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Axiomatizations of Pareto Equilibria in Multicriteria Games

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We focus on axiomatizations of the Pareto equilibrium concept in noncooperative multicriteria games based on consistency. Some axiomatizations of the Nash equilibrium concept have immediate generalizations. For strategic games it was shown that there exist no consistent refinements of the Nash equilibrium concept that satisfy individual rationality and nonemptiness on a reasonably large class of games. We show that such refinements of the Pareto equilibrium concept on multicriteria games do exist. *Journal of Economic Literature* Classification Number: C72. © 1999 Academic Press

1. INTRODUCTION

In a recent manifesto, Bouyssou *et al.* (1993) observe that within multicriteria decision making “(a) systematic axiomatic analysis of decision procedures and algorithms is yet to be carried out.” In this paper, several axiomatizations of the Pareto equilibrium concept for multicriteria games are provided.

In multicriteria games, a player can have more than one criterion function. Shapley (1959) introduced (Pareto) equilibrium points for the type of games that are a straightforward generalization of the Nash equilibrium concept for unicriterion games.

Axiomatic properties of the Nash equilibrium concept based on the notion of consistency have been studied in several articles, including Peleg and Tijs (1996), Peleg, Potters, and Tijs (1996), and Norde *et al.* (1996). Informally, consistency requires that if a strategy combination x is a solution of a game with player set N , and players outside a coalition S of players commit to playing according to $x_{N \setminus S}$, i.e., the strategy combination restricted to the players in $N \setminus S$, then x_S is a solution of the reduced

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game. Several of these axiomatizations carry over to multicriteria games. The strong result of Norde *et al.* (1996), characterizing the Nash equilibrium concept on the set of mixed extensions of finite strategic form games by nonemptiness, the selection of utility maximizing strategies in one person games, and consistency, does not have such an analogon in multicriteria games: We show that nonemptiness, consistency and an immediate extension of utility maximization are not sufficient to axiomatize the Pareto equilibrium concept. An additional property is provided to establish an axiomatization.

2. PRELIMINARIES

A finite *multicriteria game* is a tuple $G = \langle N^G, (X_i)_{i \in N^G}, (u_i)_{i \in N^G} \rangle$, where $N^G \subset \mathbb{N}$ is a finite set of players, X_i is the finite set of pure strategies of player $i \in N^G$, and for each player $i \in N^G$, the function $u_i: \prod_{j \in N^G} X_j \rightarrow \mathbb{R}^{r(i)}$ maps each strategy combination to a point in $r(i)$ -dimensional Euclidean space. The interpretation of this last function is that player $i \in N^G$ considers not just one, but $r(i) \in \mathbb{N}$ different criteria. For notational convenience, the set of finite multicriteria games is denoted Γ_{finite} .

The payoff functions are extended to mixed strategies in the obvious way. The set of mixed extensions of finite multicriteria games is denoted by Γ . This set contains the set $\Gamma_{\text{strategic}}$ of mixed extensions of finite games in strategic form, since these are simply multicriteria games in which each player has only one criterion.

For notational convenience, let $\Delta(X)$ denote the set of probability measures on a finite set X , and for $m \in \mathbb{N}$, $\Delta_m := \{\mu \in \mathbb{R}_+^m \mid \sum_{i=1}^m \mu_i = 1\}$ is the unit simplex in $\mathbb{R}^m$.

Let $G = \langle N^G, (X_i)_{i \in N^G}, (u_i)_{i \in N^G} \rangle \in \Gamma$ be a multicriteria game, let $x \in \prod_{i \in N^G} \Delta(X_i)$ be a strategy profile in G , and let $S \in 2^{N^G} \setminus \{\emptyset, N^G\}$ be a proper subcoalition of the player set N^G . The *reduced game* $G^{S,x}$ of G with respect to S and x is the multicriteria game in which

- the player set is S ;
- each player $i \in S$ has the same set X_i of pure strategies as in G ;
- the payoff functions $(u'_i)_{i \in S}$ are defined by $u'_i(y_S) := u_i(y_S, x_{N^G \setminus S})$ for all $y_S \in \prod_{i \in S} \Delta(X_i)$.

Notice that this is the game that arises if the players in $N^G \setminus S$ commit to playing according to $x_{N^G \setminus S}$, the strategy combination restricted to the players in $N^G \setminus S$. Definitions for reduced games on Γ_{finite} and $\Gamma_{\text{strategic}}$ are completely analogous.

A *solution concept* on Γ is a function ρ which assigns to each element $G \in \Gamma$ a subset $\rho(G) \subseteq \prod_{i \in N^G} \Delta(X_i)$ of strategy combinations. Analogously one defines a solution concept on $\Gamma_{\text{strategic}}$ or Γ_{finite} .

For strategic form games, we recall the following axioms. A solution concept ρ on $\Gamma_{\text{strategic}}$ satisfies:

- **Nonemptiness (NEM)**, if $\rho(G) \neq \emptyset$ for all $G \in \Gamma_{\text{strategic}}$;
- **Utility Maximization (UM)**, if for each one player game $G = \langle \{i\}, X_i, u_i \rangle \in \Gamma_{\text{strategic}}$ we have that $\rho(G) \subseteq \{x \in \Delta(X_i) \mid u_i(x) \geq u_i(y) \forall y \in \Delta(X_i)\}$;
- **Consistency (CONS)**, if for each game $G \in \Gamma_{\text{strategic}}$, each proper subcoalition $S \in 2^{N^G} \setminus \{\emptyset, N^G\}$, and each element $x \in \rho(G)$, we have that $x_S \in \rho(G^{S,x})$.

Norde *et al.* (1996) prove:

PROPOSITION 2.1. *A solution concept ρ on $\Gamma_{\text{strategic}}$ satisfies NEM, UM, and CONS if and only if $\rho = NE$, the Nash equilibrium concept.*

Equilibrium points for multicriteria games are introduced by Shapley (1959). Let $G = \langle N^G, (X_i)_{i \in N^G}, (u_i)_{i \in N^G} \rangle \in \Gamma$ be a multicriteria game. A *Pareto equilibrium* is a strategy combination $x \in \prod_{i \in N^G} \Delta(X_i)$ such that for each $i \in N^G$, there does not exist an $\tilde{x}_i \in \Delta(X_i)$ such that:

$$u_i(\tilde{x}_i, x_{-i}) > u_i(x_i, x_{-i}),$$

where for two vectors $a, b \in \mathbb{R}^m$, we write $a > b$ if $a_j > b_j$ for all $j = 1, \dots, m$. The solution concept on Γ assigning to each $G \in \Gamma$ the set of Pareto equilibria is denoted by PE. The Pareto equilibrium concept PE on Γ_{finite} is, of course, defined in a similar way by restricting attention to pure, rather than mixed, strategies.

Consider a multicriteria game $G \in \Gamma$ in which player i has $r(i) \in \mathbb{N}$ criteria. For each $i \in N^G$, let $\lambda_i \in \Delta_{r(i)}$ be a vector of weights for the criteria, $\lambda := (\lambda_i)_{i \in N^G}$. The λ -weighted game G_λ is the strategic form game with player set N^G , mixed strategy spaces $(\Delta(X_i))_{i \in N^G}$, and payoff functions $(v_i)_{i \in N^G}$ defined for all $i \in N^G$ and $x \in \prod_{i \in N^G} \Delta(X_i)$ by $v_i(x) = \langle \lambda_i, u_i(x) \rangle = \sum_{k=1}^{r(i)} \lambda_{ik} u_{ik}(x)$. If each player assigns equal weight to all his criteria, i.e., $\lambda_i = \frac{1}{r(i)}(1, \dots, 1) \in \mathbb{R}^{r(i)}$ for all $i \in N^G$, the weighted game is denoted by G_e . Shapley (1959) proves that all Pareto equilibria can be found by a suitable weighing of the criteria of the players:

LEMMA 2.2. *For each $G \in \Gamma$: $x \in PE(G)$ if and only if there exists for each $i \in N^G$ a vector of weights $\lambda_i \in \Delta_{r(i)}$ such that $x \in NE(G_\lambda)$.*

As a corollary, Pareto equilibria always exist in mixed extensions of finite multicriteria games, since for any vector of weights the game G_λ has Nash equilibria in mixed strategies.

3. FINITE MULTICRITERIA GAMES

Peleg and Tijs (1996) and Peleg, Potters, and Tijs (1996) provide several axiomatizations of the Nash equilibrium concept for finite strategic form games. In this section two of these axiomatizations are extended to finite multicriteria games.

We use the following axioms. A solution concept ρ on Γ_{finite} satisfies:

- **Restricted Nonemptiness (r-NEM)**, if for every $G \in \Gamma_{\text{finite}}$ with $\text{PE}(G) \neq \emptyset$ we have $\rho(G) \neq \emptyset$;
- **One Person Efficiency (OPE)**, if for each one player game $G = \langle \{i\}, X_i, u_i \rangle \in \Gamma_{\text{finite}}$ we have that $\rho(G) = \{x \in X_i \mid \nexists y \in X_i: u_i(y) > u_i(x)\}$;
- **Consistency (CONS)**, if for each $G \in \Gamma_{\text{finite}}$, each proper subcoalition $S \in 2^{N^G} \setminus \{\emptyset, N^G\}$, and each element $x \in \rho(G)$, we have that $x_S \in \rho(G^{S,x})$;
- **Converse Consistency (COCONS)**, if for each $G \in \Gamma_{\text{finite}}$ with at least two players, we have that $\tilde{\rho}(G) \subseteq \rho(G)$, where

$$\tilde{\rho}(G) = \left\{ x \in \prod_{i \in N^G} X_i \mid \forall S \in 2^{N^G} \setminus \{\emptyset, N^G\}: x_S \in \rho(G^{S,x}) \right\}.$$

According to restricted nonemptiness, the solution concept provides a nonempty set of strategies whenever Pareto equilibria exist. One person efficiency claims that in games with only one player, the solution concept picks out all strategies which yield a maximal payoff with respect to the $>$ order. Consistency means that a solution x of a game is also a solution of each reduced game in which the players that leave the game play according to the strategies in x . Converse consistency prescribes that a strategy combination which gives rise to a solution in every reduced game is also a solution of the original game.

Our first result indicates that the axiomatization of the Nash equilibrium concept on finite strategic games of Peleg, Potters, and Tijs (1996, Theorem 3) in terms of restricted nonemptiness, one person rationality and consistency can be generalized to multicriteria games.

THEOREM 3.1. *A solution concept ρ on Γ_{finite} satisfies r-NEM, OPE, and CONS if and only if $\rho = \text{PE}$.*

Proof. It is clear that PE satisfies the axioms. Let ρ be a solution concept on Γ_{finite} satisfying r-NEM, OPE, and CONS. Let $G = \langle N^G, (X_i)_{i \in N^G}, (u_i)_{i \in N^G} \rangle \in \Gamma_{\text{finite}}$. We first show that $\rho(G) \subseteq \text{PE}(G)$. Let $x \in \rho(G)$. If $|N^G| = 1$, then $x \in \text{PE}(G)$ by OPE. If $|N^G| > 1$, then CONS

implies that for each $i \in N^G$: $x_i \in \rho(G^{(i),x})$, so $x_i \in \{y_i \in X_i \mid \nexists z_i \in X_i: u_i(z_i, x_{-i}) > u_i(y_i, x_{-i})\}$ by OPE. Hence, x is a Pareto equilibrium: $x \in \text{PE}(G)$. Since $G \in \Gamma_{\text{finite}}$ was chosen arbitrarily, conclude that $\rho \subseteq \text{PE}$. To prove the converse inclusion, i.e., that $\text{PE} \subseteq \rho$, again let $G = \langle N^G, (X_i)_{i \in N^G}, (u_i)_{i \in N^G} \rangle \in \Gamma_{\text{finite}}$ and let $\hat{x} \in \text{PE}(G)$. Construct a finite multicriteria game H as follows:

- let $m \in \mathbb{N} \setminus N^G$; the player set is $N^G \cup \{m\}$;
- players $i \in N^G$ have the same strategy set X_i as in G ;
- player m has strategy set $\{0, 1\}$;
- payoff functions v_i to players $i \in N^G$ are defined, for all $(x_m, x) \in \{0, 1\} \times \prod_{i \in N^G} X_i$, by:

$$v_i(x_m, x) = \begin{cases} u_i(x) & \text{if } x_m = 1 \\ -e^{r(i)} & \text{if } x_m = 0, x_i \neq \hat{x}_i \\ e^{r(i)} & \text{if } x_m = 0, x_i = \hat{x}_i \end{cases}$$

where $e^{r(i)} \in \mathbb{R}^{r(i)}$ is the vector with each component equal to one.

- the payoff function v_m to player m is defined, for all $(x_m, x) \in \{0, 1\} \times \prod_{i \in N^G} X_i$, by

$$v_m(x_m, x) = \begin{cases} 0 & \text{if } x_m = 0 \\ -1 & \text{if } x_m = 1, x \neq \hat{x} \\ 1 & \text{if } x_m = 1, x = \hat{x} \end{cases}$$

Simple verification indicates that $(1, \hat{x})$ is the unique Pareto equilibrium of H . Since $\rho(H) \subseteq \text{PE}(H)$ by the previous part of the proof, we conclude by r -NEM that $(1, \hat{x}) \in \rho(H)$. Then by CONS, $\hat{x} \in \rho(H^{N^G, (1, \hat{x})}) = \rho(G)$, since by definition $H^{N^G, (1, \hat{x})} = G$. Hence $\hat{x} \in \rho(G)$, finishing our proof. ■

Our second result shows that the axiomatization of the Nash equilibrium concept on finite strategic games of Peleg and Tijs (1996, Theorem 2.12) in terms of one person rationality, consistency and converse consistency can also be generalized to multicriteria games.

THEOREM 3.2. *A solution concept ρ on Γ_{finite} satisfies OPE, CONS, and COCONS if and only if $\rho = \text{PE}$.*

Proof. PE satisfies the axioms. Let ρ be a solution concept on Γ_{finite} that also satisfies them. As in the proof of Theorem 3.1, we have that $\rho(G) \subseteq \text{PE}(G)$ for each $G \in \Gamma_{\text{finite}}$ by OPE and CONS. To prove that

$PE(G) \subseteq \rho(G)$ for each $G \in \Gamma_{\text{finite}}$, we use induction on the number of players. In one player games, the claim follows from OPE. Now assume the claim holds for all finite multicriteria games with at most n players and let $G \in \Gamma_{\text{finite}}$ be an $(n + 1)$ -player game. By CONS of PE: $PE(G) \subseteq \widetilde{PE}(G)$. By induction: $\widetilde{PE}(G) \subseteq \tilde{\rho}(G)$. By COCONS of ρ : $\tilde{\rho}(G) \subseteq \rho(G)$. Combining these three inclusions: $PE(G) \subseteq \rho(G)$. ■

These results seem to illustrate that the axiomatizations that exist in the literature for the Nash equilibrium concept generalize to the Pareto equilibrium concept for multicriteria games. This analogy, however, breaks down when we consider mixed extensions of finite multicriteria games, as is done in the next section.

4. MIXED EXTENSIONS OF FINITE MULTICRITERIA GAMES

Norde *et al.* (1996) characterize the Nash equilibrium concept on mixed extensions of finite strategic form games by nonemptiness, utility maximization, and consistency (cf. Proposition 2.1). In this section it is shown that analogons of these properties are not sufficient to characterize the Pareto equilibrium concept in mixed extensions of finite multicriteria games.

First, we list some of the axioms used in this section. A solution concept ρ on Γ satisfies:

- **Nonemptiness (NEM)**, if $\rho(G) \neq \emptyset$ for each $G \in \Gamma$;
- **Weak One Person Efficiency (WOPE)**, if for each one player game $G = \langle \{i\}, X_i, u_i \rangle \in \Gamma$ we have that $\rho(G) \subseteq \{x \in \Delta(X_i) \mid \nexists y \in \Delta(X_i): u_i(y) > u_i(x)\}$;
- **Consistency (CONS)**, if for each $G \in \Gamma$, each proper subcoalition $S \in 2^{N^G} \setminus \{\emptyset, N^G\}$, and each element $x \in \rho(G)$, we have that $x_S \in \rho(G^{S,x})$;
- **Converse Consistency (COCONS)**, if for each $G \in \Gamma$ with at least two players, we have that $\tilde{\rho}(G) \subseteq \rho(G)$, where

$$\tilde{\rho}(G) = \left\{ x \in \prod_{i \in N^G} \Delta(X_i) \mid \forall S \in 2^{N^G} \setminus \{\emptyset, N^G\}: x_S \in \rho(G^{S,x}) \right\}.$$

It is easy to see that PE on Γ satisfies NEM (See Lemma 2.2), WOPE, and CONS.

LEMMA 4.1. *If a solution concept ρ on Γ satisfies WOPE and CONS, then $\rho \subseteq PE$. Hence, PE is the unique maximal-under-inclusion solution concept that satisfies WOPE and CONS.*

Proof. Let ρ be a solution concept on Γ , satisfying WOPE and CONS. Let $G \in \Gamma$ and $x \in \rho(G)$. If $|N^G| = 1$, then $x \in \text{PE}(G)$ by WOPE. If $|N^G| > 1$, then for each player $i \in N^G$: $x_i \in \rho(G^{(i),x})$ by CONS, so $x_i \in \{y_i \in \Delta(X_i) \mid \nexists z_i \in \Delta(X_i): u_i(z_i, x_{-i}) > u_i(y_i, x_{-i})\}$ by WOPE. Hence, x is a Pareto equilibrium: $x \in \text{PE}(G)$. ■

Obviously, PE is the largest solution concept on Γ satisfying NEM, WOPE, and CONS, but not the only one, as our next result shows.

THEOREM 4.2. *There exists a solution concept ρ on Γ which satisfies NEM, WOPE, and CONS, such that $\rho \neq \text{PE}$.*

Proof. Define ρ as follows. Let $G = \langle N^G, (X_i)_{i \in N^G}, (u_i)_{i \in N^G} \rangle \in \Gamma$.

- If $|N^G| = 1$, take $\rho(G) = \{x \in \Delta(X_i) \mid \nexists y \in \Delta(X_i): u_i(y) > u_i(x)\} = \text{PE}(G)$. This guarantees that ρ satisfies WOPE.

- If $|N^G| > 1$, take

$$\rho(G) := \left\{ x \in \prod_{i \in N^G} \Delta(X_i) \mid \forall i \in N^G: \nexists y_i \in \Delta(X_i) \text{ such that } u_i(y_i, x_{-i}) \geq u_i(x_i, x_{-i}) \right\},$$

where for $a, b \in \mathbb{R}^m$ we write $a \geq b$ if $a_j \geq b_j$ for all $j = 1, \dots, m$ and $a \neq b$. Shapley (1959) calls this the set of strong equilibrium points and provides an existence theorem similar to Lemma 2.2, thereby establishing NEM.

It is easy to see that ρ is also consistent. To show that $\rho \neq \text{PE}$, consider the game G in Fig. 1, where both players have two pure strategies and two criteria. Obviously $(B, L) \in \text{PE}(G)$, but $(B, L) \notin \rho(G)$, since $u_1(T, L) \geq u_1(B, L)$. ■

In order to arrive at an axiomatization of PE, we require an additional axiom. A solution concept ρ on Γ satisfies:

- **WEIGHT** if for every game $G \in \Gamma$ and each vector $(\lambda_i)_{i \in N^G} \in \prod_{i \in N^G} \Delta_{r(i)}$ of weights: $\rho(G_\lambda) \subseteq \rho(G)$.

	L	R
T	(1,1),(1,0)	(0,0),(0,2)
B	(1,0),(0,0)	(0,0),(0,0)

FIGURE 1

The solution concept ρ satisfies WEIGHT if for every weight vector, the solutions of the associated weighted strategic form game are solutions of the underlying multicriteria game.

Our main result, using the strong theorems of Norde *et al.* (1996) and Shapley (1959), shows that the Pareto equilibrium concept is the unique solution concept on Γ satisfying NEM, WOPE, CONS, and WEIGHT.

THEOREM 4.3. *A solution concept ρ on Γ satisfies NEM, WOPE, CONS, and WEIGHT if and only if $\rho = PE$.*

Proof. Straightforward verification and application of Lemma 2.2 indicates that PE indeed satisfies the four axioms. Now let ρ be a solution concept on Γ satisfying NEM, WOPE, CONS, and WEIGHT. By Lemma 4.1, $\rho \subseteq PE$. Now let $G \in \Gamma$, and $x \in PE(G)$. Remains to show that $x \in \rho(G)$. By Lemma 2.2, there exists a vector $\lambda = (\lambda_i)_{i \in N^G} \in \prod_{i \in N^G} \Delta_{r(i)}$ of weights such that $x \in NE(G_\lambda)$. Notice that ρ restricted to $\Gamma_{\text{strategic}}$, the set of mixed extensions of strategic form games, satisfies NEM, UM, and CONS, and hence by Proposition 2.1, $\rho(H) = NE(H)$ for all $H \in \Gamma_{\text{strategic}}$. Consequently, $\rho(G_\lambda) = NE(G_\lambda) \ni x$. So by WEIGHT: $x \in \rho(G)$. ■

Finally, without proof, we mention that the analogon of Theorem 3.2 also holds when we consider mixed extensions:

THEOREM 4.4. *A solution concept ρ on Γ satisfies OPE, CONS, and COCONS if and only if $\rho = PE$.*

It is an easy exercise to show that the axioms used in our theorems are logically independent.

Remark 4.5. In the proof of Theorem 4.2 we mentioned the strong Pareto equilibrium concept. By slight modifications in the axioms (in particular, to (weak) one person strong efficiency and a weight axiom concerning strictly positive, rather than nonnegative, weights), all axiomatizations in Secs. 3 and 4 have analogons for the strong Pareto equilibrium concept. Also, a result analogous to Theorem 4.2 holds. To see this, define a solution concept ψ on Γ as follows. Let $G = \langle N^G, (X_i)_{i \in N^G}, (u_i)_{i \in N^G} \rangle \in \Gamma$.

- If $|N^G| = 1$, take $\psi(G) = \{x \in \Delta(X_i) \mid \exists y \in \Delta(X_i): u_i(y) \geq u_i(x)\}$. This guarantees that ψ satisfies (weak) one person strong efficiency.

- If $|N^G| > 1$, take $\psi(G) = NE(G_e)$, the set of Nash equilibria of the scalarized game in which the players assign equal weight to their criteria. By the existence of Nash equilibria in mixed extensions, ψ satisfies NEM.

It is easy to see that ψ is also consistent. To show that ψ is not equal to the strong Pareto equilibrium concept, refer again to the game G in Fig. 1.

(T, L) is a strong Pareto equilibrium of G , but the weighted payoff to player 2 increases from $\frac{1+0}{2}$ to $\frac{0+2}{2}$ if he deviates to R , indicating that $(T, L) \notin \psi(G) = NE(G_e)$.

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