution property. Let $(N, v) \in \Gamma$, let $x \in \sigma(N, v)$, and let $S \in 2^N \setminus \{\emptyset\}$. Then there is $i \in S$ such that

$$-x_i \leq \frac{v^*(N) - v^*(N \setminus S)}{|S|} = \frac{v(N \setminus N) - v(N) - v(S) + v(N)}{|S|} = -\frac{v(S)}{|S|}.$$

This means that $x_i \geq \frac{v(S)}{|S|}$. Hence, σ satisfies equal division stability. $\square$

Given an axiomatic characterization of a solution on a domain closed under antiduality, the antidual solution is characterized on this domain by the corresponding antidual axioms. To see this, note that the antidual solution satisfies the antidual axioms by definition. It is the unique one, since otherwise the antidual of the other solution would satisfy the antiduals of these antidual axioms, i.e. the original axioms, which contradicts the characterization of the original solution. Moreover, independence of the axioms in the antidual characterization follows from independence of the axioms in the original characterization by the same logic.

We apply this logic to the axiomatic characterizations of the Dutta and Ray (1989) solution on the class of exact partition games provided by Dietzenbacher and Yanovskaya (2020). These characterizations are generally based on consistency properties. Suppose that some players leave with their allocated payoffs and the remaining players reevaluate their payoffs by applying the same solution to a reduced game. The solution is consistent if it assigns to this reduced game the same payoffs for the remaining players as in the original game. However, such reduced game can be defined in many ways and the context of the underlying allocation problem determines which one is most appropriate.

Davis and Maschler (1965) proposed a reduced game in which the worth of a coalition is defined as the maximal joint worth with any subgroup of leaving players subtracted by their payoffs. The corresponding axiom which requires consistent allocations when the richest players leave together, to which we refer as rich-restricted max-consistency, characterizes the Dutta and Ray (1989) solution on the class of exact partition games in combination with feasible richness and equal division stability.

Rich-restricted max-consistency

for each $(N, v) \in \Gamma$ and each $x \in \sigma(N, v)$ with $R_1^x \neq N$,

$$(N \setminus R_1^x, v_{r-max}^x) \in \Gamma \text{ and } x_{N \setminus R_1^x} \in \sigma(N \setminus R_1^x, v_{r-max}^x),$$

where

$$v_{r-max}^x(S) = \begin{cases} v(N) - x(R_1^x) & \text{if } S = N \setminus R_1^x; \\ \max_{Q \subseteq R_1^x} \{v(S \cup Q) - x(Q)\} & \text{if } \emptyset \subset S \subset N \setminus R_1^x; \\ 0 & \text{if } S = \emptyset. \end{cases}$$

Theorem (cf. Dietzenbacher and Yanovskaya (2020))

The Dutta and Ray (1989) solution is the unique solution on Γ_{exp} satisfying feasible richness, equal division stability, and rich-restricted max-consistency.

Rich-restricted max-consistency is antidual to poor-restricted max-consistency, which requires consistent allocations based on the same reduced game when the poorest players leave together. The corresponding proof is provided in Lemma A.1 in the appendix. Using antiduality, a characterization of the Dutta and Ray (1989) solution on the class of exact partition games in terms of limited poorness, the average contribution property, and poor-restricted max-consistency follows immediately from Theorem 1, Lemma 2, and Lemma A.1.

Poor-restricted max-consistency

for all $(N, v) \in \Gamma$ and each $x \in \sigma(N, v)$ with $P_1^x \neq N$,

$$(N \setminus P_1^x, v_{p-max}^x) \in \Gamma \text{ and } x_{N \setminus P_1^x} \in \sigma(N \setminus P_1^x, v_{p-max}^x),$$

where

$$v_{p-max}^x(S) = \begin{cases} v(N) - x(P_1^x) & \text{if } S = N \setminus P_1^x; \\ \max_{Q \subseteq P_1^x} \{v(S \cup Q) - x(Q)\} & \text{if } \emptyset \subset S \subset N \setminus P_1^x; \\ 0 & \text{if } S = \emptyset. \end{cases}$$

Theorem 2

The Dutta and Ray (1989) solution is the unique solution on Γ_{exp} satisfying limited poorness, the average contribution property, and poor-restricted max-consistency.

Since the Dutta and Ray (1989) solution is also self-antidual on the subclass of convex games, this characterization is also valid on the class of convex games, repairing and strenghtening the corresponding axiomatization of Oishi et al. (2016) antidual to the result of Klijn et al. (2000).

Theorem 3

The Dutta and Ray (1989) solution is the unique solution on Γ_{conv} satisfying limited poorness, the average contribution property, and poor-restricted max-consistency.

Moulin (1985) proposed a reduced game in which the worth of a coalition is defined as the joint worth with all leaving players substracted by their payoffs. The corresponding axiom which requires consistent allocations when the richest players leave together, to which we refer as rich-restricted complement-consistency, characterizes the Dutta and Ray (1989) solution on the class of exact partition games in combination with feasible richness and equal division stability.

Rich-restricted complement-consistency

for each $(N, v) \in \Gamma$ and each $x \in \sigma(N, v)$ with $R_1^x \neq N$,

$$(N \setminus R_1^x, v_{comp}^x) \in \Gamma \text{ and } x_{N \setminus R_1^x} \in \sigma(N \setminus R_1^x, v_{comp}^x),$$

where

$$v_{comp}^x(S) = \begin{cases} v(S \cup R_1^x) - x(R_1^x) & \text{if } \emptyset \subset S \subseteq N \setminus R_1^x; \\ 0 & \text{if } S = \emptyset. \end{cases}$$

Theorem (cf. Dietzenbacher and Yanovskaya (2020))

The Dutta and Ray (1989) solution is the unique solution on Γ_{exp} satisfying feasible richness, equal division stability, and rich-restricted complement-consistency.

Rich-restricted complement-consistency is antidual to poor-restricted projection-consistency, which requires consistent allocations when the poorest players leave together based on the reduced game where the worth of a coalition equals its worth in the original game. Funaki and Yamato (2001) used projection-consistency in a characterization of the core. Bhattacharya (2004) used projection-consistency in a characterization of the equal division core. Using antiduality, a characterization of the Dutta and Ray (1989) solution on the class of exact partition games in terms of limited poorness, the average contribution property, and poor-restricted projection-consistency follows immediately from Theorem 1, Lemma 2, and Lemma A.2.

Poor-restricted projection-consistency

for each $(N, v) \in \Gamma$ and each $x \in \sigma(N, v)$ with $P_1^x \neq N$,

$$(N \setminus P_1^x, v_{proj}^x) \in \Gamma \text{ and } x_{N \setminus P_1^x} \in \sigma(N \setminus P_1^x, v_{proj}^x),$$

where

$$v_{proj}^x(S) = \begin{cases} v(N) - x(P_1^x) & \text{if } S = N \setminus P_1^x; \\ v(S) & \text{if } S \subset N \setminus P_1^x. \end{cases}$$

Theorem 4

The Dutta and Ray (1989) solution is the unique solution on Γ_{exp} satisfying limited poorness, the average contribution property, and poor-restricted projection-consistency.

Since the Dutta and Ray (1989) solution is also self-antidual on the subclass of convex games, this characterization is also valid on the class of convex games.

Theorem 5

The Dutta and Ray (1989) solution is the unique solution on Γ_{conv} satisfying limited poorness, the average contribution property, and poor-restricted projection-consistency.

Klijn et al. (2000) proposed a reduced game in which the worth of a coalition is defined as the joint marginal contribution to the leaving players. The corresponding axiom which requires consistent allocations when the richest players leave together, to which we refer as rich-restricted marginal-consistency, characterizes the Dutta and Ray (1989) solution on the class of exact partition games in combination with equal division stability.

Rich-restricted marginal-consistency

for each $(N, v) \in \Gamma$ and each $x \in \sigma(N, v)$ with $R_1^x \neq N$,

$$(N \setminus R_1^x, v_{\text{marg}}^x) \in \Gamma \text{ and } x_{N \setminus R_1^x} \in \sigma(N \setminus R_1^x, v_{\text{marg}}^x),$$

where

$$v_{\text{marg}}^x(S) = v(S \cup R_1^x) - v(R_1^x) \text{ for all } S \subseteq N \setminus R_1^x.^4$$

Theorem (cf. Dietzenbacher and Yanovskaya (2020))

The Dutta and Ray (1989) solution is the unique solution on Γ_{exp} satisfying equal division stability and rich-restricted marginal-consistency.

Rich-restricted marginal-consistency is antidual to poor-restricted subgame-consistency, which requires consistent allocations based on the subgame when the poorest players leave together. Using antiduality, a characterization of the Dutta and Ray (1989) solution on the class of exact partition games in terms of the average contribution property and poor-restricted subgame-consistency follows immediately from Theorem 1, Lemma 2, and Lemma A.3.

Poor-restricted subgame-consistency

for each $(N, v) \in \Gamma$ and each $x \in \sigma(N, v)$ with $P_1^x \neq N$,

$$(N \setminus P_1^x, v_{\text{sub}}^x) \in \Gamma \text{ and } x_{N \setminus P_1^x} \in \sigma(N \setminus P_1^x, v_{\text{sub}}^x),$$

where

$$v_{\text{sub}}^x(S) = v(S) \text{ for all } S \subseteq N \setminus P_1^x.^5$$

Theorem 6

The Dutta and Ray (1989) solution is the unique solution on Γ_{exp} satisfying the average contribution property and poor-restricted subgame-consistency.

Since the Dutta and Ray (1989) solution is also self-antidual on the subclass of convex games, this characterization is also valid on the class of convex games, repairing and strengthening the corresponding axiomatization of Oishi et al. (2016) antidual to the result of Klijn et al. (2000).

Theorem 7

The Dutta and Ray (1989) solution is the unique solution on Γ_{conv} satisfying the average contribution property and poor-restricted subgame-consistency.

⁴Klijn et al. (2000) called this property *max-consistency* and Oishi et al. (2016) called this property *max-restricted contribution-consistency*.

⁵Oishi et al. (2016) called this property *min-restricted projection-consistency*.

4 Concluding Remarks

Using antiduality, this note provides several new axiomatic characterizations of the Dutta and Ray (1989) solution on the class of exact partition games and repairs and strengthens several related characterizations on the subclass of convex games. For other games than exact partition games, existence of the Dutta and Ray (1989) solution is not guaranteed. However, several extensions to the class of balanced games and the class of all games are proposed.

On the class of balanced games, the solution which assigns the Lorenz undominated allocations in the core (cf. Hougaard et al. (2001)) is self-antidual. The $Lmin$ solution (cf. Arin and Iñarra (2001)) and $Lmax$ solution (cf. Arin et al. (2003)) are antidual. The least squares solution (cf. Arin et al. (2008)) is self-antidual. Future research could use antiduality to further study these solutions.

On the class of all games, neither the equal split-off set (cf. Branzei et al. (2006)) nor the procedural egalitarian solution (cf. Dietzenbacher et al. (2017)) is self-antidual. However, these solutions are contained in some self-antidual solutions. Future research could further explore this.

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Lemma A.1

Rich-restricted max-consistency and poor-restricted max-consistency are antidual.

Proof. Let σ and σ^* be two antidual solutions on Γ . Assume that σ^* satisfies rich-restricted max-consistency. Let $(N, v) \in \Gamma$ and let $x \in \sigma(N, v)$ be such that $P_1^x \neq N$. Then

$$(N \setminus R_1^{-x}, (v^*)_{r-max}^{-x}) \in \Gamma \text{ and } -x_{N \setminus R_1^{-x}} \in \sigma^*(N \setminus R_1^{-x}, (v^*)_{r-max}^{-x}).$$

This implies that

$$(N \setminus P_1^x, ((v^*)_{r-max}^{-x})^*) \in \Gamma \text{ and } x_{N \setminus P_1^x} \in \sigma(N \setminus P_1^x, ((v^*)_{r-max}^{-x})^*).$$

If $S = N \setminus P_1^x$, then

$$\begin{aligned} ((v^*)_{r-max}^{-x})^*(S) &= (v^*)_{r-max}^{-x}((N \setminus P_1^x) \setminus S) - (v^*)_{r-max}^{-x}(N \setminus P_1^x) = -(v^*)_{r-max}^{-x}(N \setminus R_1^{-x}) \\ &= -v^*(N) - x(R_1^{-x}) = -v(N \setminus N) + v(N) - x(R_1^{-x}) = v(N) - x(P_1^x) \\ &= v_{p-max}^x(S). \end{aligned}$$

If $\emptyset \subset S \subset N \setminus P_1^x$, then

$$\begin{aligned} ((v^*)_{r-max}^{-x})^*(S) &= (v^*)_{r-max}^{-x}((N \setminus P_1^x) \setminus S) - (v^*)_{r-max}^{-x}(N \setminus P_1^x) \\ &= (v^*)_{r-max}^{-x}((N \setminus R_1^{-x}) \setminus S) - (v^*)_{r-max}^{-x}(N \setminus R_1^{-x}) \\ &= \max_{Q \subseteq R_1^{-x}} \{v^*((N \setminus R_1^{-x}) \setminus S) \cup Q + x(Q)\} - v^*(N) - x(R_1^{-x}) \\ &= \max_{Q \subseteq R_1^{-x}} \{v(S \cup (R_1^{-x} \setminus Q)) - v(N) + x(Q)\} - v(N \setminus N) + v(N) - x(R_1^{-x}) \\ &= \max_{Q \subseteq P_1^x} \{v(S \cup (P_1^x \setminus Q)) - x(P_1^x \setminus Q)\} \\ &= \max_{Q \subseteq P_1^x} \{v(S \cup Q) - x(Q)\} \\ &= v_{p-max}^x(S). \end{aligned}$$

If $S = \emptyset$, then

$$((v^*)_{r-max}^{-x})^*(S) = (v^*)_{r-max}^{-x}((N \setminus P_1^x) \setminus S) - (v^*)_{r-max}^{-x}(N \setminus P_1^x) = 0 = v_{p-max}^x(S).$$

This means that $((v^*)_{r-max}^{-x})^* = v_{p-max}^x$. Hence, σ satisfies poor-restricted max-consistency.

Conversely, assume that σ^* satisfies poor-restricted max-consistency. Let $(N, v) \in \Gamma$ and let $x \in \sigma(N, v)$ be such that $R_1^x \neq N$. Then

$$(N \setminus P_1^{-x}, (v^*)_{p-max}^{-x}) \in \Gamma \text{ and } -x_{N \setminus P_1^{-x}} \in \sigma^*(N \setminus P_1^{-x}, (v^*)_{p-max}^{-x}).$$

This implies that

$$(N \setminus R_1^x, ((v^*)_{p-max}^{-x})^*) \in \Gamma \text{ and } x_{N \setminus R_1^x} \in \sigma(N \setminus R_1^x, ((v^*)_{p-max}^{-x})^*).$$

If $S = N \setminus R_1^x$, then

$$\begin{aligned} ((v^*)_{p-max}^{-x})^*(S) &= (v^*)_{p-max}^{-x}((N \setminus R_1^x) \setminus S) - (v^*)_{p-max}^{-x}(N \setminus R_1^x) = -(v^*)_{p-max}^{-x}(N \setminus P_1^{-x}) \\ &= -v^*(N) - x(P_1^{-x}) = -v(N \setminus N) + v(N) - x(P_1^{-x}) = v(N) - x(R_1^x) \\ &= v_{r-max}^x(S). \end{aligned}$$

If $\emptyset \subset S \subset N \setminus R_1^x$, then

$$\begin{aligned}
((v^*)_{p-max}^{-x})^*(S) &= (v^*)_{p-max}^{-x}((N \setminus R_1^x) \setminus S) - (v^*)_{p-max}^{-x}(N \setminus R_1^x) \\
&= (v^*)_{p-max}^{-x}((N \setminus P_1^{-x}) \setminus S) - (v^*)_{p-max}^{-x}(N \setminus P_1^{-x}) \\
&= \max_{Q \subseteq P_1^{-x}} \{v^*((N \setminus P_1^{-x}) \setminus S) \cup Q + x(Q)\} - v^*(N) - x(P_1^{-x}) \\
&= \max_{Q \subseteq P_1^{-x}} \{v(S \cup (P_1^{-x} \setminus Q)) - v(N) + x(Q)\} - v(N \setminus N) + v(N) - x(P_1^{-x}) \\
&= \max_{Q \subseteq R_1^x} \{v(S \cup (R_1^x \setminus Q)) - x(R_1^x \setminus Q)\} \\
&= \max_{Q \subseteq R_1^x} \{v(S \cup Q) - x(Q)\} \\
&= v_{r-max}^x(S).
\end{aligned}$$

If $S = \emptyset$, then

$$((v^*)_{p-max}^{-x})^*(S) = (v^*)_{p-max}^{-x}((N \setminus R_1^x) \setminus S) - (v^*)_{p-max}^{-x}(N \setminus R_1^x) = 0 = v_{r-max}^x(S).$$

This means that $((v^*)_{p-max}^{-x})^* = v_{r-max}^x$. Hence, σ satisfies rich-restricted max-consistency. $\square$

Lemma A.2

Rich-restricted complement-consistency and poor-restricted projection-consistency are antidual.

Proof. Let σ and σ^* be two antidual solutions on Γ . Assume that σ^* satisfies rich-restricted complement-consistency. Let $(N, v) \in \Gamma$ and let $x \in \sigma(N, v)$ be such that $P_1^x \neq N$. Then

$$(N \setminus R_1^{-x}, (v^*)_{comp}^{-x}) \in \Gamma \text{ and } -x_{N \setminus R_1^{-x}} \in \sigma^*(N \setminus R_1^{-x}, (v^*)_{comp}^{-x}).$$

This implies that

$$(N \setminus P_1^x, ((v^*)_{comp}^{-x})^*) \in \Gamma \text{ and } x_{N \setminus P_1^x} \in \sigma(N \setminus P_1^x, ((v^*)_{comp}^{-x})^*).$$

If $S = N \setminus P_1^x$, then

$$\begin{aligned}
((v^*)_{comp}^{-x})^*(S) &= (v^*)_{comp}^{-x}((N \setminus P_1^x) \setminus S) - (v^*)_{comp}^{-x}(N \setminus P_1^x) = -(v^*)_{comp}^{-x}(N \setminus R_1^{-x}) \\
&= -v^*(N) - x(R_1^{-x}) = -v(N \setminus N) + v(N) - x(R_1^{-x}) = v(N) - x(P_1^x) \\
&= v_{proj}^x(S).
\end{aligned}$$

If $\emptyset \subset S \subset N \setminus P_1^x$, then

$$\begin{aligned}
((v^*)_{comp}^{-x})^*(S) &= (v^*)_{comp}^{-x}((N \setminus P_1^x) \setminus S) - (v^*)_{comp}^{-x}(N \setminus P_1^x) \\
&= (v^*)_{comp}^{-x}((N \setminus R_1^{-x}) \setminus S) - (v^*)_{comp}^{-x}(N \setminus R_1^{-x}) \\
&= v^*(N \setminus S) + x(R_1^{-x}) - v^*(N) - x(R_1^{-x}) \\
&= v(S) - v(N) - v(N \setminus N) + v(N) \\
&= v(S) = v_{proj}^x(S).
\end{aligned}$$

If $S = \emptyset$, then

$$((v^*)_{comp}^{-x})^*(S) = (v^*)_{comp}^{-x}((N \setminus P_1^x) \setminus S) - (v^*)_{comp}^{-x}(N \setminus P_1^x) = 0 = v(S) = v_{proj}^x(S).$$

This means that $((v^*)_{comp}^{-x})^* = v_{proj}^x$. Hence, σ satisfies poor-restricted projection-consistency.

Conversely, assume that σ^* satisfies poor-restricted projection-consistency. Let $(N, v) \in \Gamma$ and let $x \in \sigma(N, v)$ be such that $R_1^x \neq N$. Then

$$(N \setminus P_1^{-x}, (v^*)_{proj}^{-x}) \in \Gamma \text{ and } -x_{N \setminus P_1^{-x}} \in \sigma^*(N \setminus P_1^{-x}, (v^*)_{proj}^{-x}).$$

This implies that

$$(N \setminus R_1^x, ((v^*)_{proj}^{-x})^*) \in \Gamma \text{ and } x_{N \setminus R_1^x} \in \sigma(N \setminus R_1^x, ((v^*)_{proj}^{-x})^*).$$

If $S = N \setminus R_1^x$, then

$$\begin{aligned} ((v^*)_{proj}^{-x})^*(S) &= (v^*)_{proj}^{-x}((N \setminus R_1^x) \setminus S) - (v^*)_{proj}^{-x}(N \setminus R_1^x) = -(v^*)_{proj}^{-x}(N \setminus P_1^{-x}) \\ &= -v^*(N) - x(P_1^{-x}) = -v(N \setminus N) + v(N) - x(P_1^{-x}) = v(N) - x(R_1^x) \\ &= v(S \cup R_1^x) - x(R_1^x) = v_{comp}^x(S). \end{aligned}$$

If $\emptyset \subset S \subset N \setminus R_1^x$, then

$$\begin{aligned} ((v^*)_{proj}^{-x})^*(S) &= (v^*)_{proj}^{-x}((N \setminus R_1^x) \setminus S) - (v^*)_{proj}^{-x}(N \setminus R_1^x) \\ &= (v^*)_{proj}^{-x}((N \setminus P_1^{-x}) \setminus S) - (v^*)_{proj}^{-x}(N \setminus P_1^{-x}) \\ &= v^*((N \setminus P_1^{-x}) \setminus S) - v^*(N) - x(P_1^{-x}) \\ &= v(S \cup P_1^{-x}) - v(N) - v(N \setminus N) + v(N) - x(P_1^{-x}) \\ &= v(S \cup R_1^x) - x(R_1^x) = v_{comp}^x(S). \end{aligned}$$

If $S = \emptyset$, then

$$((v^*)_{proj}^{-x})^*(S) = (v^*)_{proj}^{-x}((N \setminus R_1^x) \setminus S) - (v^*)_{proj}^{-x}(N \setminus R_1^x) = 0 = v_{comp}^x(S).$$

This means that $((v^*)_{proj}^{-x})^* = v_{comp}^x$. Hence, σ satisfies rich-restricted complement-consistency. $\square$

Lemma A.3

Rich-restricted marginal-consistency and poor-restricted subgame-consistency are antidual.

Proof. Let σ and σ^* be two antidual solutions on Γ . Assume that σ^* satisfies rich-restricted marginal-consistency. Let $(N, v) \in \Gamma$ and let $x \in \sigma(N, v)$ be such that $P_1^x \neq N$. Then

$$(N \setminus R_1^{-x}, (v^*)_{marg}^{-x}) \in \Gamma \text{ and } -x_{N \setminus R_1^{-x}} \in \sigma^*(N \setminus R_1^{-x}, (v^*)_{marg}^{-x}).$$

This implies that

$$(N \setminus P_1^x, ((v^*)_{marg}^{-x})^*) \in \Gamma \text{ and } x_{N \setminus P_1^x} \in \sigma(N \setminus P_1^x, ((v^*)_{marg}^{-x})^*).$$

Let $S \subseteq N \setminus P_1^x$. Then

$$\begin{aligned} ((v^*)_{marg}^{-x})^*(S) &= (v^*)_{marg}^{-x}((N \setminus P_1^x) \setminus S) - (v^*)_{marg}^{-x}(N \setminus P_1^x) \\ &= (v^*)_{marg}^{-x}((N \setminus R_1^{-x}) \setminus S) - (v^*)_{marg}^{-x}(N \setminus R_1^{-x}) \\ &= v^*(N \setminus S) - v^*(R_1^{-x}) - v^*(N) + v^*(R_1^{-x}) \\ &= v(S) - v(N) - v(N \setminus N) + v(N) \\ &= v(S) = v_{sub}^x(S). \end{aligned}$$

This means that $((v^*)_{marg}^{-x})^* = v_{sub}^x$. Hence, σ satisfies poor-restricted subgame-consistency.

Conversely, assume that σ^* satisfies poor-restricted subgame-consistency. Let $(N, v) \in \Gamma$ and let $x \in \sigma(N, v)$ be such that $R_1^x \neq N$. Then

$$(N \setminus P_1^{-x}, (v^*)_{sub}^{-x}) \in \Gamma \text{ and } -x_{N \setminus P_1^{-x}} \in \sigma^*(N \setminus P_1^{-x}, (v^*)_{sub}^{-x}).$$

This implies that

$$(N \setminus R_1^x, ((v^*)_{sub}^{-x})^*) \in \Gamma \text{ and } x_{N \setminus R_1^x} \in \sigma(N \setminus R_1^x, ((v^*)_{sub}^{-x})^*).$$

Let $S \subseteq N \setminus R_1^x$. Then

$$\begin{aligned} ((v^*)_{sub}^{-x})^*(S) &= (v^*)_{sub}^{-x}((N \setminus R_1^x) \setminus S) - (v^*)_{sub}^{-x}(N \setminus R_1^x) \\ &= (v^*)_{sub}^{-x}((N \setminus P_1^{-x}) \setminus S) - (v^*)_{sub}^{-x}(N \setminus P_1^{-x}) \\ &= v^*((N \setminus P_1^{-x}) \setminus S) - v^*(N \setminus P_1^{-x}) \\ &= v(S \cup P_1^{-x}) - v(N) - v(P_1^{-x}) + v(N) \\ &= v(S \cup R_1^x) - v(R_1^x) = v_{marg}^x(S). \end{aligned}$$

This means that $((v^*)_{sub}^{-x})^* = v_{marg}^x$. Hence, σ satisfies rich-restricted marginal-consistency. $\square$