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van Buggenum, Hugo; Uras, Burak

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**MONEY, ASSES MARKETS AND
EFFICIENCY OF CAPITAL FORMATION**

By

Hugo van Buggenum, Burak Uras

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Money, Asset Markets and Efficiency of Capital Formation*

Hugo van Buggenum[†]

Tilburg University

Burak R. Uras[‡]

Tilburg University

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Abstract

Holdings of money and illiquid assets are likely to be determined jointly. Therefore, frictions that give rise to a need for money may affect capital formation, resulting in either too much or too little investment. Existing models of money and capital however tend to overlook that both types of investment inefficiencies can be equilibrium outcomes. Building upon insights from the New-Monetarist literature, we construct a model in which preference heterogeneity between agents implies that both over- and under-investment can arise. We use our framework to study whether monetary policy can effectively resolve both types of investment inefficiencies, and find that increasing inflation could resolve under-investment inefficiencies while reducing inflation could curb over-investment inefficiencies.

Keywords: Optimal Monetary Policy, Asset Markets, Under-investment, Over-investment.

JEL Classification: E22, E41, E44, E52, O16.

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[†]E-mail: h.j.vanbuggenum@uvt.nl

[‡]E-mail: r.b.uras@uvt.nl

1 Introduction

When holdings of liquidity and illiquid capital are jointly determined, inflation might have welfare consequences through its influence on capital formation.¹ Optimal policy design should then respect whether there is “too little” or “too much” capital investment in the society. However, existing models of money tend to overlook the fact that depending on the economic fundamentals both types of capital inefficiency could be equilibrium outcomes. Our aim in this paper is to understand the determinants of under-investment and over-investment equilibria and whether monetary policy could effectively correct both of them.

In order to provide microfounded answers to the questions of our interest we develop a New-Monetarist model, where money not only facilitates goods market exchange but can also be used to re-balance capital positions in an asset market. In this environment we study the equilibrium interactions between money, asset market trade and the efficiency of capital formation. Different from existing studies of money and capital, our approach allows us to reconcile the conditions that give rise to *over-investment* and *under-investment* equilibria within a comprehensive framework. Based on this structure we uncover that while a deflationary monetary policy curbs the over-investment inefficiency, the optimal policy is likely to exhibit positive inflation if under-investment prevails in equilibrium.

The key feature of our model is the heterogeneity of consumption preferences and discount factor shocks. As in most models of monetary-search, we assume that agents are heterogeneous with respect to their consumption preferences in a (decentralized) market where (outside) money is the only recognizable object that facilitates trade. As another important element we also introduce heterogeneity in consumption preferences in a frictionless (centralized) market that meets after the decentralized transactions. This additional layer of preference heterogeneity is isomorphic to modeling heterogeneity in discount factor shocks, such that those agents who value decentralized consumption relatively more discount future consumption heavily and thereby exhibit *impatience*.²

To allow for joint determination of liquidity holdings and capital formation, we suppose that preference-types are stochastic and insurable only ex-post: prior to the realization of idiosyncratic preference shocks agents have to decide on the fraction of their wealth that they wish to carry forward as liquidity (money). The remaining fraction gets invested in an illiquid capital investment.

¹See for example Ferraris and Watanabe (2008), Aruoba, Waller and Wright (2011), and Andolfatto, Berentsen and Waller (2016).

²This is an explicit property of finite-horizon banking models that build upon Diamond and Dybvig (1983).

Capital cannot substitute money in decentralized exchange, while it yields a positive net return in the consecutive centralized market.

Since idiosyncratic shocks are realized after portfolio decisions are undertaken, re-balancing portfolios could improve welfare. To serve this purpose we introduce a competitive asset market based on which agents can exchange money for capital. In this market the available supply of money determines asset prices, like in the framework of Allen and Gale (1994). Our structure differs though from that of Allen and Gale's in one essential policy relevant aspect: in our model liquidity is a nominal object, whose rate of return is influenced by the inflation rate, while in Allen and Gale (1994) liquidity is a real good.

We uncover that monetary policy improves welfare when it is consistently implemented based on the type of capital formation inefficiency that prevails in equilibrium. On the one hand, inefficiencies associated with over-investment can be reduced with a deflationary policy, because inefficient investment in capital occurs when the equilibrium asset price is too high and that happens when the demand of patient agents for decentralized consumption is substantially low. Under-investment on the other hand is caused by too high of valuations of liquidity, which suppresses investment in capital below to its socially efficient level. This situation occurs when impatient types discount the future heavily and wish to sell a lot of capital in the asset market. In turn low asset prices in the market then imply high demand for liquidity ex-ante and thus originating capital at the socially efficient level becomes unprofitable. We show that low pricing of assets in the market, and as a result the inefficiencies associated with under-investment, can in some cases be corrected by running a positive rate of inflation.

Related Literature. Our model speaks to several strands of literature. First, we relate to the line of research that aims to understand the role of money in promoting efficient capital formation. Lagos and Rocheteau (2008), for instance, develop a model where cash and capital are competing media of exchange and show that a deflationary monetary policy (at Friedman rule) would maintain investment efficiency. In Aruoba, Waller and Wright (2011), who develop a quantitative search framework, capital produced in a centralized market is used as an input in decentralized market production. In that set-up, depending on the bargaining protocol between buyers and sellers, reducing the inflation rate from 10% to the Friedman rule could stimulate capital formation by 7% - with substantial implications on aggregate welfare.

As an important difference compared to our framework, these studies do not analyze the implications of money on capital formation through its impact on financial markets. There are other studies,

like ours, which consider this important extension. For instance, Ferraris and Watanabe (2008) study the role of banks' credit provision on capital formation when cash is essential for decentralized trade. However, their model lacks the type of discount factor heterogeneity that we incorporate and therefore the authors do not recover the conditions that would generate under-investment equilibria. Therefore, over-investment is the only type of capital efficiency that monetary policy can address in Ferraris and Watanabe (2008).

Closely related to ours is a paper by Andolfatto, Berentsen, and Martin (2018) who consider, amongst banks, financial markets in a New-Monetarist environment with fiat money. Our approach nevertheless differs from theirs in two respects. First, the preference structure in Andolfatto et al. (2018) represents early and late consumers while we consider patient and impatient consumers. This difference is critical for the effects of monetary policy. Since there are no inter-period discount factor shocks in Andolfatto et al. (2018) the distribution of asset holdings is inconsequential for welfare while this does not hold true in our model. Second, because of our preference structure we can treat under- and over-investment in a tractable way for all parameter specifications. Compared to Andolfatto et al. (2018), this gives us novel insights about the welfare consequences of inflation given either under- or over-investment, and how these effects relate to the importance of the asset distribution.

As we have agents with heterogeneous discount factors, our model is also related to that of Boel and Camera(2006), and Boel and Waller (2015). These papers study infinitely lived agents with heterogeneous discount factors in a new-monetarist setup, and find that deviating from the Friedman rule may be optimal. The key difference between our approach and that of Boel and Camera (2006), and Boel and Waller (2015), is our timing structure and our investment technology. We are therefore able to study how inflation can positively affect welfare through its impact on financial markets and investment.

There are also several papers which study cash and other forms of liquid assets as competing media of exchange and inquire the role of monetary policy on asset prices. Geromichalos, Licari, and Suarez-Lledo (2007), Lagos (2011), Lester, Postlewaite, and Wright (2012), and Geromichalos and Herrenbrueck (2016) argue that inflation could induce real assets to include liquidity premia, because holding assets helps either to facilitate exchange in the goods market or to avoid the inflation tax associated with carrying cash. In our framework we also endogenously determine asset price reactions to monetary policy. But different from the studies in this strand we are primarily interested in understanding formation of capital and how asset prices transmit the effects of inflation on investment

efficiency.

Finally, we also relate to the banking literature, which studies the joint determination of liquidity and asset origination in models of finitely lived agents, such as Allen and Gale (1994 and 2005). Although insights from our framework share similarities with these models, we differ in one very important aspect. Our monetarist approach towards modeling liquidity allows us to explore the consequences of inflation on capital formation (in)efficiency, such that we can draw conclusions on monetary optimal policy design.

The rest of the paper is organized as follows. Section 2 develops a benchmark model, where for expositional purposes we utilize an overlapping-generations framework. Sections 3 and 4 characterize the market solution of this model, first and second-best allocations and the conditions that give rise to the optimality of inflation (and deflation) policy. Section 5 extends the benchmark into a set-up where agents are infinitely lived and shows that the key theoretical findings of the paper are not sensitive to assuming finitely-lived agents with overlapping-generations.

2 Model Environment

We study an overlapping generations version of the two sector model by Lagos and Wright (2005). Time in the model is discrete and extends to infinity: $t = 0, 1, \dots, \infty$. Each time period t is divided in two sub-periods that we call as *Day* and *Night*. There are four tradable objects in the economy. Two of these objects are fully perishable real goods: *general good* and *special good*. The other two objects are *money* and *assets*. Money is a non-interest bearing nominal government liability, and hence cannot be consumed, but is portable, perfectly divisible, and can be stored over time. Assets are one-period real bonds backed by an investment opportunity, which pays off in terms of the general good.³ We assume these bonds take the form of accounting entries, and are therefore perfectly divisible though not portable.

In any time period t the economy is inhabited by two-period lived agents, who overlap and trade with the two adjacent cohorts - as to be delineated in the timing of events below. We denote the first-period of an agent's life-cycle as *young* and the second period as *old*. Each cohort consists of a continuum of agents of measure 1. Cohorts in the economy are populated by agents of two-

³Since the assets in our economy are not used as a medium-of-exchange to purchase special goods, they do not classify as money. Therefore we can label cash, i.e. the non-interest bearing nominal government liability, as money without causing confusion.

exogenously determined types: *buyers* and *sellers*. While both buyers and sellers have preferences to consume the general good when they are old, only buyers are interested in consuming the special good - and only when young. Detailed characteristics of agents are as follows.

Sellers do not have any endowment when they are born. However, they can produce the special good in the Night of their youth period, for which they have no consumption preference - unlike for the case of buyers. Letting Y denote the consumption of general goods by a seller in the Day of the old-period, and Q the production of special good by the seller in the Night of the youth-period, life-time preferences of a seller born in period t are represented by:

$$U_t^s = -c(Q_t) + \beta Y_{t+1},$$

where $c(Q_t) = Q_t$ is the linear disutility associated with special good production and $\beta \in (0, 1]$ is the inter-temporal discount factor.

Buyers cannot produce special goods and also have no endowment of special goods when they are born. However, they are endowed with h_t units of the general good at the beginning of their life-time. In addition, in the Day sub-period of their youth, buyers have access to a divisible investment opportunity that returns R_t units of general good in the Night for each unit of general good invested in the Day - with $R_t > 1$.

Unlike for sellers, buyers have consumption preferences for the special good in the Night sub-period of their youth. Buyers are ex-post heterogeneous in the sense that they can be patient or impatient with respect to consuming the special good. Let y^P denote the consumption of general good by patient buyers and y^I the consumption of general good by impatient buyers. Furthermore, suppose that q^P denotes the consumption of special good by patient buyers and q^I the consumption of special good by impatient buyers. Life-time utility of an impatient buyer born in period t is given by

$$U_t^{b,I} = u(q_t^I) + \delta \beta y_{t+1}^I,$$

while the life-time utility of a patient buyer born in period t is

$$U_t^{b,P} = \varepsilon u(q_t^P) + \beta y_{t+1}^P,$$

where $\varepsilon \in [0, 1]$ and $\delta \in [0, 1]$. Also, we assume $u' > 0$, $u'' < 0$, $\lim_{c \rightarrow 0} u'(c) = \infty$ and $\lim_{c \rightarrow \infty} u'(c) = 0$. Buyers learn their patience type before the special good market opens in the Night of their youth. The probability of becoming impatient is denoted with $\pi \in (0, 1)$.

Before proceeding, consider the role of ε and δ . If we let $\delta = 1$ and $\varepsilon < 1$, then agents are heterogeneous *only* because they derive different flow utility from the consumption of special goods.

If instead we let $\delta < 1$ and $\varepsilon = 1$, then agents are heterogeneous *only* because they discount the future differently. Finally, if $\varepsilon < 1$ and $\delta < 1$ then patient and impatient agents differ in two respects: patient buyers derive less flow utility from special goods consumption and impatient buyers and have a higher discount factor.

There are three markets in the economy.

1. *Centralized general consumption good market (CM)*: In this market, organized during the day, money is traded for general consumption good, where money will be supplied by the old agents to the young buyers in return for general consumption good. The general good market is frictionless and we denote the price of money in terms of general goods with ϕ_t .
2. *Asset Market (AM)*: In this market, which takes place during the Night and during which record-keeping is possible, money can be traded for assets. We denote the price of money in terms of assets with ϕ_t^a . The asset market is competitive, meaning that agents take prices as given. Credit extension is not feasible in this market; if agents are old they have no endowments so they lack the means to repay their creditors.
3. *Decentralized special consumption good market (DM)*: In this market, which also takes place during the night, each buyer is randomly matched to a seller. We assume that agents are anonymous and that there is no record keeping technology available. All transactions must thus be settled with a portable and durable object, so money is the only instrument that buyers can use to obtain special goods from sellers. We denote the price of special goods in terms of money with p_t , and this price will be determined by bargaining.

The following timing of events completes the description of the model environment.

Day sub-period of period t (inter-generational trade):

- i. The young cohort of period- t enters the economy. By the time the young agents are born they know whether they are buyers or sellers. Young buyers are endowed with h_t units of the general good at this stage.
- ii. The general consumption good market of period t opens and the old cohort of period t exchanges money with young buyers in turn for general consumption good. In these exchanges the young buyers provide fractions of the general good endowment of h_t that they are born with in order to buy money from the old.

- iii. Young buyers invest fractions of the general good remaining from the centralized general good market exchange into a tradable asset that will pay out in the Day of period $t + 1$.
- iv. Old agents consume general goods obtained from centralized market exchange and from the proceedings of the asset investment made in the Day of period $t - 1$.
- v. Young buyers move forward to the Night sub-period period with money and asset holdings.

Night sub-period of period t (intra-generational trade):

- ii. Young buyers learn whether they are patient or impatient.
- iii. Young buyers (patient and impatient) meet in the asset market to exchange money for assets.
- iv. Young buyers and young sellers meet in the decentralized special good market and exchange money for special goods. In these exchanges buyers make take-it-or-leave-it (TIOLI) offers to sellers.⁴
- v. Young buyers consume special goods.
- vi. Young buyers move to the next period with money and asset holdings, while young sellers move to the next period with money holdings.

Old agents remain inactive throughout the Night sub-period.

3 Solution

We first express the maximization problems for buyers in each stage of the timing tree and then derive the optimizing conditions. Having derived the optimizing conditions we characterize equilibria. Since sellers in the model are relatively passive we skip their optimization programs for brevity.

⁴If sellers are assumed to have some bargaining power, then buyers may have an incentive to hide their true preferences in bargaining. This issue has not been raised before in the new-monetarist literature, and we do not want it to influence our results. Because sellers are homogeneous, assuming that buyers make TIOLI offers does not give rise to complications related to hiding private information in bargaining.

3.1 Optimization Programs

Stage-IV: Centralized Market for the old. For an old-buyer of type $j \in \{I, P\}$, P denoting the patient-type and I denoting the impatient-type, we have:

$$\begin{aligned} W_{t+1}^{j,b}(m_{t+1}, a_{t+1}) &= \max_{y_{t+1}^j} y_{t+1}^j, \\ \text{s.t. } y_{t+1}^j &\leq \phi_{t+1} m_{t+1}^{j,b} + R_{t+1} a_{t+1}^{j,b}, \quad (\zeta_{t+1}^{j,b}) \end{aligned} \quad (1)$$

where $W_{t+1}^{j,b}(m_{t+1}, a_{t+1})$ is the value function in the centralized market, and $m_{t+1}^{j,b}, a_{t+1}^{j,b}$ are the holdings of money and assets at the beginning of the centralized market for an old agent of type j . Inequality (1) is the budget constraint of an old agent in the centralized (day) market. We specify the Lagrange multiplier(s) associated with (each) constraint in parentheses.

Stage-III: Decentralized Market for the young. For a young patient-buyer we have

$$\begin{aligned} V_t^{P,b}(m_t, a_t) &= \max_{q_t^P, m_{t+1}^P} \varepsilon u(q_t^P) + \beta W_{t+1}^{P,b}(m_{t+1}, a_{t+1}) \\ \text{s.t. } q_t^P &\leq \beta \phi_{t+1} (m_t^{P,b} - m_{t+1}^{P,b}) \quad (\lambda_t^{P,b}) \end{aligned} \quad (2)$$

$$m_{t+1}^{P,b} \geq 0, \quad (\mu_{m,t}^{P,b}) \quad (3)$$

$$q_t^P \geq 0, \quad (\mu_{q,t}^{P,b}) \quad (4)$$

$$a_t^{P,b} = a_{t+1}^{P,b}, \quad (5)$$

where $V_t^{P,b}(m_t, a_t)$ is the value function in the decentralized market, and $m_t^{P,b}, a_t^{P,b}$ are the holdings of money and assets at the beginning of the decentralized market for the young agent of type P . The constraints in DM are seller's participation constraint - stemming from buyer's take-it-or-leave-it offer (2), non-negativity constraints associated with money to be carried to the day market in the next period (3) and DM consumption (4), and finally the constraint that specifies that asset carried into DM cannot be spent (disposed) in DM (5).

Similarly, for a young impatient buyer:

$$V_t^{I,b}(m_t, a_t) = \max_{q_t^I, m_{t+1}^I} u(q_t^I) + \delta \beta W_{t+1}^{I,b}(m_{t+1}, a_{t+1})$$

$$\text{s.t. } q_t^I \leq \beta \phi_{t+1} (m_t^{I,b} - m_{t+1}^{I,b}), \quad (\lambda_t^{I,b})$$

$$m_{t+1}^{I,b} \geq 0, \quad (\mu_{m,t}^{I,b})$$

$$q_t^I \geq 0, \quad (\mu_{q,t}^{I,b})$$

$$a_t^{I,b} = a_{t+1}^{I,b},$$

where $V_t^{I,b}(m_t, a_t)$ is the value function in the decentralized market, and $m_t^{I,b}, a_t^{I,b}$ are the holdings of money and assets at the beginning of the decentralized market for the young agent of type I .

Stage-II: Asset Market for the young. For a young buyer of type- j :

$$\begin{aligned} \Omega_t^{j,b}(\tilde{m}_t, \tilde{a}_t) &= \max_{m_t^{j,b}, a_t^{j,b}} V_t^{j,b}(m_t, a_t), \\ \text{s.t. } \phi_t^a m_t^{j,b} + a_t^{j,b} &\leq \phi_t^a \tilde{m}_t^{j,b} + \tilde{a}_t^{j,b}, \quad (\theta_t^{j,b}) \end{aligned} \quad (6)$$

$$m_t^{j,b} \geq 0, \quad (v_{m,t}^{j,b}), \quad (7)$$

$$a_t^{j,b} \geq 0, \quad (v_{a,t}^{j,b}), \quad (8)$$

where $\Omega_t^{j,b}(\tilde{m}_t, \tilde{a}_t)$ is the value function in the asset market, and $\tilde{m}_t^{j,b}, \tilde{a}_t^{j,b}$ are the holdings of money and assets at the beginning of the asset market for a young agent of type j . In the asset market optimization program, (6) is the asset market budget constraint of a trader of type j , while (7) and (8) are non-negativity constraints associated with money and asset holdings respectively.

Stage-I: Centralized Market for the young. For a young buyer:

$$\begin{aligned} W_t^b(h_t) &= \max_{\tilde{m}_t, \tilde{a}_t} (1 - \pi) \Omega_t^{P,b}(\tilde{m}_t, \tilde{a}_t) + \pi \Omega_t^{I,b}(\tilde{m}_t, \tilde{a}_t), \\ \text{s.t. } \phi_t \tilde{m}_t + \tilde{a}_t &\leq h_t, \quad (\eta_t^b) \end{aligned} \quad (9)$$

$$\tilde{m}_t \geq 0, \quad (\chi_{m,t}^b), \quad (10)$$

$$\tilde{a}_t \geq 0, \quad (\chi_{a,t}^b), \quad (11)$$

where $W_t^b(h_t)$ is the value function in the centralized market, and h_t is the general good endowment of a young buyer at the beginning of the centralized market. In this first-stage optimization, (9) is the budget constraint of the young, while (10) and (11) are non-negativity constraints associated with money and asset holdings respectively.

3.2 Optimizing Behavior

First, we recursively solve for the optimizing behavior for patient and impatient buyers in Stages IV and III. From the solution of **Stage-IV** of a type- j , we have the following FOC

$$y_{t+1}^j : 1 - \zeta_{t+1}^{j,b} = 0, \quad (12)$$

together with dual feasibility and complementary-slackness conditions:

$$\begin{aligned} \zeta_{t+1}^{j,b} &\geq 0, \\ \zeta_{t+1}^{j,b} [\phi_{t+1} m_{t+1}^{j,b} + R_{t+1} a_{t+1}^{j,b} - y_{t+1}^j] &= 0, \end{aligned}$$

and also the envelope conditions:

$$\begin{aligned}\frac{\partial W_{t+1}^{j,b}}{\partial m_{t+1}} &= \zeta_{t+1}^{j,b} \phi_{t+1}, \\ \frac{\partial W_{t+1}^{j,b}}{\partial a_{t+1}} &= \zeta_{t+1}^{j,b} R_{t+1}.\end{aligned}$$

From (12) we get $\zeta_{t+1}^{j,b} = 1$ and hence

$$y_{t+1}^j = \phi_{t+1} m_{t+1}^{j,b} + R_{t+1} a_{t+1}^{j,b}, \quad (13)$$

holds for $j \in \{P, I\}$.

Then using (13) in the **Stage-III** problem for a patient buyer we have the following FOCs:

$$q_t^P : \quad \varepsilon u'(q_t^P) - \lambda_t^{P,b} + \mu_{q,t}^{P,b} = 0, \quad (14)$$

$$m_{t+1}^P : \quad \beta \phi_{t+1} - \lambda_t^{P,b} \beta \phi_{t+1} + \mu_{m,t}^{P,b} = 0, \quad (15)$$

together with dual feasibility and complementary-slackness conditions:

$$\begin{aligned}\lambda_t^{P,b}, \mu_{m,t}^{P,b}, \mu_{q,t}^{P,b} &\geq 0, \\ \lambda_t^{P,b} [\beta \phi_{t+1} (m_t^{P,b} - m_{t+1}^{P,b}) - q_t^P] &= 0, \\ \mu_{q,t}^{P,b} q_t^P &= 0, \\ \mu_{m,t}^{P,b} m_{t+1}^{P,b} &= 0,\end{aligned}$$

and also the envelope conditions:

$$\begin{aligned}\frac{\partial V_t^{P,b}}{\partial m_t} &= \lambda_t^{P,b} \phi_{t+1} \beta, \\ \frac{\partial V_t^{P,b}}{\partial a_t} &= \frac{\partial V_t^{P,b}}{\partial a_{t+1}} = \beta \frac{\partial W_{t+1}^{P,b}}{\partial a_{t+1}} = \beta R_{t+1}.\end{aligned}$$

Similarly using (13) in the **Stage-III** problem of an impatient buyer we have the following FOCs:

$$\begin{aligned}q_t^I : \quad & u'(q_t^I) - \lambda_t^{I,b} + \mu_{q,t}^{I,b} = 0, \\ m_{t+1}^I : \quad & \delta \beta \phi_{t+1} - \lambda_t^{I,b} \beta \phi_{t+1} + \mu_{m,t}^{I,b} = 0,\end{aligned} \quad (16)$$

together with dual feasibility and complementary-slackness conditions:

$$\begin{aligned}\lambda_t^{I,b}, \mu_{m,t}^{I,b}, \mu_{q,t}^{I,b} &\geq 0, \\ \lambda_t^{I,b} [\delta \beta \phi_{t+1} (m_t^{I,b} - m_{t+1}^{I,b}) - q_t^I] &= 0, \\ \mu_{q,t}^{I,b} q_t^I &= 0, \\ \mu_{m,t}^{I,b} m_{t+1}^{I,b} &= 0,\end{aligned}$$

and also the envelope conditions:

$$\begin{aligned}\frac{\partial V_t^{I,b}}{\partial m_t} &= \lambda_t^{I,b} \phi_{t+1} \beta, \\ \frac{\partial V_t^{I,b}}{\partial a_t} &= \frac{\partial V_t^{I,b}}{\partial a_{t+1}} = \beta \frac{\partial W_{t+1}^{I,b}}{\partial a_{t+1}} = \delta \beta R_{t+1}.\end{aligned}$$

Using the solutions to the stage III and IV problems we can already note that there are two cases - for both patient and impatient buyers - characterized by the money balances carried over in between the youth and old periods.

Case-1 - Patient. $\mu_{m,t}^{P,b} = 0$ and $m_{t+1}^{P,b} \geq 0$. Then from (15) we get:

$$\lambda_t^{P,b} = 1.$$

Furthermore, there are two sub-cases to consider:

1. $\varepsilon = 0$, which implies $q_t^P = 0$ (follows from $\mu_{q,t}^{P,b} = \lambda_t^{P,b} = 1$, which can be derived from (14)).
2. $\varepsilon > 0$, which implies $q_t^P > 0$ and $\mu_{q,t}^{P,b} = 0$, because otherwise - given $\lim_{q_t^P \rightarrow 0} u'(q_t^P) = \infty$ - condition (14) won't hold.

Assuming $\varepsilon > 0$, and hence $\mu_{q,t}^{P,b} = 0$, it follows from (14) that the optimal q_t^P satisfies

$$\varepsilon u'(q_t^P) = 1.$$

Before we move on with case-2 define:

$$\hat{q}_t^P \equiv u'^{-1}(\varepsilon^{-1})$$

and observe that

$$m_{t+1}^{P,b} \geq 0 \Leftrightarrow \hat{q}_t^P \leq \beta \phi_{t+1} m_t^{P,b}.$$

Case-2 - Patient. $\mu_{m,t}^{P,b} > 0$ and $m_{t+1}^{P,b} = 0$. Then, if $\varepsilon > 0$, $\mu_{q,t}^{P,b} = 0$ and the optimal q_t^P satisfies

$$q_t^P = \beta \phi_{t+1} m_t^{P,b} < \hat{q}_t^P,$$

and if $\varepsilon = 0$, $\mu_{q,t}^{P,b} > 0$ and the optimal $q_t^P = 0$. Then, (14) and (15) yield:

$$\begin{aligned}\lambda_t^{P,b} &= \varepsilon u'(\beta \phi_{t+1} m_t^{P,b}) > 1, \\ \mu_{m,t}^{P,b} &= \beta \phi_{t+1} [\varepsilon u'(\beta \phi_{t+1} m_t^{P,b}) - 1] > 0.\end{aligned}$$

Finally, using what we obtained for the cases of 1 & 2 for the patient buyer, we can express:

$$\lambda_t^{P,b} = \max\{\varepsilon u'(\beta\phi_{t+1}m_t^{P,b}), 1\},$$

$$\frac{\partial V_t^{P,b}}{\partial m_t} = \beta\phi_{t+1} \max\{\varepsilon u'(\beta\phi_{t+1}m_t^{P,b}), 1\}.$$

Case-1 - Impatient. $\mu_{m,t}^{I,b} = 0$ and $m_{t+1}^{I,b} \geq 0$. Then from (16) we get:

$$\lambda_t^{I,b} = \delta.$$

Furthermore, $q_t^I > 0$ and $\mu_{q,t}^{I,b} = 0$, because otherwise - given $\lim_{q_t^I \rightarrow 0} u'(q_t^I) = \infty$ - condition (14) won't hold. Then, the optimal q_t^I satisfies

$$u'(q_t^I) = \delta.$$

Before we move on with case-2, let us define:

$$\hat{q}_t^I \equiv u'^{-1}(\delta)$$

and observe that

$$m_{t+1}^{I,b} \geq 0 \Leftrightarrow \hat{q}_t^I \leq \beta\phi_{t+1}m_t^{I,b}.$$

Case-2 - Impatient. $\mu_{m,t}^{I,b} > 0$ and $m_{t+1}^{I,b} = 0$. Then, $\mu_{q,t}^{I,b} = 0$ and the optimal q_t^I satisfies

$$q_t^I = \beta\phi_{t+1}m_t^{I,b} < \hat{q}_t^I.$$

Equations (14) and (15) then yield:

$$\lambda_t^{I,b} = u'(\beta\phi_{t+1}m_t^{I,b}) > \delta,$$

$$\mu_{m,t}^{I,b} = \beta\phi_{t+1}[u'(\beta\phi_{t+1}m_t^{I,b}) - \delta] > 0.$$

Finally, using what we obtained for the cases of 1 & 2 for the impatient buyer, we can express:

$$\lambda_t^{I,b} = \max\{u'(\beta\phi_{t+1}m_t^{I,b}), \delta\},$$

$$\frac{\partial V_t^{I,b}}{\partial m_t} = \beta\phi_{t+1} \max\{u'(\beta\phi_{t+1}m_t^{I,b}), \delta\}.$$

We then move on with the solution to the **Stage-II** problem for a type-j buyer, which gives the following FOCs

$$m_t^j : \quad \frac{\partial V_t^{j,b}}{\partial m_t} - \theta_t^{j,b}\phi_t^a + v_{m,t}^{j,b} = 0,$$

$$a_t^j : \quad \frac{\partial V_t^{j,b}}{\partial a_t} - \theta_t^{j,b} + v_{a,t}^{j,b} = 0,$$

together with dual feasibility and complementary-slackness conditions

$$\begin{aligned}\theta_t^{j,b}, v_{m,t}^{j,b}, v_{a,t}^{j,b} &\geq 0, \\ \theta_t^{j,b}[\phi_t^a \tilde{m}_t^{j,b} + \tilde{a}_t^{j,b} - \phi_t^a m_t^{j,b} - a_t^{j,b}] &= 0, \\ v_{m,t}^{j,b} m_t^{j,b} &= 0, \\ v_{a,t}^{j,b} a_t^{j,b} &= 0,\end{aligned}$$

and the envelope conditions,

$$\frac{\partial \Omega_t^{j,b}}{\partial \tilde{m}_t} = \theta_t^{j,b} \phi_t^a, \quad (17)$$

$$\frac{\partial \Omega_t^{j,b}}{\partial \tilde{a}_t} = \theta_t^{j,b}. \quad (18)$$

Then using the envelope conditions that we derived for Stage-III in the Stage-II problem we obtain for a patient buyer

$$\frac{\partial \Omega_t^{P,b}}{\partial m_t} = \beta \phi_{t+1} \max\{\varepsilon u'(\beta \phi_{t+1} m_t^{P,b}), 1\} - \theta_t^{P,b} \phi_t^a + v_{m,t}^{P,b}, \quad (19)$$

$$\frac{\partial \Omega_t^{P,b}}{\partial a_t} = \beta R_{t+1} - \theta_t^{P,b} + v_{a,t}^{P,b}, \quad (20)$$

and for an impatient buyer

$$\frac{\partial \Omega_t^{I,b}}{\partial m_t} = \beta \phi_{t+1} \max\{u'(\beta \phi_{t+1} m_t^{I,b}), \delta\} - \theta_t^{I,b} \phi_t^a + v_{m,t}^{I,b}, \quad (21)$$

$$\frac{\partial \Omega_t^{I,b}}{\partial a_t} = \delta \beta R_{t+1} - \theta_t^{I,b} + v_{a,t}^{I,b}. \quad (22)$$

Finally, the FOCs of the **Stage-I** problem are expressed as

$$\begin{aligned}\tilde{m}_t : \quad & (1 - \pi) \frac{\partial \Omega_t^{P,b}}{\partial \tilde{m}_t} + \pi \frac{\partial \Omega_t^{I,b}}{\partial \tilde{m}_t} - \eta_t^b \phi_t + \chi_{m,t}^b = 0, \\ \tilde{a}_t : \quad & (1 - \pi) \frac{\partial \Omega_t^{P,b}}{\partial \tilde{a}_t} + \pi \frac{\partial \Omega_t^{I,b}}{\partial \tilde{a}_t} - \eta_t^b + \chi_{a,t}^b = 0,\end{aligned}$$

together with dual feasibility and complementary-slackness conditions

$$\begin{aligned}\eta_t^b, \chi_{m,t}^b, \chi_{a,t}^b &\geq 0, \\ \eta_t^b[h_t - \phi_t \tilde{m}_t - \tilde{a}_t] &= 0, \\ \chi_{m,t}^b \tilde{m}_t^b &= 0, \\ \chi_{a,t}^b \tilde{a}_t^b &= 0.\end{aligned}$$

3.3 Characterizing Equilibria

There are two general equilibria (solutions) to consider, the *interior* solution (with capital investment and money holdings) and the *corner* solution (with only money holdings).

Interior Solution. $\chi_{m,t}^b = \chi_{a,t}^b = 0$, such that both assets and money are supplied in the asset market. Then utilizing (17) and (18) in centralized market FOCs for a young buyer we get:

$$\begin{aligned}(1 - \pi)\theta_t^{P,b}\phi_t^a + \pi\theta_t^{I,b}\phi_t^a &= \eta_t^b\phi_t, \\ (1 - \pi)\theta_t^{P,b} + \pi\theta_t^{I,b} &= \eta_t^b,\end{aligned}$$

which gives:

$$\phi_t = \phi_t^a.$$

Corner Solution. $\chi_{a,t}^b > \chi_{m,t}^b = 0$, such that no asset is brought to the asset market. Then utilizing (17) and (18) in centralized market FOCs for a young buyer we get:

$$\begin{aligned}(1 - \pi)\theta_t^{P,b}\phi_t^a + \pi\theta_t^{I,b}\phi_t^a &= \eta_t^b\phi_t, \\ (1 - \pi)\theta_t^{P,b} + \pi\theta_t^{I,b} &= \eta_t^b - \chi_{a,t}^b,\end{aligned}$$

which gives:

$$\phi_t < \phi_t^a.$$

We would like to note that there is a potential third possibility: $\chi_{m,t}^b > \chi_{a,t}^b = 0$. This a non-monetary equilibrium, which we rule out - given the essentiality of money in the decentralized market.

3.3.1 Equilibria with capital investment: interior solution

Consider next the equilibria that arise when both money and assets are carried to the asset market: $\tilde{m}_t, \tilde{a}_t > 0$. From the analysis above it follows that this requires $\phi_t^a = \phi_t$. Moreover, it also requires that $v_{a,t}^{P,b} = 0$. To demonstrate why, consider $v_{a,t}^{P,b} > 0$ so that complementary slackness implies $a_t^{P,b} = 0$. Then, to clear the asset market we need $v_{a,t}^{I,b} = 0$, $v_{m,t}^{P,b} = 0$ and given the properties of u

we can also use $v_{m,t}^{I,b} = 0$. From the optimality conditions in the asset market (19)-(22) we obtain

$$\begin{aligned}\theta_t^{P,b} &= \beta \frac{\phi_{t+1}}{\phi_t} \max \left\{ \varepsilon u' \left(\beta \phi_{t+1} \left(\tilde{m}_t + \frac{\tilde{a}_t}{\phi_t} \right) \right), 1 \right\}, \\ \theta_t^{P,b} &> \beta R_{t+1}, \\ \theta_t^{I,b} &= \delta \beta R_{t+1}, \\ \theta_t^{I,b} &= \beta \frac{\phi_{t+1}}{\phi_t} \max \left\{ u' \left(\beta \phi_{t+1} m_t^{I,b} \right), \delta \right\}.\end{aligned}$$

Combining the above four relationships, we obtain:

$$\varepsilon u' \left(\beta \phi_{t+1} \left(\tilde{m}_t + \frac{\tilde{a}_t}{\phi_t} \right) \right) > \delta^{-1} u' \left(\beta \phi_{t+1} m_t^{I,b} \right),$$

which in turn requires $m_t^{I,b} > \tilde{m}_t + \frac{\tilde{a}_t}{\phi_t}$ but contradicts $v_{a,t}^{I,b} = 0$. Therefore, we must have $v_{a,t}^{P,b} = 0$.

Given $v_{a,t}^{P,b} = 0$, use the asset market FOCs for the patient buyer, (19) and (20), to obtain

$$\begin{aligned}\beta \phi_t R_{t+1} &= \beta \phi_{t+1} \max \left\{ \varepsilon u' \left(\beta \phi_{t+1} m_t^{P,b} \right), 1 \right\} + v_{m,t}^{P,b} \\ &\geq \beta \phi_{t+1},\end{aligned}$$

so that $R_t \phi_t \geq \phi_{t+1}$. It can be shown easily that this condition represents a zero lower bound (henceforth ZLB) on the nominal interest rate earned by assets between the asset market and the next centralized market. Using this insight we formulate 4 equilibrium cases that can arise for an interior solution, i.e. $\tilde{m}_t, \tilde{a}_t > 0$.

Case-1 - At the ZLB and impatient buyers sell all assets. This case is defined by properties $R_t \phi_t = \phi_{t+1}$ and $v_{a,t}^{I,b} > 0$. Moreover, we know $v_{a,t}^{P,b} = v_{m,t}^{I,b} = 0$ and $v_{m,t}^{P,b} \geq 0$. From FOCs (20) and (21) we obtain

$$\begin{aligned}\theta_t^{P,b} &= \beta R_{t+1}, \\ \theta_t^{I,b} &= \beta \frac{\phi_{t+1}}{\phi_t} \max \left\{ u' \left(\beta \phi_{t+1} \left(\tilde{m}_t + \frac{\tilde{a}_t}{\phi_t} \right) \right), \delta \right\},\end{aligned}$$

and to ensure $v_{a,t}^{I,b} > 0$ it follows from FOC (22) that we need

$$u' \left(\beta \phi_{t+1} \left(\tilde{m}_t + \frac{\tilde{a}_t}{\phi_t} \right) \right) > \delta$$

which, given that $\phi_t \tilde{m}_t + \tilde{a}_t = h_t$, reduces to

$$\beta R_{t+1} h_t < u'^{-1}(\delta). \tag{23}$$

Next, consider asset market clearance. For the patient buyer we can use the FOC w.r.t. money holdings (19) to obtain

$$\beta\phi_{t+1} \max \left\{ \varepsilon u' \left(\beta\phi_{t+1} m_t^{P,b} \right), 1 \right\} - \beta\phi_t R_{t+1} + v_{m,t}^{P,b} = 0.$$

Given that $v_{m,t}^{P,b} = 0$ if $\varepsilon > 0$ and $q^I = 0$ if $\varepsilon = 0$, it follows that the following relationship holds with regard to the amount of money that patient buyers wish to carry to the decentralized market.

$$\beta\phi_{t+1} m_t^{P,b} \geq u'^{-1}(\varepsilon^{-1}).$$

To ensure clearance of the asset market we therefore need

$$\pi \frac{\tilde{a}_t}{\phi_t} \leq (1 - \pi) \left[\tilde{m} - \frac{u'^{-1}(\varepsilon^{-1})}{\beta\phi_{t+1}} \right],$$

where the LHS captures the net-demand for money by impatient buyers and the RHS captures the maximum net-supply of money by patient buyers. The above condition can be rearranged to obtain

$$x_t h_t \geq \pi h_t + (1 - \pi) \beta^{-1} R_{t+1}^{-1} u'^{-1}(\varepsilon^{-1})$$

where we define $x_t \equiv \frac{\phi_t \tilde{m}_t}{h_t}$. To ensure an interior solution, that means $\tilde{m}_t, \tilde{a}_t > 0$, we need $x_t < 1$ which reduces to:

$$\beta R_{t+1} h_t > u'^{-1}(\varepsilon^{-1}). \quad (24)$$

Summarizing, existence requires that (23) and (24) are satisfied.

To finalize the analysis, observe that aggregate welfare of the buyers satisfies

$$\begin{aligned} \mathcal{W}_t &= \pi_t V_t^{P,b} + (1 - \pi) V_t^{I,b} \\ &= \pi u(\beta R_{t+1} h_t) + (1 - \pi) \varepsilon u(\hat{q}_t^P) + (1 - \pi) \beta R_{t+1} h_t - (1 - \pi) \hat{q}_t^P. \end{aligned}$$

Case-2 - Away from the ZLB and impatient buyers sell all assets. This case is defined by properties $R_t \phi_t > \phi_{t+1}$ and $v_{a,t}^{I,b} > 0$. Moreover, we know $v_{a,t}^{P,b} = v_{m,t}^{I,b} = 0$ and $v_{m,t}^{P,b} \geq 0$. FOCs (20) and (21) therefore imply

$$\begin{aligned} \theta_t^{P,b} &= \beta R_{t+1}, \\ \theta_t^{I,b} &= \beta \frac{\phi_{t+1}}{\phi_t} \max \left\{ u' \left(\beta \phi_{t+1} \left(\tilde{m}_t + \frac{\tilde{a}_t}{\phi_t} \right) \right), \delta \right\}. \end{aligned}$$

Using the FOC for asset holdings by impatient agents (22) we can then obtain

$$\delta \beta R_{t+1} \frac{\phi_t}{\phi_{t+1}} = \beta \max \left\{ u' \left(\beta \phi_{t+1} \left(\tilde{m}_t + \frac{\tilde{a}_t}{\phi_t} \right) \right), \delta \right\} - v_{a,t}^{I,b} \frac{\phi_t}{\phi_{t+1}},$$

which implies that

$$\begin{aligned} u' \left(\beta \phi_{t+1} \left(\tilde{m}_t + \frac{\tilde{a}_t}{\phi_t} \right) \right) &> \delta, \\ \Rightarrow \beta \frac{\phi_{t+1}}{\phi_t} h_t &< u'^{-1}(\delta). \end{aligned} \quad (25)$$

To ensure that $v_{a,t}^{I,b} > 0$ indeed holds we then need

$$\begin{aligned} \beta u' \left(\beta \phi_{t+1} \left(\tilde{m}_t + \frac{\tilde{a}_t}{\phi_t} \right) \right) &> \delta \beta R_{t+1} \frac{\phi_t}{\phi_{t+1}}, \\ \Rightarrow \beta \frac{\phi_{t+1}}{\phi_t} h_t &< u'^{-1} \left(\frac{R_{t+1} \phi_t}{\phi_{t+1}} \delta \right). \end{aligned} \quad (26)$$

It follows that when (26) holds then (25) also holds, hence the latter is redundant.

Consider next asset market clearance, using the FOC for patient agents w.r.t. money carried to the decentralized market (19) we obtain:

$$\beta \phi_{t+1} \max \left\{ \varepsilon u' \left(\beta \phi_{t+1} m_t^{P,b} \right), 1 \right\} - \beta \phi_t R_{t+1} + v_{m,t}^{P,b} = 0.$$

If $\varepsilon > 0$ then the properties of u imply $v_{m,t}^{P,b} = 0$. Moreover, if $\varepsilon = 0$ we have $v_{m,t}^{P,b} > 0$. Therefore we can write

$$\beta \phi_{t+1} m_t^{P,b} = u'^{-1} \left(\frac{R_{t+1} \phi_t}{\phi_t} \varepsilon^{-1} \right).$$

Asset market clearance therefore implies

$$\pi \frac{\tilde{a}_t}{\phi_t} = (1 - \pi) \left[\tilde{m}_t - \frac{u'^{-1} \left(\frac{R_{t+1} \phi_t}{\phi_t} \varepsilon^{-1} \right)}{\beta \phi_{t+1}} \right],$$

where the LHS captures the net-demand for money of impatient buyers and the RHS captures the net-supply of money by patient buyers. Rearrange the above to obtain

$$x_t h_t = \pi h_t + (1 - \pi) \frac{\phi_t}{\phi_{t+1}} \beta^{-1} u'^{-1} \left(\frac{R_t \phi_t}{\phi_{t+1}} \varepsilon^{-1} \right).$$

To ensure an interior solution, i.e. $x_t < 1$, we need

$$\beta \frac{\phi_{t+1}}{\phi_t} h_t > u'^{-1} \left(\frac{R_{t+1} \phi_t}{\phi_{t+1}} \varepsilon^{-1} \right). \quad (27)$$

Summarizing, existence requires that (26) and (27) hold.

To finalize the analysis, consider aggregate welfare of the buyers:

$$\begin{aligned}
\mathcal{W}_t &= \pi V_t^{I,b} + (1 - \pi) V_t^{P,b} \\
&= \pi u(q_t^I) + (1 - \pi) \varepsilon u(q_t^P) - \pi q_t^I - (1 - \pi) q_t^P + \beta \phi_{t+1} \tilde{m}_t + \beta R_{t+1} \tilde{a}_t \\
&= \pi u\left(\beta \frac{\phi_{t+1}}{\phi_t} h_t\right) + (1 - \pi) \varepsilon u\left(u'^{-1}\left(\frac{R_{t+1} \phi_t}{\phi_{t+1}} \varepsilon^{-1}\right)\right) + \beta (1 - \pi) R_{t+1} h_t \\
&\quad - (1 - \pi) \frac{R_{t+1} \phi_t}{\phi_{t+1}} u'^{-1}\left(\frac{R_{t+1} \phi_t}{\phi_{t+1}} \varepsilon^{-1}\right).
\end{aligned}$$

Case-3 - At the ZLB and impatient buyers do not sell all assets. This case is defined by properties $R_t \phi_t = \phi_{t+1}$ and $v_{a,t}^{I,b} = 0$. Moreover, we know $v_{a,t}^{P,b} = v_{m,t}^{P,b} = 0$ and $v_{m,t}^{P,b} \geq 0$. From the FOCs for assets acquired in the asset market, (20) and (22), we obtain

$$\begin{aligned}
\theta_t^{P,b} &= \beta R_{t+1}, \\
\theta_t^{I,b} &= \delta \beta R_{t+1}.
\end{aligned}$$

Using $\theta_t^{I,b}$ in the FOC for money carried into the decentralized market by impatient agents (21) we obtain

$$\begin{aligned}
\beta \phi_{t+1} \max \left\{ u' \left(\beta \phi_{t+1} m_t^{I,b} \right), \delta \right\} &= \delta \beta \phi_t R_{t+1}, \\
\Rightarrow \max \left\{ u' \left(\beta \phi_{t+1} m_t^{I,b} \right), \delta \right\} &= \delta \\
\Rightarrow \beta \phi_{t+1} m_t^{I,b} &\geq u'^{-1}(\delta),
\end{aligned}$$

To ensure that $a_t^{I,b} \geq 0$ we need

$$\begin{aligned}
\beta R_{t+1} h_t &= \beta \frac{\phi_{t+1}}{\phi_t} \left(\tilde{m}_t + \frac{\tilde{a}_t}{\phi_t} \right) \\
&\geq \beta \phi_{t+1} m_t^{I,b} \\
&\geq u'^{-1}(\delta).
\end{aligned} \tag{28}$$

Analogously, we can use $\theta_t^{P,b}$ in the FOC for money carried into the decentralized market by patient agents (19) to obtain

$$\begin{aligned}
\beta \phi_{t+1} \max \left\{ \varepsilon u' \left(\beta \phi_{t+1} m_t^{P,b} \right), 1 \right\} &= \beta \phi_t R_{t+1} - v_{a,t}^{P,b}, \\
\Rightarrow \max \left\{ \varepsilon u' \left(\beta \phi_{t+1} m_t^{P,b} \right), 1 \right\} &= 1, \\
\Rightarrow \beta \phi_{t+1} \left(\tilde{m}_t + \frac{\tilde{a}_t}{\phi_t} \right) &\geq u'^{-1}(\varepsilon^{-1}).
\end{aligned}$$

To ensure that $a_t^{P,b} \geq 0$ we need

$$\begin{aligned}\beta R_{t+1} h_t &= \beta \phi_{t+1} \left(\tilde{m}_t + \frac{\tilde{a}_t}{\phi_t} \right) \\ &\geq \beta \phi_{t+1} m_t^{P,b} \\ &\geq u'^{-1}(\varepsilon^{-1}).\end{aligned}\tag{29}$$

It follows that if (28) holds then (29) holds as well, hence the latter is redundant.

Next, to clear the asset market we need

$$\pi_t \left(\frac{u'^{-1}(\delta)}{\beta \phi_{t+1}} - \tilde{m}_t \right) \leq (1 - \pi) \left(\tilde{m}_t - \frac{u'^{-1}(\varepsilon^{-1})}{\beta \phi_{t+1}} \right),$$

where the LHS captures the minimum net-demand for money by patient buyers and the RHS captures the maximum net-supply of money by impatient buyers. The above can be rearranged to obtain

$$\beta R_{t+1} h_t x_t \geq \pi u'^{-1}(\delta) + (1 - \pi) u'^{-1}(\varepsilon^{-1}).$$

To ensure an interior solution, i.e. $x_t < 1$, we need

$$\beta R_{t+1} h_t \geq \pi u'^{-1}(\delta) + (1 - \pi) u'^{-1}(\varepsilon^{-1}).\tag{30}$$

If (28) holds then (30) holds as well, so the latter is redundant. Summarizing, existence requires (28).

To finalize the analysis, consider aggregate welfare of the buyers. For this we need relationships for the amounts of assets and money carried over by buyers to the centralized market and their value in the period $t + 1$ centralized market. It can be shown that

$$\begin{aligned}R_{t+1} a_{t+1}^{P,b} + \phi_{t+1} m_{t+1}^{P,b} &= R_{t+1} \left(a_t^{P,b} + \phi_t m_{t+1}^{P,b} \right) \\ &= R_{t+1} \left(\phi_t \tilde{m}_t + \tilde{a}_t - \frac{\phi_t u'^{-1}(\varepsilon^{-1})}{\beta \phi_{t+1}} \right) \\ &= R_{t+1} h_t - \beta^{-1} u'^{-1}(\varepsilon^{-1}),\end{aligned}\tag{31}$$

$$\begin{aligned}R_{t+1} a_{t+1}^{I,b} + \phi_{t+1} m_{t+1}^{I,b} &= R_{t+1} \left(a_t^{I,b} + \phi_t m_{t+1}^{I,b} \right) \\ &= R_{t+1} \left(\phi_t \tilde{m}_t + \tilde{a}_t - \frac{\phi_t u'^{-1}(\delta)}{\beta \phi_{t+1}} \right) \\ &= R_{t+1} h_t - \beta^{-1} u'^{-1}(\delta).\end{aligned}\tag{32}$$

Use (31) and (32) to obtain for buyer welfare:

$$\begin{aligned}\mathcal{W}_t &= \pi V_t^{I,b} + (1 - \pi) V_t^{P,b} \\ &= \pi u(\hat{q}_t^P) + (1 - \pi) \varepsilon u(\hat{q}_t^I) + \pi \delta \beta R_{t+1} [h_t - \beta^{-1} R_{t+1}^{-1} u'^{-1}(\delta)] \\ &\quad + (1 - \pi) \beta R_{t+1} [h_t - \beta^{-1} R_{t+1}^{-1} u'^{-1}(\varepsilon^{-1})].\end{aligned}$$

Case-4 - Away from the ZLB and impatient buyers do not sell all assets. This case is defined by properties $R_t \phi_t > \phi_{t+1}$ and $v_{a,t}^{I,b} = 0$. Moreover, we know $v_{a,t}^{P,b} = v_{m,t}^{P,b} = 0$ and $v_{m,t}^{P,b} \geq 0$. From the FOCs for assets acquired in the asset market, (20) and (22), we obtain

$$\begin{aligned}\theta_t^{P,b} &= \beta R_{t+1}, \\ \theta_t^{I,b} &= \delta \beta R_{t+1}.\end{aligned}$$

Using $\theta_t^{I,b}$ in the FOC for money carried into the decentralized market by impatient agents (21) we obtain

$$\begin{aligned}\beta \phi_{t+1} \max \left\{ u' \left(\beta \phi_{t+1} m_t^{I,b} \right), \delta \right\} &= \delta \beta \phi_t R_{t+1}, \\ \Rightarrow \max \left\{ u' \left(\beta \phi_{t+1} m_t^{I,b} \right), \delta \right\} &> \delta,\end{aligned}\tag{33}$$

where the last line follows from the fact that $R_t \phi_t > \phi_{t+1}$. Therefore, money carried to the centralized market by impatient buyers satisfies

$$\beta \phi_{t+1} m_t^{I,b} = u'^{-1} \left(\frac{R_{t+1} \phi_t}{\phi_{t+1}} \delta \right).\tag{34}$$

To ensure that $a_t^{I,b} \geq 0$ we need

$$\begin{aligned}\beta \frac{\phi_{t+1}}{\phi_t} h_t &= \phi_t \tilde{m}_t + \tilde{a}_t \\ &\geq u'^{-1} \left(\frac{R_{t+1} \phi_t}{\phi_{t+1}} \delta \right).\end{aligned}\tag{35}$$

Analogously, use $\theta_t^{P,b}$ in the FOC for money carried into the decentralized market by patient agents (19) to obtain

$$\beta \phi_{t+1} \max \left\{ \varepsilon u' \left(\beta \phi_{t+1} m_t^{P,b} \right), 1 \right\} = \beta \phi_t R_{t+1} - v_{m,t}^{P,b}.$$

Given $R_{t+1} \phi_t > \phi_{t+1}$, if $\varepsilon = 0$ then $m_t^{P,b} = 0$ and if $\varepsilon > 0$ then, since the properties of u imply $v_{m,t}^{P,b} = 0$, we have

$$\max \left\{ \varepsilon u' \left(\beta \phi_{t+1} m_t^{P,b} \right), 1 \right\} > 1.\tag{36}$$

Money carried to the centralized market by patient buyers therefore satisfies

$$\beta\phi_{t+1}m_t^{P,b} = u'^{-1} \left(\frac{R_{t+1}\phi_t}{\phi_{t+1}} \varepsilon^{-1} \right). \quad (37)$$

To ensure that $a_t^{P,b} \geq 0$ we need

$$\begin{aligned} \beta \frac{\phi_{t+1}}{\phi_t} h_t &= \phi_t \tilde{m}_t + \tilde{a}_t \\ &\geq u'^{-1} \left(\frac{R_{t+1}\phi_t}{\phi_{t+1}} \varepsilon^{-1} \right). \end{aligned} \quad (38)$$

Nevertheless, if (35) holds then (38) holds as well so the latter is redundant.

Next, consider clearance of the asset market. Using (34) and (37) we obtain

$$\pi \left[\frac{u'^{-1} \left(\frac{R_{t+1}\phi_t}{\phi_{t+1}} \delta \right)}{\beta\phi_{t+1}} - \tilde{m}_t \right] = (1 - \pi) \left[\tilde{m}_t - \frac{u'^{-1} \left(\frac{R_{t+1}\phi_t}{\phi_{t+1}} \varepsilon^{-1} \right)}{\beta\phi_{t+1}} \right],$$

where the LHS captures the net-demand for money by impatient buyers and the RHS captures the net-supply of money by patient buyers. Rearrange to obtain:

$$\beta \frac{\phi_{t+1}}{\phi_t} x_t h_t = \pi u'^{-1} \left(\frac{R_{t+1}\phi_t}{\phi_{t+1}} \delta \right) + (1 - \pi) u'^{-1} \left(\frac{R_{t+1}\phi_t}{\phi_{t+1}} \varepsilon^{-1} \right).$$

To ensure an interior solution, i.e. $x_t < 1$, we therefore need

$$\beta \frac{\phi_{t+1}}{\phi_t} h_t > \pi u'^{-1} \left(\frac{R_{t+1}\phi_t}{\phi_{t+1}} \delta \right) + (1 - \pi) u'^{-1} \left(\frac{R_{t+1}\phi_t}{\phi_{t+1}} \varepsilon^{-1} \right). \quad (39)$$

Nevertheless, if (35) holds then (39) holds as well. The latter is therefore redundant so for existence we only need (35).

To finalize the analysis, consider aggregate welfare of the buyers. Given (33) and (36) we know $m_{t+1}^{P,b} = m_{t+1}^{I,b} = 0$. Therefore, we need relationships for the amount of assets carried over by buyers to the centralized market and their value in the period $t + 1$ centralized market. It can be shown that

$$\begin{aligned} R_{t+1}a_{t+1}^{P,b} &= R_{t+1} \left(\phi_t \tilde{m}_t + \tilde{a}_t - \frac{\phi_t u'^{-1} \left(\frac{R_{t+1}\phi_t}{\phi_{t+1}} \varepsilon^{-1} \right)}{\beta\phi_{t+1}} \right) \\ &= R_{t+1} h_t - \beta^{-1} \frac{R_{t+1}\phi_t}{\phi_{t+1}} u'^{-1} \left(\frac{R_{t+1}\phi_t}{\phi_{t+1}} \varepsilon^{-1} \right), \\ R_{t+1}a_{t+1}^{I,b} &= R_{t+1} \left(\phi_t \tilde{m}_t + \tilde{a}_t - \frac{\phi_t u'^{-1} \left(\frac{R_{t+1}\phi_t}{\phi_{t+1}} \delta \right)}{\beta\phi_{t+1}} \right) \\ &= R_{t+1} h_t - \beta^{-1} \frac{R_{t+1}\phi_t}{\phi_{t+1}} u'^{-1} \left(\frac{R_{t+1}\phi_t}{\phi_{t+1}} \delta \right), \end{aligned}$$

Use (31) and (32) to obtain for buyer welfare:

$$\begin{aligned}
\mathcal{W}_t &= \pi V_t^{I,b} + (1 - \pi) V_t^{P,b} \\
&= \pi u \left(u'^{-1} \left(\frac{R_{t+1} \phi_t}{\phi_{t+1}} \delta \right) \right) + (1 - \pi) \varepsilon u \left(u'^{-1} \left(\frac{R_{t+1} \phi_t}{\phi_{t+1}} \varepsilon^{-1} \right) \right) + \\
&\quad + \pi \delta \beta R_{t+1} \left[h_t - \beta^{-1} \frac{\phi_t}{\phi_{t+1}} u'^{-1} \left(\frac{R_{t+1} \phi_t}{\phi_{t+1}} \delta \right) \right] \\
&\quad + (1 - \pi) \beta R_{t+1} \left[h_t - \beta^{-1} \frac{\phi_t}{\phi_{t+1}} u'^{-1} \left(\frac{R_{t+1} \phi_t}{\phi_{t+1}} \varepsilon^{-1} \right) \right].
\end{aligned}$$

3.3.2 Equilibria without capital investment: corner solution

This is the over-arching case of “ $x_t = 1$ ”, where no asset-origination takes place ($\phi_t^a \geq \phi_t$).

Case-1. $R_{t+1} \frac{\phi_t}{\phi_{t+1}} \leq 1$. $x_t = 1$, $\exists \phi_t^a \geq \phi_t$, when both $v_{m,t}^I = v_{m,t}^P = 0$. Then, $v_{a,t}^j \geq 0$ for $j \in \{P, I\}$ requires both

$$\begin{aligned}
\beta \phi_{t+1} \max\{u'(\beta \phi_{t+1} \tilde{m}_t), \delta\} &\geq \phi_t^a \beta \delta R_{t+1}, \\
\beta \phi_{t+1} \max\{\varepsilon u'(\beta \phi_{t+1} \tilde{m}_t), 1\} &\geq \phi_t^a \beta R_{t+1}.
\end{aligned}$$

If $\phi_t = \phi_t^a$, then two inequalities boil down to

$$\begin{aligned}
\beta \frac{\phi_{t+1}}{\phi_t} h_t &\leq u'^{-1} \left(\frac{\phi_t R_{t+1}}{\phi_{t+1} \varepsilon} \right) < u'^{-1} \left(\frac{1}{\varepsilon} \right), \\
\beta \frac{\phi_{t+1}}{\phi_t} h_t &\leq u'^{-1} \left(\frac{\delta \phi_t R_{t+1}}{\phi_{t+1}} \right),
\end{aligned} \tag{40}$$

where (40) is redundant.

Case-2. $R_{t+1} \frac{\phi_t}{\phi_{t+1}} > 1$. Then $x_t = 1$, if

$$\beta \phi_{t+1} \max\{\varepsilon u'(\beta \phi_{t+1} \tilde{m}_t), 1\} \geq \phi_t^a \beta R_{t+1},$$

which implies

$$\beta \frac{\phi_{t+1}}{\phi_t} h_t \leq u'^{-1} \left(\frac{\phi_t R_{t+1}}{\phi_{t+1} \varepsilon} \right).$$

Welfare in the corner. For the case of $R_{t+1} \frac{\phi_t}{\phi_{t+1}} > 1$, at $x_t = 1$ welfare of the buyers is given by

$$\mathcal{W}_t = u \left(\beta \frac{\phi_{t+1}}{\phi_t} h_t \right) (\pi + (1 - \pi) \varepsilon).$$

For the case of $R_{t+1} \frac{\phi_t}{\phi_{t+1}} \leq 1$, we first note that the aggregate buyer welfare had already been discussed for $R_{t+1} \frac{\phi_t}{\phi_{t+1}} = 1$, where buyers are indifferent across some spectrum of x_t that includes $x_t = 1$. When $R_{t+1} \frac{\phi_t}{\phi_{t+1}} < 1$, we always have $x_t = 1$ and there are three different sub-cases to consider for the aggregate welfare:

$$\mathcal{W}_{1,t} = u \left(\beta \frac{\phi_{t+1}}{\phi_t} h_t \right) (\pi + (1 - \pi)\varepsilon),$$

if $\beta \frac{\phi_{t+1}}{\phi_t} h_t < \hat{q}_t^P$.

$$\mathcal{W}_{2,t} = \pi u \left(\beta \frac{\phi_{t+1}}{\phi_t} h_t \right) + \varepsilon(1 - \pi)u(\hat{q}_t^P) + \beta(1 - \pi) \left(\frac{\phi_{t+1}}{\phi_t} h_t - \hat{q}_t^P \right),$$

if $\hat{q}_t^P \leq \beta \frac{\phi_{t+1}}{\phi_t} h_t < \hat{q}_t^I$. Finally,

$$\mathcal{W}_{3,t} = \pi u(\hat{q}_t^I) + \varepsilon(1 - \pi)u(\hat{q}_t^P) + \delta\beta\pi \left(\frac{\phi_{t+1}}{\phi_t} h_t - \hat{q}_t^I \right) + \beta(1 - \pi) \left(\frac{\phi_{t+1}}{\phi_t} h_t - \hat{q}_t^P \right),$$

if $\hat{q}_t^I \leq \beta \frac{\phi_{t+1}}{\phi_t} h_t$.

3.4 Summary of different equilibria

In sum we have the following possible types of equilibria:

Type-1. $R_{t+1} < \frac{\phi_{t+1}}{\phi_t}$, then $x_t = 1$.

Type-2. $R_{t+1} = \frac{\phi_{t+1}}{\phi_t}$,

i. if $\beta R_{t+1} h_t \leq u'^{-1}(1/\varepsilon)$, then $x_t = 1$.

ii. if $u'^{-1}(1/\varepsilon) < \beta R_{t+1} h_t < u'^{-1}(\delta)$, then

$$x_t \in \left[\pi + \frac{(1 - \pi)u'^{-1}(\varepsilon^{-1})}{\beta R_{t+1} h_t}, 1 \right],$$

iii. if $\beta R_{t+1} h_t \geq u'^{-1}(\delta)$, then

$$x_t \in \left[\frac{\pi u'^{-1}(\delta)}{\beta R_{t+1} h_t} + \frac{(1 - \pi)u'^{-1}(\varepsilon^{-1})}{\beta R_{t+1} h_t}, 1 \right].$$

Type-3. $R_{t+1} > \frac{\phi_{t+1}}{\phi_t}$,

i. if $\beta \frac{\phi_{t+1}}{\phi_t} h_t \leq u'^{-1} \left(\frac{\phi_t R_{t+1}}{\phi_{t+1} \varepsilon} \right)$, then

$$x_t = 1,$$

Figure 1: Equilibrium types depending on inflation and the endowment. Drawn for logarithmic utility $u = \ln(c)$.

ii. if $\frac{\phi_t}{\phi_{t+1}} u'^{-1} \left(\frac{\phi_t R_{t+1}}{\phi_{t+1} \varepsilon} \right) < \beta h_t < \frac{\phi_t}{\phi_{t+1}} u'^{-1} \left(\frac{\delta \phi_t R_{t+1}}{\phi_{t+1}} \right)$, then

$$x_t = \pi + (1 - \pi) \frac{\phi_t}{\phi_{t+1}} \beta^{-1} h_t^{-1} u'^{-1} \left(\frac{\phi_t R_{t+1}}{\phi_{t+1} \varepsilon} \right),$$

iii. $\beta h_t \geq \frac{\phi_t}{\phi_{t+1}} u'^{-1} \left(\frac{\delta \phi_t R_{t+1}}{\phi_{t+1}} \right)$, then

$$x_t = \pi \frac{\phi_t}{\phi_{t+1}} \beta^{-1} h_t^{-1} u'^{-1} \left(\frac{\delta \phi_t R_{t+1}}{\phi_{t+1}} \right) + (1 - \pi) \frac{\phi_t}{\phi_{t+1}} \beta^{-1} h_t^{-1} u'^{-1} \left(\frac{\phi_t R_{t+1}}{\phi_{t+1} \varepsilon} \right).$$

Figures 1 and 2 illustrate how inflation and the endowment affect the equilibrium type that will materialize.

3.5 Welfare Implications of Monetary Policy

We concentrate our welfare analysis on equilibrium types 2 and 3, as in these equilibria investment weakly dominates money in terms of return. Throughout we adopt a stationary inflation rate $\gamma = \frac{\phi_t}{\phi_{t+1}}$ and provide sellers with monetary injections when they are old.⁵

⁵Our results hold true if all old agents, that is buyers and sellers, are provided with monetary injections.

Figure 2: Equilibrium types depending on inflation and the endowment. Drawn for CRRA utility with a coefficient of relative risk aversion smaller than 1.

Since sellers receive the monetary injection, we note:

$$V_t^S = \beta(\gamma - 1)m_t\phi_{t+1} = \beta \left(\frac{\gamma - 1}{\gamma} \right) x_t h_t,$$

where the last equality follows from the fact that $m_t\phi_t = x_t h_t$.

Type 3-i. The welfare implications of inflation on buyers is given by

$$\frac{\partial \mathcal{W}_t}{\partial \gamma} = -[\pi + \varepsilon(1 - \pi)] u' \left(\frac{\beta h}{\gamma} \right) \frac{\beta h}{\gamma^2},$$

while the welfare implications of inflation on sellers is $\beta h/\gamma^2$. Then defining the societal social surplus (welfare) with S_t , we have:

$$\frac{\partial \mathcal{S}_t}{\partial \gamma} = \frac{\beta h}{\gamma^2} \left[1 - [\pi + \varepsilon(1 - \pi)] u' \left(\frac{\beta h}{\gamma} \right) \right] < 0.$$

The fact that this expression is negative follows from $u'(\beta h/\gamma) \geq R/\gamma$.

Type 3-ii. We first note that:

$$V_t^S = \pi \beta \left(\frac{\gamma - 1}{\gamma} \right) h_t + (1 - \pi)(\gamma - 1)u'^{-1}(R\gamma\varepsilon^{-1}).$$

The welfare implications of inflation on buyers:

$$\frac{\partial \mathcal{W}_t}{\partial \gamma} = -\pi u'(q^I) \frac{\beta h_t}{\gamma^2} - (1 - \pi) R q^P,$$

while the welfare implications of inflation on sellers:

$$\frac{\partial V_t^S}{\partial \gamma} = \beta \frac{\pi h_t}{\gamma^2} + (1 - \pi) q^P - (1 - \pi) \left(\frac{\gamma - 1}{\gamma} \right) \frac{q^P}{\sigma(q^P)},$$

where $\sigma(q) \equiv -qu''(q)/u'(q)$ is the coefficient of relative risk aversion. Then defining the societal social surplus (welfare) with S_t , we have:

$$\frac{\partial \mathcal{S}_t}{\partial \gamma} = \frac{\beta \pi h}{\gamma^2} (1 - u'(q^I)) - (1 - \pi)(R - 1)q^P - (1 - \pi) \left(\frac{\gamma - 1}{\gamma} \right) \frac{q^P}{\sigma(q^P)}, \quad (41)$$

which may be negative or positive - depending on the parameter constellations of the model.

Type 3-iii. We first note that:

$$V_t^S = (\gamma - 1)[\pi u'^{-1}(R\gamma\delta) + (1 - \pi)u'^{-1}(R\gamma\varepsilon^{-1})].$$

The welfare implications of inflation on buyers in this case:

$$\frac{\partial \mathcal{W}_t}{\partial \gamma} = -\pi R \delta q^I - (1 - \pi) R q^P,$$

while the welfare implications of inflation on sellers:

$$\frac{\partial V_t^S}{\partial \gamma} = \pi q^I + (1 - \pi) q^P - \left(\frac{\gamma - 1}{\gamma} \right) \left[\frac{\pi q^I}{\sigma(q^I)} + \frac{(1 - \pi) q^P}{\sigma(q^P)} \right].$$

Then:

$$\frac{\partial \mathcal{S}_t}{\partial \gamma} = \pi q^I (1 - \delta R) - (1 - \pi)(R - 1)q^P - \left(\frac{\gamma - 1}{\gamma} \right) \left[\frac{\pi q^I}{\sigma(q^I)} + \frac{(1 - \pi) q^P}{\sigma(q^P)} \right], \quad (42)$$

which may be negative or positive - depending on the parameter constellations of the model. We also note that the social surplus is continuous at the intersection of the parameter-space where cases 3-i and 3-ii meet, and where cases 3-ii and 3-iii meet. This observation follows from the continuity of DM consumption. Moreover, we also note that when $\gamma = 1$, the derivative of social welfare with respect to inflation is continuous at the intersection of the parameter-space where cases 3-i and 3-ii meet, and where cases 3-ii and 3-iii meet.

Proposition 1 *Equilibrium welfare is a function of the inflation rate (γ) and depending on parameter constellations it could increase as well as decrease with γ .*

We next move on with the welfare comparison of market outcomes with social planner's allocations in order to draw conclusions with respect to optimal monetary policy design.

4 Optimal Allocations and Monetary Policy

In this section we study the first- and second-best allocations chosen by a social planner and compare the welfare implications of those against that of market outcomes. Throughout we focus on a stationary solution, meaning that all real quantities are stable over time.

4.1 First Best

In the unconstrained social optimum the social planner can freely choose allocations subject to resource constraints:

$$\begin{aligned} \mathcal{S}^* &= \max_{\substack{x, q^I, q^P, \\ y^I, y^P, y^s}} \pi (u(q^I) - q^I) + (1 - \pi) (\varepsilon u(q^P) - q^P) + \beta \delta \pi y^I + \beta (1 - \pi) y^P + \beta \pi y^s \\ \text{s.t. } &\pi y^I + (1 - \pi) y^P + y^s \leq xh + (1 - x)Rh \\ &q^I, q^P, y^I, y^P, y^s \geq 0, \\ &0 \leq x \leq 1. \end{aligned}$$

The solution to the above program satisfies:

$$\begin{aligned} u'(q^I) &= 1, \\ \varepsilon u'(q^P) &= 1, \\ y^I &= 0, \\ (1 - \pi) y^P + y^s &= \max\{R, 1\}h, \\ x &= \begin{cases} 1 & \text{if } R < 1, \\ (0, 1) & \text{if } R = 1, \\ 0 & \text{if } R > 0. \end{cases} \end{aligned}$$

4.2 Second Best

Let us consider the second-best allocations that satisfy two additional constraints. First, sellers cannot be forced to produce in the DM, they must be compensated so that the utility cost of DM production are lower than the discounted utility from consumption by sellers when they are old. Second, we also impose a constraint that is satisfied in a market economy as well: the compensation for production from the sellers born in period t must come from the endowment of the agents born

in period $t + 1$. This constraint reflects the fact that in a market economy sellers born in period t produce in exchange for fiat money which in turn is bought by the generation born in period $t + 1$. We obtain the following constrained optimization problem:

$$\begin{aligned}
\mathcal{S}' = \max_{\substack{x, q^I, q^P, \\ y^I, y^P, y^s}} & \pi (u(q^I) - q^I) + (1 - \pi) (\varepsilon u(q^P) - q^P) + \beta \delta \pi y^I + \beta (1 - \pi) y^P + \beta \pi y^s \\
s.t. \quad & \beta h x \geq \pi q^I + (1 - \pi) q^P, \\
& \beta y^s = \pi q^I + (1 - \pi) q^P, \\
& \pi y^I + (1 - \pi) y^P + y^s \leq x h + (1 - x) R h \\
& q^I, q^P, y^I, y^P, y^s \geq 0, \\
& 0 \leq x \leq 1.
\end{aligned}$$

The solution to the constrained (second-best) optimization problem depends on the rate of return on productive investment and the size of endowment available to the young agents:

Case-1. In this case $R > 1$ and $\beta h \geq \pi u'^{-1}(R) + (1 - \pi) u'^{-1}(R \varepsilon^{-1})$. The solution satisfies

$$\begin{aligned}
u'(q^I) &= R, \\
\varepsilon u'(q^P) &= R, \\
\beta x h &= \pi q^I + (1 - \pi) q^P, \\
y^P &\geq x h, \\
y^I &= 0, \\
(1 - \pi) y^P + y^s &= x h + (1 - x) h R.
\end{aligned}$$

In this case investment is productive and so the social planner prefers investment over inter-generational transfers. As a result, intergenerational transfers are only used to compensate the seller for his production and the opportunity cost of consumption are given by the return on investment. The endowment is large enough to sustain the inter-generational transfers needed to compensate the seller.

Case-2. In this case $R > 1$ and $\beta h < \pi u'^{-1}(R) + (1 - \pi)u'^{-1}(R\varepsilon^{-1})$. The solution satisfies:

$$\begin{aligned} u'(q^I) &= \varepsilon u'(q^P), \\ \beta h &= \pi q^I + (1 - \pi)q^P, \\ x &= 1, \\ y^s &= h, \\ y^P &= 0, \\ y^I &= 0. \end{aligned}$$

This case is the same as the previous case, but the endowment is too small to sustain the inter-generational transfers needed to compensate the sellers when DM consumption levels are as in case 1. Therefore, the social planner uses the entire endowment as an inter-generation transfer to finance DM consumption.

Case-3. In this case $R = 1$ and $\beta h \geq \pi u'^{-1}(1) + (1 - \pi)u'^{-1}(\varepsilon^{-1})$. The optimal allocations satisfy:

$$\begin{aligned} u'(q^I) &= 1, \\ \varepsilon u'(q^P) &= 1, \\ \beta x h &\geq \pi q^I + (1 - \pi)q^P, \\ y^s &\geq \beta^{-1} (\pi q^I + (1 - \pi)q^P), \\ y^I &= 0, \\ (1 - \pi)y^P + y^s &= h. \end{aligned}$$

This case is similar to case 1, but now the social planner is indifferent between investment or inter-generational transfers. As a result, x need not be pinned down uniquely.

Case-4. In this case $R = 1$ and $\beta h < \pi u'^{-1}(1) + (1 - \pi)u'^{-1}(\varepsilon^{-1})$. The optimal allocations are the same as in case 2. The endowment is too small to sustain the inter-generational transfers that are associated with the consumption levels in case 3. As a result, the full endowment is used as an inter-generational transfer to compensate the sellers for DM production.

Case-5. In this case $R < 1$ and $\beta h \geq \pi u'^{-1}(1) + (1 - \pi)u'^{-1}(\varepsilon^{-1})$. The optimal allocations

satisfy:

$$\begin{aligned}
u'(q^I) &= 1, \\
\varepsilon u'(q^P) &= 1, \\
x &= 1, \\
y^s &\geq \beta^{-1} (\pi q^I + (1 - \pi) q^P), \\
y^I &= 0, \\
(1 - \pi) y^P + y^s &= h.
\end{aligned}$$

In this case the social planner prefers inter-generational transfers over investment. As a result, the entire endowment is used as an inter-generational transfer. Moreover, DM consumption levels are the same as in case 3 and the endowment is large enough to compensate sellers for producing these levels of consumption.

Case-6. In this case $R < 1$ and $\beta h < \pi u'^{-1}(1) + (1 - \pi) u'^{-1}(\varepsilon^{-1})$. The optimal allocations are the same as in case 2 because the endowment is too small to support the consumption levels in case 5.

4.3 Relationship to CIMP, Investment and Welfare Effects of Inflation

The second-best allocations can be compared with an alternative formulation of the model: rather than thinking of xh as an inter-generational transfer to compensate sellers, we can also interpret h as a real endowment of goods that can either be stored (and thus consumed early) or invested (but not consumed early). That means the social planner thus has access to a long-term investment opportunity that returns R , but that cannot be liquidated prematurely, and to a short-term investment opportunity that can be liquidated early or rolled over (cash in the parlance of Allen and Gale).

To understand how market solutions differ from those chosen by the constrained social planner, it is important first to understand what drives individual decisions in a market economy and how these decisions translate into aggregate investment. Assume, without loss of generality, that $\phi_t = \phi_t^a$ so that young agents are indifferent between investing in money or in assets during the CM. In other words, when agents are young they perceive their investment decisions to be irrelevant; their wealth in the AM is purely determined by the endowment h_t . Individuals therefore perceive that they can determine their desired consumption levels q_t^I and q_t^P independently from each other. When determining these consumption levels, it is the agents choosing between money holdings and asset

holdings and therefore these levels satisfy:

$$\begin{aligned} u'(q_t^I) &= \max \left\{ \beta \delta \frac{R_{t+1} \phi_t}{\phi_t}, u' \left(\beta \frac{\phi_t}{\phi_{t+1}} h_t \right) \right\}, \\ \varepsilon u'(q_t^P) &= \max \left\{ \beta \frac{\phi_t R_{t+1}}{\phi_{t+1}}, \varepsilon u' \left(\beta \frac{\phi_t}{\phi_{t+1}} h_t \right) \right\}. \end{aligned}$$

If the endowment is large enough, some buyers will therefore want to carry assets out of the AM. In equilibrium, the level of investment in the morning CM adapts to satisfy the demand for assets in the next AM.

The social planner however decides on q_t^I and q_t^P jointly, taking into account a different opportunity cost of consumption. SP solves:

$$\begin{aligned} \mathcal{S}' &= \max_{q^I, q^P} \pi u(q^I) + (1 - \pi) \varepsilon u(q^P) - R [\pi q^I + (1 - \pi) q^P] \\ \text{s.t. } & \pi q^I + (1 - \pi) q^P \leq \beta h. \end{aligned}$$

Two important differences show up. First, the social planner considers an aggregate resource constraint. Second, the social planner takes into account different opportunity costs of consumption - he takes the rate of return on the liquid technology to be one as it represents an inter-generational transfer - and he does not take into account the additional discounting factor δ for impatient agents, because the social planner allocates all the proceeds from productive investment to patient agents.

Our model is a generalization of standard CIMP models in the sense that decisions made in the market economy are affected by inflation while this does not hold for the constraint-efficient allocations (moreover, we also take into account the welfare benefits of inflation). To understand what can drive over- and/or under-investment it is useful to compare market allocations against constrained-efficient allocations for $\phi_t = \phi_{t+1}$.

Case-1 No investment. This case arises when $\beta h \leq u'^{-1}(R\varepsilon^{-1})$. The endowment is so small that in a market economy even the patient agents do not carry any assets out of the DM. Moreover, the social planner also faces a binding budget constraint and therefore does not invest.

Case-2 Investment by the market, but no investment by the social planner. This case arises when $u'^{-1}(R\varepsilon^{-1}) < \beta h \leq \pi u'^{-1}(R) + (1 - \pi) u'^{-1}(R\varepsilon^{-1})$ and exists only if $\varepsilon < 1$. The endowment is large enough so that patient buyers wish to carry assets out of the AM and therefore investment must take place in the market economy. However, the social planner does not want to invest: the marginal utility of consumption for an impatient agent would then exceed the marginal benefit of investment, and therefore the social planner uses the endowment to finance DM consumption rather

than investment. When $\varepsilon = 1$, this case does not exist since, for given a given consumption level, the marginal benefits of consumption for both types are equal. Summarizing, in the market economy agents must invest relatively much in assets - if they invest the socially optimal level then there is too much cash available so that the AM can only clear if ϕ_a goes up. The latter can however not constitute an equilibrium, as it would then be attractive for all agents to carry only assets to the AM in order to obtain cash at a discount when they turn out to be impatient.

Case-3 Investment by the market and the social planner, but over-investment. This case arises when $\pi u'^{-1}(R) + (1 - \pi)u'^{-1}(R\varepsilon^{-1}) < \beta h < u'(R)$ and exists only if $\varepsilon < 1$. The endowment is large enough to also make the social planner willing to invest. However, the amount invested in the market economy is always larger:

$$\begin{aligned}\beta x h &= \beta \pi h + (1 - \pi)u'^{-1}(R\varepsilon^{-1}) \\ &< \pi u'^{-1}(R) + (1 - \pi)(R\varepsilon^{-1}) \\ &= \beta x' h,\end{aligned}$$

where $1 - x$ and $1 - x'$ are the shares of h invested in the productive technology by the market economy and the constrained social planner respectively. The over-investment problem becomes less severe as $\beta h \rightarrow u'(R)$. The reason is that in the market economy, the endowment is still too small to have the impatient agent carry assets out of the DM. As a result an increasing endowment only promotes investment in the market economy because the patient agents want to carry more assets out of the AM, the impatient agents will use the increased endowment for additional DM consumption. The social planner however does not want to further increase the consumption level of the impatient agents; the endowment not consumed by the patient agent is large enough to ensure that the marginal utility of consumption by patient agents equals that of the impatient agents. Hence, additional endowment will be fully invested in the productive asset by the social planner.

Case-4 Same investment by market and social planner. This case arises when $\beta h = u'^{-1}R$, and the market economy implements the constrained-efficient allocations. In the market economy, the impatient agents do not carry any assets out of the AM - they consume their entire endowment in the DM which induces a marginal utility that equals the rate of return on productive assets. The level of investment is the same as that chosen by the constrained social planner.

Case-5 Underinvestment by the market economy, patient agents do not carry assets out of AM. This case arises when $u'^{-1}(R) < \beta h \leq u'^{-1}(R\delta)$ and exists only if $\delta < 1$. The endowment is still too small to induce impatient buyers to carry assets out of the AM. Though the endowment is more than enough to set q^I to the constrained-efficient level even in the market

economy, the impatient agent continues to consume his full endowment in the DM. The reason lies in the fact that the impatient agent has a lower discount factor than the social planner, and as a result the impatient agent finds it more attractive to consume in the DM rather than in the next CM when he is old. Under-investment thus arises:

$$\begin{aligned}\beta x h &= \beta \pi h + (1 - \pi) u'^{-1}(R \varepsilon^{-1}) \\ &> \pi u'^{-1}(R) + (1 - \pi) (R \varepsilon^{-1}) \\ &= \beta x' h,\end{aligned}$$

Agents must invest a relatively large part of their endowment in cash when young, if not there would be too little cash available in the afternoon to meet the demand for cash by impatient agents in which case the AM price of asset in terms of money would go down. The latter can however not constitute an equilibrium, as it would then be attractive for all agents to carry only cash to the AM in order to obtain assets at a discount when an agent turns out to be patient.

Case-6 Potential underinvestment by the market economy, patient agents do carry assets out of the DM. This case arises when $\beta h \geq u'^{-1}(R \delta)$, under investment in this case exists if and only if $\delta < 1$. The endowment is now large enough to induce impatient buyers to carry assets out of the AM. If $\delta = 1$, the impatient agent chooses the constrained-efficient level of consumption and the market economy is characterized by exactly the same allocations as chosen by the constrained social planner. However, if $\delta < 1$ the impatient buyers again have a lower discount factor than the constrained social planner. Though the impatient buyer still carries assets out of the AM, he consumes too much in the DM as his opportunity cost of consumption are too low. Hence, for $\delta < 1$ the economy is characterized by underinvestment

$$\begin{aligned}\beta x h &= \pi u'^{-1}(R \delta) + (1 - \pi) u'^{-1}(R \varepsilon^{-1}) \\ &> \pi u'^{-1}(R) + (1 - \pi) (R \varepsilon^{-1}) \\ &= \beta x' h,\end{aligned}$$

Again agents must invest a relatively large amount of their endowment in cash when young because of the same reason as discussed in case 5.

Figure 3 illustrates the different cases graphically, which is also summarized in the following proposition.

Proposition 2 *Under-investment (over-investment) can exist only if $\delta < 1$ ($\varepsilon < 1$). Optimal monetary policy depends on the size of the endowment (h), where high h induces under-investment*

Figure 3: Under- and over-investment given the size of the endowment when $\gamma = 1$. The share of the endowment devoted to inter-generational transfers by the social planner is denoted with x' (fraction of endowment devoted to capital is therefore $1 - x'$). The share of the endowment devoted to inter-generational transfers in a market economy is denoted with x (fraction of endowment devoted to capital is $1 - x$).

whereas low h induces over-investment. Lowering the intertemporal value of cash, and generating inflation, would curb the under-investment inefficiency while increasing the intertemporal value of cash, and running a inflation, would curb the over-investment inefficiency.

We are now in a position to discuss the welfare effects of the inflationary policy introduced earlier. Specifically, we study the welfare effects of changing inflation when $\gamma = 1$, as money is then equivalent to the intergenerational transfers that a social planner can implement. First, we note that for the degenerate case $u'(\beta h) = R$, constrained efficient allocations are implemented by the market economy if $\gamma = 1$. Clearly, the optimal monetary policy for this degenerate case is to set the gross inflation rate equal to one, which is equivalent to keeping the money stock constant. A similar argument applies when the economy is in case 1, that means $u'(\beta h) \geq R\varepsilon^{-1}$, as then both the market economy as well as the constrained efficient allocations are characterized by no investment.

Second, we notice that in case 2 and case 3, the proposed inflationary policy is always welfare reducing. The reason is that the inflationary policy reduces DM consumption by the buyers, and increases CM consumption by old sellers. In cases 2 and 3, consumption by impatient buyers is however relatively low as they consume their entire endowment, which in turn is relatively small. The marginal welfare loss associated with reducing DM consumption by buyers is thus large and does not outweigh the marginal welfare benefit associated with increasing CM consumption by old

Figure 4: Share of the endowment devoted to investment by the market economy and by the constrained social planner when $\gamma = 1$. If $\beta h \rightarrow \infty$, then both fractions approach 1. Parametrization: $u(c) = 8.25 \ln(c)$, $R = 1.1$, $\pi = 0.5$, $\varepsilon = 0.5$ and $\delta = 0.8$.

sellers. In fact, this observation holds true if we assume that the monetary injections are distributed to all old agents.

Therefore, we find that inflation is welfare reducing whenever the market economy is characterized by over-investment, which happens when the initial endowment is relatively small. We mentioned before that a small endowment implied that investment by patient buyers yielded lower social returns than DM consumption by impatient buyers. In addition, the high social returns from DM consumption by impatient buyers imply that inflation is welfare reducing, as inflation reduces DM consumption of impatient buyers. With over-investment, a deflationary policy is thus optimal, but this requires the ability to levy taxes.

Third, in cases 5 and 6 inflation may improve welfare. Evaluating equations (41) and (42), in the limit case inflation is for sure welfare improving when $R \rightarrow 1$. The reason for this result is that in this segment of the parameter space, the impatient agents consume too much in DM, so that the social returns from DM consumption of impatient agents are too low. By increasing the inflation rate, DM consumption is reduced and CM consumption by old sellers is increased. Because the social returns from DM consumption are low, this reallocation of consumption can be welfare improving under certain parameter constellations and also if monetary injections are distributed proportionally to all old agents.

Summarizing, we thus find that starting from a benchmark case with a constant money stock, $\gamma = 1$, inflation can be welfare improving, but only if the economy is characterized by under-investment. The main reason for this effect is that the endowment must be relatively large for under-investment to occur, as otherwise the impatient agents do not over-consume in DM.

5 Extension: A Model with Infinitely-Lived Agents

The key results that we have obtained in the previous section are not artifacts of the finitely-lived agents assumption that we have imposed for ease of exposition.⁶ In order to highlight the general nature of a key part of our findings, in this section we consider a production economy with infinitely lived buyers and sellers with *discount factor shocks* and reconsider the interaction between monetary policy and under-investment. At the Friedman rule, our modified economy is characterized by under-investment. When the inflation rate is increased, impatient agents may have to sell their assets at a

⁶In general the mechanism that we highlight in this paper is quite different than the mechanism of search-based OLG models of money such as Zhu (2008) and Hiraguchi (2017).

discount to patient agents in the asset market. As a result, resources are transferred from impatient to patient agents in the next CM implying higher social welfare. For an increase in inflation to have a negative effect on the price of assets, these assets must be relatively abundant in supply at the Friedman rule, as otherwise the economy remains stuck at the ZLB when inflation is increased.

5.1 Model Environment

The extended model is characterized by the same set of tradable objects from the benchmark: general good, special good, money and assets. The key difference from the benchmark is that we replace the two-period lived buyers and sellers with infinitively lived buyers and sellers. Sellers have preferences represented by:

$$U_{j,t}^s = E_t \sum_{t'=t}^{\infty} \beta^{t'-t} [y_{j,t'} - c(q_{j,t'})],$$

where $y_{j,t}$ is consumption of general good in period t by seller $j \in [0, 1]$, $q_{j,t}$ is production of special good in period t and $c(q) = q$.

Buyers have preferences represented by:

$$U_{i,t}^b = E_t \sum_{t'=t}^{\infty} \beta^{t'-t} \zeta_{i,t',t} [y_{i,t'} + u(q_{i,t'})],$$

where $y_{i,t}$ is consumption of general good by buyer $i \in [0, 1]$ and $q_{i,t}$ is consumption of special good by buyer $i \in [0, 1]$. To model idiosyncratic shocks to time preferences we introduce $\zeta_{i,t',t}$, which is defined as:

$$\zeta_{i,t',t} \equiv \begin{cases} \prod_{\tau=t}^{\tau=t'-1} \zeta_{i,\tau} & \text{if } t' > t, \\ 1 & \text{if } t' = t. \end{cases}$$

In this expression $\zeta_{i,t}$ represents an idiosyncratic shock that is independently and identically distributed across sellers and across time according to:

$$\zeta_{i,t} = \begin{cases} \delta^h & \text{with probability } \pi, \\ \delta^l & \text{with probability } 1 - \pi, \end{cases}$$

where $\delta^h \geq \delta^l$ and $\pi\delta^h + (1 - \pi)\delta^l = 1$.

As another difference from the benchmark specification we assume that the general good can be produced by buyers *and* sellers according to a linear technology from labor. We assume that exerting labor effort generates linear dis-utility and we also assume that labor endowments are large

enough so that constraints regarding a maximum amount of hours worked never bind. To ensure that under- or over-investment is easy to identify we choose to endogenize the rate of return R_{t+1} on assets. We do this by assuming that buyers can produce $f(k_t)$ units of general good in the day of period $t+1$ from k_t units of capital good produced in the day of period t . We assume that $f(0) = 0$, $f' > 0$, $f'' < 0$, $\lim_{k \rightarrow 0} f'(k) = \infty$ and $\lim_{k \rightarrow \infty} f'(k) = 0$.

As before we have two sub-periods, *Day* and *Night*. Also, we assume again that three markets are organized in the following order:

1. *CM*. In this market, organized during the day of period t , money, capital produced in period $t-1$, and general goods are traded. The price of money in terms of general goods is again ϕ_t , and the price of capital produced in period $t-1$ is R_t .
2. *AM*. In this market, organized during the night of period t , buyers can trade capital goods, produced in the day of period t , and money. We denote the price of money in terms of these capital goods with ϕ_t^a .
3. *DM*. In this market, organized during the night of period $t-1$, each buyer is randomly matched to a seller. Sellers only accept money as a payment instrument and buyers will make TIOLI offers to sellers.⁷

Further timing of events is as follows:

Day sub-period of period t :

- i. The CM opens. Sellers can produce general goods from labor, buyers can produce general goods from labor and capital goods produced in period $t-1$. Buyers and sellers exchange money, capital goods and general goods. Buyers produce new capital goods from labor.
- ii. Sellers and buyers consume general goods obtained in the CM

Night sub-period of period t :

- i. Buyers learn $\zeta_{i,t}$ as private information.
- ii. The AM opens, buyers meet to exchange capital goods for money.

⁷Again, we do not want to face complications related to incentives of agents to hide private information in bargaining. Therefore, we assume that buyers make TIOLO offers to the sellers.

- iii. Buyers are randomly matched to sellers in the DM. Sellers produce special good for the buyers they are matched to, and the buyers consume this special good.

Before proceeding, we shortly want to mention here how our approach deviates from that of Boel and Camera (2006), and Boel and Waller (2015). First, the former do not use idiosyncratic shocks to discount factors but instead assume that agents are born with different discount factors. Second, the latter do have idiosyncratic shocks to discount factors, but assume that they are drawn at the beginning of the CM. In both papers, the return on money is therefore limited by the discount factor of patient agents. It is therefore impossible to compensate impatient agents for the cost of holding money, which is not the case in our setup. Moreover, because preferences are known in the CM, there is no role for an asset market in the setup of Boel and Camera (2006), and Boel and Waller (2015), while asset markets are welfare improving in our setup.

5.2 Individual Decision Making

We solve the model recursively. To begin with, we let $W_t^b(\bar{m}_{t-1}, \bar{k}_{t-1})$ denote the value function of entering the CM with $\bar{m}_{t-1}$ units of money and $\bar{k}_{t-1}$ units of capital. Because of the quasi-linear utility structure we conjecture that $(\partial W_t^b)/(\partial \bar{m}_{t-1}) = \phi_t$ and $(\partial W_t^b)/(\partial \bar{k}_{t-1}) = R_t$. Similarly, let $W_t^s(\bar{m}_t)$ denote the value function of entering the CM of period t with $\bar{m}_t$ units of money for the seller.⁸ We conjecture that $(\partial W_t^s)/(\partial \bar{m}_{t-1}) = \phi_t$.

Decentralized Market. Buyers make TIOLI offer to sellers. Therefore,

$$\begin{aligned} V_t^b(\tilde{m}_t, \tilde{k}_t, \zeta_t) &= \max_{q_t, \bar{m}_t} \left[u(q_t) + \zeta_t \beta W_{t+1}^b(\bar{m}_t, \bar{k}_t) \right], \\ \text{s.t. } q_t &\leq \beta \phi_{t+1}(\tilde{m}_t - \bar{m}_t), \\ \bar{m}_t &\geq 0, \quad q_t \geq 0, \quad \tilde{k}_t = \bar{k}_t, \end{aligned}$$

where $V_t^b(\tilde{m}_t, \tilde{k}_t, \zeta_t)$ is the value of entering the DM with $\tilde{m}_t$ units of money and $\tilde{k}_t$ units of capital, given a shock ζ_t . It can easily be shown that:

$$\begin{aligned} q_t &= \min \left\{ \beta \phi_{t+1} \bar{m}_t, u'^{-1}(\zeta_t) \right\}, \\ \tilde{m}_t - \bar{m}_t &= \min \left\{ \tilde{m}_t, u'^{-1}(\zeta_t)/(\beta \phi_{t+1}) \right\}. \end{aligned}$$

⁸We exclude capital here as the sellers will never acquire it.

It is clear that the solution depends on the realization of discount factor shock ζ_t . Also observe that:

$$\begin{aligned}\frac{\partial V_t}{\partial \tilde{m}_t} &= \beta \phi_{t+1} \max \{u'(\beta \phi_{t+1} \tilde{m}_t), \zeta_t\}, \\ \frac{\partial V_t}{\partial \tilde{k}_t} &= \zeta_t \beta R_{t+1}.\end{aligned}$$

Asset Market. Given that ϕ_t^a is the price of asset in terms of money we obtain that:

$$\begin{aligned}\Omega_t^b(m_t, k_t, \zeta_t) &= \max_{\tilde{m}_t, \tilde{k}_t} V_t(\tilde{m}_t, \tilde{k}_t, \zeta_t) \\ s.t. \quad &\phi_t^a \tilde{m}_t + \tilde{k}_t \leq \phi_t^a m_t + k_t, \\ &\tilde{m}_t \geq 0, \quad \tilde{k}_t \geq 0.\end{aligned}$$

Here, $\Omega_t^b(m_t, k_t, \zeta_t)$ is the value function of entering the AM of period t with m_t units of money and k_t units of capital, given discount factor shock ζ_t . It can be verified that:

$$\tilde{m}_t \begin{cases} = m_t + k_t / \phi_t^a & \text{if } \frac{R_{t+1} \phi_t^a}{\phi_{t+1}} < 1, \\ \in \left[\min \left\{ m_t + k_t / \phi_t^a, \frac{u'^{-1}(\zeta_t)}{\beta \phi_{t+1}} \right\}, m_t + k_t / \phi_t^a \right] & \text{if } \frac{R_{t+1} \phi_t^a}{\phi_{t+1}} = 1, \\ = \min \left\{ m_t + k_t / \phi_t^a, \frac{u'^{-1} \left(\frac{\zeta_t R_{t+1} \phi_t^a}{\phi_{t+1}} \right)}{\beta \phi_{t+1}} \right\} & \text{if } \frac{R_{t+1} \phi_t^a}{\phi_{t+1}} > 1, \end{cases}$$

and $\tilde{k}_t = k_t - \phi_t^a(\tilde{m}_t - m_t)$. Observe that when $R_{t+1} \phi_t^a = \phi_{t+1}$ the solution need not be pinned down uniquely. Nevertheless we do find:

$$\begin{aligned}\frac{\partial \Omega_t^b}{\partial m_t} &= \begin{cases} \max \{ \beta \phi_{t+1} u'(\beta \phi_{t+1} (m_t + k_t / \phi_t^a)), \beta \zeta_t \phi_{t+1} \} & \text{if } \frac{R_{t+1} \phi_t^a}{\phi_{t+1}} \leq 1, \\ \max \{ \beta \phi_{t+1} u'(\beta \phi_{t+1} (m_t + k_t / \phi_t^a)), \beta \zeta_t R_{t+1} \phi_t^a \} & \text{if } \frac{R_{t+1} \phi_t^a}{\phi_{t+1}} > 1, \end{cases} \\ \frac{\partial \Omega_t^b}{\partial k_t} &= \frac{1}{\phi_t^a} \frac{\partial \Omega_t^b}{\partial m_t}.\end{aligned}$$

Centralized Market. In this market we have:

$$\begin{aligned}W_t^b(\bar{m}_{t-1}, \bar{k}_{t-1}) &= \max_{m_t, k_t, \hat{k}_{t-1}, y_t} \left[y_t + \pi \Omega_t^b(m_t, k_t, \delta^l) + (1 - \pi) \Omega_t^b(m_t, k_t, \delta^h) \right] \\ s.t. \quad &y_t + \phi_t m_t + k_t + R_t \hat{k}_{t-1} + \tau \leq \phi_t \bar{m}_{t-1} + R_t \bar{k}_{t-1} + f(\hat{k}_{t-1}), \\ &m_t \geq 0, \quad k_t \geq 0, \quad \hat{k}_{t-1} \geq 0,\end{aligned}$$

where $\hat{k}_{t-1}$ is the amount of capital produced in the day of period $t - 1$, that is put to work by the buyer to produce $f(\hat{k}_{t-1})$ units of general good, and τ is a lump-sum tax levied upon the buyer. The

first order conditions are given by:

$$f'(\hat{k}_{t-1}) = R_t, \quad (43)$$

$$\frac{\phi_t}{\beta\phi_{t+1}} = E \max \left\{ u'(\beta\phi_{t+1}(m_t + k_t/\phi_t^a)), \frac{\zeta_t R_{t+1} \phi_t^a}{\phi_{t+1}} \right\}, \quad (44)$$

$$\phi_t = \phi_t^a. \quad (45)$$

It follows that all buyers devote the same level of capital goods to technology f .

5.3 General equilibrium

First, consider the partial equilibrium in the asset market. Without loss of generality we can focus on a situation in which all buyers enter the asset market with identical money and capital holdings. Moreover, we conjecture that buyers carry strictly positive amounts of money into the AM, which we will confirm to be true. As a result $R_{t+1}\phi_t^a \geq \phi_{t+1}$ must hold, as otherwise all buyers want to sell their capital holdings. For market clearance we need:

$$m_t \geq \pi \min \left\{ m_t + k_t/\phi_t^a, \frac{u'^{-1} \left(\frac{\delta^l R_{t+1} \phi_t^a}{\phi_{t+1}} \right)}{\beta\phi_{t+1}} \right\} + (1 - \pi) \min \left\{ m_t + k_t/\phi_t^a, \frac{u'^{-1} \left(\frac{\delta^h R_{t+1} \phi_t^a}{\phi_{t+1}} \right)}{\beta\phi_{t+1}} \right\}, \quad (46)$$

and it must hold with equality if $R_{t+1}\phi_t^a > \phi_{t+1}$. There always exists a unique $\phi_t^a \in [\phi_{t+1}/R_{t+1}, \infty)$ that clears the asset market.

In what follows we will focus on a steady-state equilibrium in which the value of money is constant over time. If the money stock grows at rate γ it follows that $\phi_t/\phi_{t+1} = \gamma$. Letting $z \equiv \phi_t m_t$ denote the real value of money carried out of the CM by the buyers, using first-order-conditions (43), (44), and (45), and asset market clearance (46), we define a steady-state equilibrium as follows

Definition 1 *A steady-state equilibrium is a pair (k, z) that solves the system:*

$$\begin{aligned} \gamma/\beta &= \pi \max \{ u'(\beta/\gamma(z+k)), \delta^l f'(k)\gamma \} + (1-\pi) \max \{ u'(\beta/\gamma(z+k)), \delta^h f'(k)\gamma \}, \\ z &\geq \pi \min \{ z+k, \gamma/\beta u'^{-1}(\delta^l f'(k)\gamma) \} + \pi \min \{ z+k, \gamma/\beta u'^{-1}(\delta^h f'(k)\gamma) \}, \\ 0 &= (f'(k)\gamma - 1) [z - \pi \min \{ z+k, \gamma/\beta u'^{-1}(\delta^l f'(k)\gamma) \} - \pi \min \{ z+k, \gamma/\beta u'^{-1}(\delta^h f'(k)\gamma) \}], \\ 1 &\leq f'(k)\gamma. \end{aligned}$$

A unique equilibrium exists for $\gamma > \beta$. This equilibrium can be of three different types:

1. *Equilibrium with plentiful capital:* In this equilibrium for all buyers the non-negativity constraint regarding capital carried out of the AM does not bind. Hence, $\beta f'(k) = 1$. This

equilibrium exists if and only if:

$$\beta/\gamma f'^{-1}(1/\beta) \geq (1-\pi) \left[u'^{-1}(\delta^l \gamma/\beta) - u'^{-1}(\delta^h \gamma/\beta) \right].$$

2. *Equilibrium with scarce capital away from ZLB:* In this equilibrium the non-negativity constraint regarding capital carried out of the AM binds for some buyers. The nominal interest rate earned by capital is however strictly positive, so: $\gamma/\beta > f'(k)\gamma > 1$. This equilibrium exists if and only if:

$$(1-\pi) \left[u'^{-1} \left(\frac{\gamma/\beta - (1-\pi)\delta^h}{\pi} \right) - u'^{-1}(\delta^h) \right] \leq \frac{\beta}{\gamma} f'^{-1} \left(\frac{1}{\gamma} \right) < (1-\pi) \left[u'^{-1} \left(\frac{\delta^l \gamma}{\beta} \right) - u'^{-1} \left(\frac{\delta^h \gamma}{\beta} \right) \right].$$

3. *Equilibrium with scarce capital at the ZLB:* In this equilibrium the non-negativity constraint regarding capital carried out of the AM binds for some buyers. The nominal interest rate earned by capital is zero, so: $f'(k)\gamma = 1$. This equilibrium exists if and only if:

$$\beta/\gamma f'^{-1}(1/\gamma) < (1-\pi) \left[u'^{-1} \left(\frac{\gamma/\beta - (1-\pi)\delta^h}{\pi} \right) - u'^{-1}(\delta^h) \right].$$

For $\gamma = \beta$, z is not pinned down uniquely but without loss of generality we can assume $z = \min \{ u'^{-1}(\delta^l) - f'^{-1}(1/\beta), \pi u'^{-1}(\delta^l) + (1-\pi)u'^{-1}(\delta^h) \}$. What matters for our result regarding optimal deviations from the Friedman rule is the following lemma:

Lemma 1 *For $\gamma = \beta$ equilibrium DM consumption levels satisfy $u'(q_i) = \zeta_i, \forall i$ and production of capital goods satisfies $\beta f'(k) = 1$, Also $R + \gamma \frac{\partial R}{\partial \gamma} \Big|_{\beta=\gamma} > 0$ if $f'^{-1}(1/\beta) > (1-\pi) [u'^{-1}(\delta^l) - u'^{-1}(\delta^h)]$.*

To support a money growth rate γ we assume that the government levies a real lump-sum tax in the day that is equal for all buyers and sellers. It follows that $2\tau_t = z(\gamma-1)/\gamma, \forall t \geq 1$ and $2\tau_0 = -z$, where the latter is the initial injection of cash into the economy in the day of period $t = 0$.⁹

5.4 Welfare and Optimal Deviation from the Friedman Rule

To construct welfare, observe that because of the TIOLI offers welfare of a seller satisfies:

$$W_t^s(0) = -\tau_t + \beta W_{t+1}^s(0).$$

Welfare of a buyers satisfies:

$$\begin{aligned} W_t^b(z/\phi_t, k_{t-1}) - f(k_{t-1}) &= -\tau_t - k - \pi u(q^l) + (1-\pi)u(q^h) + \pi\beta\delta^l \left[\phi_{t+1}\bar{m}_t^l + R(\bar{k}^l - k) \right] \\ &\quad + (1-\pi)\beta\delta^h \left[\phi_{t+1}\bar{m}_t^h + R(k^l - k) \right] \\ &\quad + \beta \left[f(k) - z + W_{t+1}^b(z/\phi_{t+1}, k) \right], \end{aligned}$$

⁹The joint mass of the buyers and sellers is two, hence 2τ instead of simply τ on the LHS of the equations.

where we index buyers that have $\zeta_{i,t} = \delta^h$ with h and buyers that have $\zeta_{i,t} = \delta^l$ with l , as period t decisions by buyers differ only because of heterogeneity in $\zeta_{i,t}$. Observe that in the steady-state equilibrium $W_t^b(z/\phi_t, k) = W_{t+1}^b(z/\phi_{t+1}, k)$, $\forall t \geq 1$.¹⁰

Additionally, using that $\phi_{t+1}(\tilde{m}_t^j - \bar{m}^j) = q^j$, $k - k^j = \phi_t(\tilde{m}_t^j - m_t)$, $2\tau_t = z(\gamma - 1)/\gamma$, $\forall t \geq 1$, $2\tau_0 = z$, and $\pi k^l + (1 - \pi)k^h = k$ we obtain:

$$(1 - \beta) \left[W_0^s(0) + W_0^b(0, k_{-1}) - f(k_{-1}) \right] = \pi[u(q^l) - \delta^l q^l] + (1 - \pi)[u(q^h) - \delta^h q^h] + \beta f(k) - k \\ + \pi(1 - \pi)(\delta^h - \delta^l)(k^h - k^l)\beta R(1 - 1/(R\gamma)),$$

It follows that $\mathcal{W} = W_0^s(0) + W_0^b(0, k_{-1}) - f(k_{-1})$ is the utilitarian welfare measure at the beginning of period 0 if the economy starts without any capital goods, i.e. $k_{-1} = 0$.¹¹

We are now in a position to formulate the following proposition:

Proposition 1 *If $f'^{-1}(1/\beta) > (1 - \pi)[u'^{-1}(\delta^l) - u'^{-1}(\delta^h)]$ and $\delta^h > \delta^l$, then it is optimal to deviate from the Friedman rule and set $\gamma > \beta$.*

Proof Consider the first-order-derivative of $\mathcal{W}$ w.r.t. γ :

$$\frac{\partial \mathcal{W}}{\partial \gamma} = \pi \delta^l \left[u'(q^l) - \delta^l \right] \frac{\partial q^l}{\partial \gamma} + (1 - \pi) \delta^h \left[u'(q^h) - \delta^h \right] \frac{\partial q^h}{\partial \gamma} + [\beta f'(k) - k] \frac{\partial k}{\partial \gamma} \\ + \pi(1 - \pi)(\delta^h - \delta^l) \left[\left(\frac{\partial k^h}{\partial \gamma} - \frac{\partial k^l}{\partial \gamma} \right) \beta R + (k^h - k^l) \beta \frac{\partial R}{\partial \gamma} \right] \left(1 - \frac{1}{R\gamma} \right) \\ + \pi(1 - \pi)(\delta^h - \delta^l) \beta R \left(\frac{1}{R\gamma} \right)^2 \left(R + \gamma \frac{\partial R}{\partial \gamma} \right).$$

We know from lemma 1 that at the Friedman rule $u'(q^h) = \delta^h$, $u'(q^l) = \delta^l$, $R\gamma = 1$, $k^h > k^l$, $\beta f'(k) = 1$ and, given that $f'^{-1}(1/\beta) > (1 - \pi)[u'^{-1}(\delta^l) - u'^{-1}(\delta^h)]$, $R + \gamma \frac{\partial R}{\partial \gamma} > 0$. Therefore, $\frac{\partial \mathcal{W}}{\partial \gamma} \Big|_{\gamma=\beta} > 0$.

First, it is important to observe that an under-investment problem exists in this economy. Let us consider what happens if we would implement the Friedman rule. Then, according to Lemma 1 production of capital goods satisfies $\beta f'(k) = 1$. From the perspective of a social planner, this level of investment is too low. The reason is that the returns from investment are realized one period later, so that it is better for welfare to give these returns only to patient agents. A social planner

¹⁰This does not hold for $t = 0$ because $\tau_0 \neq \tau_t$ for $t \neq 0$.

¹¹If $k_{-1} > 0$, this does not have consequences for the welfare effects of inflation as k_{-1} is predetermined and will thus not respond to the level of inflation determined in period $t = 0$

would therefore set $\beta\delta^h = f'(k)$, taking into account the discount factor faced by patient agents rather than the average discount factor across all agents. Because of decreasing-returns-to-scale, a social planner will then find it attractive to invest more than the market economy.

In addition, proposition 1 demonstrates that if agents have access to a financial market on which they can trade capital goods, deviating from the Friedman rule is optimal when doing so affects the nominal interest rate. The reason is that positive nominal interest rates are good for welfare in an economy with heterogeneous time preferences. Because impatient agents care less about the future, they want to spend more money in the DM. A financial market allows them to do so: they can sell assets to impatient agents in exchange for more money. Alternatively, one can see this as a form of collateralized credit, where those who want to borrow money can do so by pledging capital as collateral in the AM. This credit perspective explains why positive nominal interest rates are beneficial: they imply an ex-post transfer of resources from impatient agents to patient agents. The impatient buyers borrow from the patient buyers in the AM because the former have a stronger desire for DM consumption than the latter. In the next CM, the impatient agents will however need to pay back what they have borrowed, plus interest, to the patient agents. The impatient agents will thus have to work more than patient agents in the next CM, which is good for welfare.

Importantly, the direct benefit of implementing inflation is not that it alleviates the under-investment problem. In fact, if we assume away investment, the argument above leads to the conjecture that if AM credit would not require collateral then deviating from the Friedman rule is still optimal. Nevertheless, under-investment and non-optimality of the Friedman rule are indirectly related to each other because both are caused by heterogeneity in discount factors. We thus find that the relationship between heterogeneous discount factors, under-investment and the optimality of implementing inflation is robust. It does not only arise in an OLG setup, but also in a setup with infinitely lived agents.

6 Conclusion

We have studied the interactions between money, asset markets and capital formation efficiency in a New-Monetarist framework, which allows for both under-investment and over-investment equilibria. The benchmark model captures that while preference shocks over decentralized market consumption could generate over-investment, discount factor shocks can be associated with under-investment. Monetary policy improves welfare when it is consistently implemented based on the prevailing capital

formation inefficiency. In case of over-investment a deflationary policy is optimal, while in case of under-investment running a positive rate of inflation may be attractive. Even though the benchmark analysis relies on an OLG framework, using an extended model with infinitely-lived agents we also show that the benchmark theoretical findings are not artifacts of having assumed finitely-lived agents.

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