Tilburg University

Determinateness of strategic games with a potential

Norde, H.W.; Tijs, S.H.

Published in:
Mathematical Methods of Operations Research

Publication date:
1998

[Link to publication in Tilburg University Research Portal](#)

Citation for published version (APA):
Norde, H. W., & Tijs, S. H. (1998). Determinateness of strategic games with a potential. *Mathematical Methods of Operations Research*, 48(3), 377-385.

General rights

Copyright and moral rights for the publications made accessible in the public portal are retained by the authors and/or other copyright owners and it is a condition of accessing publications that users recognise and abide by the legal requirements associated with these rights.

- Users may download and print one copy of any publication from the public portal for the purpose of private study or research.
- You may not further distribute the material or use it for any profit-making activity or commercial gain
- You may freely distribute the URL identifying the publication in the public portal

Take down policy

If you believe that this document breaches copyright please contact us providing details, and we will remove access to the work immediately and investigate your claim.

Determinateness of strategic games with a potential*

Henk Norde, Stef Tijs

Department of Econometrics, Tilburg University, P.O. Box 90153, 5000 LE Tilburg, The Netherlands (e-mail: h.norde@kub.nl)

Received October 1997/Revised version April 1998

Abstract. Finite potential games are determined, i.e. have Nash equilibria in pure strategies. In this paper we investigate the determinateness of potential games, in which one or more players have infinitely many pure strategies.

Key words: Potential game, approximate equilibria, determinateness

1 Introduction

It is a well-known fact that strategic games, in which every player has only finitely many strategies, need not be determined, i.e. need not have Nash equilibria. However, if mixed strategies are allowed, such games are determined. For two-person matrix games this was shown by von Neumann [5] and for general n -person games by Nash [4]. For two-person games, where one player has infinitely (but countably) many pure strategies, Norde and Potters [6] proved (weak) determinateness by showing that (approximate) equilibria in mixed strategies always exist. However games, in which two or more players have infinitely many pure strategies, need not be determined, even if mixed strategies are allowed. A famous example is the following $\infty \times \infty$ -bimatrix game, given by Wald [10], with payoff matrices

$$\begin{pmatrix} 0 & -1 & -1 & \dots \\ 1 & 0 & -1 & \dots \\ 1 & 1 & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 1 & 1 & \dots \\ -1 & 0 & 1 & \dots \\ -1 & -1 & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

* We are indebted to Fioravante Patrone, Mark Voorneveld and an anonymous referee for their useful comments.

(i.e. the two-person game where both players choose a natural number and the player, choosing the smallest number, pays one dollar to the other player).

In [3] Monderer and Shapley introduced potential games, i.e. strategic games with a potential function. In fact, such functions were first used by Rosenthal [7] and the idea of a potential function is also present in [8]. Potential games have the nice property that they possess Nash equilibria in pure strategies, if the strategy spaces are finite. So, for these games, mixing strategies is not necessary in order to get determinateness. However, if one or more players have infinitely many pure strategies, (approximate) equilibria in pure strategies need not exist. The main aim of this paper is to provide sufficient conditions for certain classes of potential games, which guarantee (weak) determinateness.

In section 2 some definitions, concerning potential games and the concept of weak determinateness, are provided. Moreover, a decomposition theorem for potential games is given. Weak determinateness for a class of potential games, in which one player has a large action space, is shown in section 3. In section 4 we deal with $\infty \times \infty$ -bimatrix games. For these games a remarkable phenomenon occurs: there are two $\infty \times \infty$ -bimatrix games, which have the same potential function, whereas one is weakly determined and the other is not. A characterization of the potentials, which are such that *any* game with this potential is weakly determined, is provided. Concluding remarks are presented in section 5.

Notation. For a strategic n -person game $\langle N, (X_i)_{i \in N}, (u_i)_{i \in N} \rangle$ with player set $N = \{1, \dots, n\}$ the set $X := \prod_{i \in N} X_i$ denotes the set of all strategy-tuples and, for every $i \in N$, the set X_{-i} is defined by $X_{-i} := \prod_{j \neq i} X_j$. For a strategy-tuple $x_{-i} \in X_{-i}$ and a strategy $x_i \in X_i$ the vector $x = (x_{-i}, x_i) \in X$ denotes the strategy-tuple, in which player i plays strategy x_i and the other players play according to x_{-i} .

2 Potential games and determinateness

A strategic game $\langle N, (X_i)_{i \in N}, (u_i)_{i \in N} \rangle$ is called a *potential game* if there is a *potential* $P : X \rightarrow \mathbb{R}$ such that $\forall i \in N, \forall x_{-i} \in X_{-i}, \forall x_i, x'_i \in X_i$ we have

$$u_i(x_{-i}, x_i) - u_i(x_{-i}, x'_i) = P(x_{-i}, x_i) - P(x_{-i}, x'_i).$$

So, in potential games the change in payoff for a unilaterally deviating player is measured by the potential P .

Special classes of potential games are the class of *coordination games* and the class of *dummy games*. A game $\langle N, (X_i)_{i \in N}, (u_i)_{i \in N} \rangle$ is a *coordination game* if there is a $u : X \rightarrow \mathbb{R}$ such that $u_i = u$ for every $i \in N$ (every player has the same payoff function, which evidently is a potential for this game). A game $\langle N, (X_i)_{i \in N}, (u_i)_{i \in N} \rangle$ is a *dummy game* if $\forall i \in N, \forall x_{-i} \in X_{-i}, \forall x_i, x'_i \in X_i$ we have $u_i(x_{-i}, x_i) - u_i(x_{-i}, x'_i) = 0$. Clearly, the zero function is a potential for dummy games. In [1] Facchini, van Megen, Borm and Tijs proved the following proposition, which states that every potential game is the sum of a coordination game and a dummy game. A similar result can be found in Slade [8].

Proposition 1. A game $\Gamma = \langle N, (X_i)_{i \in N}, (u_i)_{i \in N} \rangle$ is a potential game iff

$$u_i = c_i + d_i \quad \text{for every } i \in N,$$

where the functions $c_i, d_i, i \in N$, are such that $\langle N, (X_i)_{i \in N}, (c_i)_{i \in N} \rangle$ is a coordination game and $\langle N, (X_i)_{i \in N}, (d_i)_{i \in N} \rangle$ is a dummy game.

Proof. For the if-part it is sufficient to observe that the sum of two potential games is again a potential game. For the only-if part we see that the functions $c_i = P, d_i = u_i - P, i \in N$, where P is a potential of Γ , satisfy the required properties. ■

Let $\varepsilon > 0, k \in \mathbb{R}$. A strategy $x_i \in X_i$ of player i is called an ε -best response to $x_{-i} \in X_{-i}$ if

$$u_i(x_{-i}, x_i) \geq \sup_{x'_i \in X_i} u_i(x_{-i}, x'_i) - \varepsilon$$

and a k -guaranteeing response to x_{-i} if

$$u_i(x_{-i}, x_i) \geq k.$$

If x_i is an ε -best response or a k -guaranteeing response (or both) to x_{-i} then x_i is called an (ε, k) -best response to x_{-i} . Note that the set of ε -best responses to x_{-i} as well as the set of k -guaranteeing responses to x_{-i} may be empty, but that the set of (ε, k) -best responses is always non-empty. A strategy-tuple $x = (x_1, \dots, x_n) \in X$ is called an ε -equilibrium of Γ if x_i is an ε -best response to $x_{-i} := (x_j)_{j \in N, j \neq i}$ for every $i \in N$, a k -equilibrium if x_i is a k -guaranteeing response to x_{-i} for every $i \in N$, and an (ε, k) -equilibrium if x_i is an (ε, k) -best response to x_{-i} for every $i \in N$. So in an (ε, k) -equilibrium every player is reasonably satisfied, since he either receives a (large) amount of at least k or he can gain no more than ε (a small amount) by deviating unilaterally. Clearly, a Nash equilibrium of a strategic game is an ε -equilibrium for every $\varepsilon > 0$. However, the following example shows that the set of Nash equilibria may be empty. For such games ε -equilibria, k -equilibria and (ε, k) -equilibria become important.

Example. The $1 \times \infty$ -bimatrix game (A_1, B_1) with

$$A_1 = [0 \ 0 \ 0 \ 0 \ \dots] \quad \text{and} \quad B_1 = [\frac{1}{2} \ \frac{2}{3} \ \frac{3}{4} \ \frac{4}{5} \ \dots]$$

has an ε -equilibrium for every $\varepsilon > 0$, the $1 \times \infty$ -bimatrix game (A_2, B_2) with

$$A_2 = [0 \ 0 \ 0 \ 0 \ \dots] \quad \text{and} \quad B_2 = [1 \ 2 \ 3 \ 4 \ \dots]$$

has a k -equilibrium for every $k \in \mathbb{R}$ and the $\infty \times \infty$ -bimatrix game (A_3, B_3) with

$$A_3 = \begin{pmatrix} 0 & 0 & 0 & \dots \\ 1 & 1 & 1 & \dots \\ 2 & 2 & 2 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \quad \text{and} \quad B_3 = \begin{pmatrix} \frac{1}{2} & \frac{2}{3} & \frac{3}{4} & \dots \\ \frac{1}{2} & \frac{2}{3} & \frac{3}{4} & \dots \\ \frac{1}{2} & \frac{2}{3} & \frac{3}{4} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

has an (ε, k) -equilibrium for every $\varepsilon > 0$ and $k \in \mathbb{R}$. Obviously, all these games are potential games without Nash equilibria.

The set of ε -equilibria, k -equilibria, and (ε, k) -equilibria of Γ will be denoted by $E^\varepsilon(\Gamma)$, $E^k(\Gamma)$, and $E^{(\varepsilon, k)}(\Gamma)$ respectively. Clearly, $E^\varepsilon(\Gamma) \subset E^{(\varepsilon, k)}(\Gamma)$ and $E^k(\Gamma) \subset E^{(\varepsilon, k)}(\Gamma)$. The game Γ is called *weakly determined* if

$$E^{(\varepsilon, k)}(\Gamma) \neq \emptyset$$

for every $\varepsilon > 0$, $k \in \mathbb{R}$. For two-person games, this definition of weak determinateness coincides with the one given by Lucchetti, Patrone and Tijs in [2].

The following theorem shows that any potential game with an upper bounded potential is weakly determined. In fact, such games have ε -equilibria for every $\varepsilon > 0$. This result is very similar to Lemma 4.1 in Monderer and Shapley [3], where the weak determinateness of potential games with bounded payoff functions is shown.

Theorem 1. *Let $\Gamma = \langle N, (X_i)_{i \in N}, (u_i)_{i \in N} \rangle$ be a potential game with an upper bounded potential P . Then Γ is weakly determined.*

Proof. Let $\varepsilon > 0$. Choose $\hat{x} \in X$ such that $P(\hat{x}) > \sup_{x \in X} P(x) - \varepsilon$. Then $\hat{x} \in E^\varepsilon(\Gamma)$. ■

The following example shows that potential games, having a potential which is not upper bounded, need not be weakly determined.

Example. Let Γ be the $\infty \times \infty$ -bimatrix game (i.e. a game with two players and strategy space $\mathbb{N}$ for both players) with payoff functions given by

$$u_1(i, j) = \begin{cases} 0 & \text{if } j = i \\ -1 & \text{if } j = i + 1 \\ -2j + 1 & \text{elsewhere} \end{cases}$$

and

$$u_2(i, j) = \begin{cases} -1 & \text{if } j = i \\ 0 & \text{if } j = i + 1 \\ -2i & \text{elsewhere} \end{cases}$$

This game is a potential game with potential

$$P(i, j) = \begin{cases} 2i - 1 & \text{if } j = i \\ 2i & \text{if } j = i + 1 \\ 0 & \text{elsewhere} \end{cases}$$

Note that the payoff functions u_1 and u_2 are upper bounded but that the potential P is not upper bounded. Let $\varepsilon \in (0, 1)$, $k > 0$. If $(i_0, j_0) \in E^{(\varepsilon, k)}(\Gamma)$ then j_0 is an ε -best response to i_0 since $u_2(i_0, j_0) \leq 0$ and i_0 is an ε -best response to j_0 since $u_1(i_0, j_0) \leq 0$. Therefore $j_0 = i_0 + 1$ and $i_0 = j_0$ which yields a con-

tradiction. So $E^{(\varepsilon, k)}(\Gamma) = \emptyset$ for every $\varepsilon \in (0, 1)$ and $k > 0$ and hence Γ is not weakly determined.

3 Potential games with one player having a large action space

The determinateness of potential games, where all players have compact action spaces and continuous payoff functions, is shown in Lemma 4.3 of Monderer and Shapley [3]. In the following theorem we show that potential games, where at most one player has a non-compact action space and the payoff functions satisfy some continuity properties, are still weakly determined.

Theorem 2. *Let $\Gamma = \langle N, (X_i)_{i \in N}, (u_i)_{i \in N} \rangle$ be a potential game with potential P . Suppose that $X_1, \dots, X_{n-1}$ are compact topological spaces. Suppose moreover that $x_{-n} \mapsto u_i(x_{-n}, x_n)$ is continuous for all $i \in N \setminus \{n\}$, $x_n \in X_n$ and that $x_{-n} \mapsto u_n(x_{-n}, x_n)$ is lower semi-continuous for every $x_n \in X_n$. Then Γ is weakly determined.*

Proof. According to Theorem 1 it is sufficient to concentrate on a potential P , which is not upper bounded. Let $x_n \in X_n$ and choose $(y_1, \dots, y_{n-1}) \in X_{-n}$ arbitrarily. Then

$$\begin{aligned} P(x_1, \dots, x_n) &= u_1(x_1, x_2, \dots, x_n) - u_1(y_1, x_2, \dots, x_n) \\ &\quad + u_2(y_1, x_2, \dots, x_n) - u_2(y_1, y_2, \dots, x_n) \\ &\quad \vdots \\ &\quad + u_{n-1}(y_1, y_2, \dots, x_{n-1}, x_n) - u_{n-1}(y_1, y_2, \dots, y_{n-1}, x_n) \\ &\quad + P(y_1, \dots, y_{n-1}, x_n) \end{aligned}$$

for every $(x_1, \dots, x_{n-1}) \in X_{-n}$, which shows that $x_{-n} \mapsto P(x_{-n}, x_n)$ is continuous. Let $k \in \mathbb{R}$ and define $d_n := u_n - P$. Then $x_{-n} \mapsto d_n(x_{-n}, x_n)$ is lower semi-continuous for every $x_n \in X_n$. Since, moreover, d_n is constant in the n -th coordinate, we may define $l := \min_{x \in X} d_n(x)$. Choose $x' = (x'_1, \dots, x'_n) \in X$ such that $P(x') \geq k - l$. Since $x_{-n} \mapsto P(x_{-n}, x'_n)$ is continuous and X_{-n} compact we may choose $x''_{-n} \in X_{-n}$ such that $P(x''_{-n}, x'_n) = \max_{x_{-n} \in X_{-n}} P(x_{-n}, x'_n)$. Let $\hat{x} := (x''_{-n}, x'_n)$. Since every player $i \in N$, $i \neq n$ cannot improve at all upon $\hat{x}$ and $u_n(\hat{x}) = P(\hat{x}) + d_n(\hat{x}) \geq P(x') + l \geq k$ we have $\hat{x} \in E^{(\varepsilon, k)}(\Gamma)$ for every $\varepsilon > 0$. ■

If we consider potential games, in which all but one player have a finite action space, and if we equip these finite spaces with the discrete topology, the following result is an immediate consequence of Theorem 2.

Corollary 1. *Let $\Gamma = \langle N, (X_i)_{i \in N}, (u_i)_{i \in N} \rangle$ be a potential game with potential P . Suppose $X_1, \dots, X_{n-1}$ are finite sets. Then Γ is weakly determined.*

Compare the above results with Theorems 4.1 and 4.2 in [9], where Tijs proved, under some mild assumptions, the weak determinateness in mixed

strategies of games, in which all but one player have finite and compact metric action spaces, respectively.

4 Potential games with two players having a large action space

In this section we consider potential games where two players have a large (but countable) action space. In the following examples a remarkable phenomenon occurs: two games are presented, which have the same potential function, whereas one is weakly determined and the other is not.

Example. Let Γ_1 be the $\infty \times \infty$ -bimatrix game with payoff functions given by $u_1(i, j) = u_2(i, j) = i + j$, $i, j \in \mathbb{N}$. Clearly, this coordination game is a potential game with potential $P(i, j) = i + j$, $i, j \in \mathbb{N}$, and $(k, 1) \in E^k(\Gamma)$ for every $k \in \mathbb{N}$. So Γ_1 is weakly determined.

Example. Let Γ_2 be the (zero-sum) $\infty \times \infty$ -bimatrix game with payoff functions given by $u_1(i, j) = i - j$, $u_2(i, j) = j - i$, $i, j \in \mathbb{N}$ (the player choosing the highest natural number wins the difference with the other player, compare with the Wald-example in the introduction). Clearly, this game is a potential game with potential $P(i, j) = i + j$, $i, j \in \mathbb{N}$ which is not weakly determined.

Theorem 1 states that potential games with an upper bounded potential are weakly determined. The assumption, that the potential P should be upper bounded, can be weakened a little bit. In fact, it is sufficient to assume that the corresponding coordination game, with payoff function P for every player, has an ε -equilibrium for every $\varepsilon > 0$. In other words, if a coordination game has ε -equilibria for every $\varepsilon > 0$ this property is inherited by every potential game with the same potential. The above examples showed however, that for a coordination game which has k -equilibria for every k , this property need not be inherited by every potential game with the same potential. A natural question now is the following: which potential functions P are such that every game with potential P is weakly determined? For $\infty \times \infty$ -bimatrix games we give a complete answer to this question. In order to do so we need the following definition.

Definition. A (potential) function $P : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{R}$ is **determined** if for every $\varepsilon > 0$ there is an $i \in \mathbb{N}$ with

$$\sup_{j \in \mathbb{N} : i \in B_1^\varepsilon(j)} P(i, j) \geq \sup_{j \in \mathbb{N}} P(i, j) - \varepsilon, \quad (1)$$

or there is a $j \in \mathbb{N}$ with

$$\sup_{i \in \mathbb{N} : j \in B_2^\varepsilon(i)} P(i, j) \geq \sup_{i \in \mathbb{N}} P(i, j) - \varepsilon, \quad (2)$$

or both, where $B_1^\varepsilon(j)$ denotes the set of ε -best responses of player 1 to strategy j and $B_2^\varepsilon(i)$ the set of ε -best responses of player 2 to strategy i (here we use the convention that $\sup S = -\infty$ if $S = \emptyset$ and $\sup S = +\infty$ if S is not upper bounded). Here, B_1^ε and B_2^ε are the best response multifunctions corresponding to the coordination game with payoff function P for both players.

If player 1 plays a strategy $i \in \mathbb{N}$, satisfying (1), then it is not a great disadvantage for player 2, if he is restricted to choose a strategy j for which strategy i is an ε -best response. If a potential P is determined, then the corresponding coordination game has, for every $\varepsilon > 0$ and $k \in \mathbb{R}$, an (ε, k) -equilibrium in which at least one of the players chooses an ε -best response.

Example. Let $P : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{R}$ be defined by $P(i, j) := i - \frac{1}{j+1}$, $i, j \in \mathbb{N}$ and let $\varepsilon > 0$. For every $j \in \mathbb{N}$ we have $B_1^\varepsilon(j) = \emptyset$ and for every $i \in \mathbb{N}$ we have $B_2^\varepsilon(i) = \{j \in \mathbb{N} : j \geq \varepsilon^{-1} - 1\}$. Therefore, there is no $i \in \mathbb{N}$ which satisfies (1), but any $j \in \mathbb{N}$ with $j \geq \varepsilon^{-1} - 1$ satisfies (2). So P is determined.

Example. Let $P : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{R}$ be defined by $P(i, j) := i - \frac{i}{j+1}$, $i, j \in \mathbb{N}$ and let $\varepsilon > 0$. For every $j \in \mathbb{N}$ we have, again, $B_1^\varepsilon(j) = \emptyset$ and for every $i \in \mathbb{N}$ we get $B_2^\varepsilon(i) = \{j \in \mathbb{N} : j \geq i\varepsilon^{-1} - 1\}$. Clearly, there is no $i \in \mathbb{N}$ satisfying (1). For every $j \in \mathbb{N}$ we have $\{i \in \mathbb{N} : j \in B_2^\varepsilon(i)\} = \{i \in \mathbb{N} : i \leq \varepsilon(j+1)\}$, which is a finite (maybe empty) set and therefore,

$$\sup_{i \in \mathbb{N} : j \in B_2^\varepsilon(i)} P(i, j) \in [-\infty, +\infty).$$

On the other hand,

$$\sup_{i \in \mathbb{N}} P(i, j) = +\infty$$

for every $j \in \mathbb{N}$. So, there is no $j \in \mathbb{N}$, which satisfies (2), and hence P is not determined.

Note that the potential in the example of section 2 is not determined either. The following theorem characterizes all potentials P which are such that every $\infty \times \infty$ -bimatrix game with potential P is weakly determined.

Theorem 3. *Let $P : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{R}$ be a (potential) function.*

- a) *If P is determined then every potential $\infty \times \infty$ -bimatrix game with potential P is weakly determined.*
- b) *If P is not determined there is a potential game $\infty \times \infty$ -bimatrix game with potential P which is not weakly determined.*

Proof. a) Let Γ be a potential $\infty \times \infty$ -bimatrix game with determined potential P and let $\varepsilon > 0$, $k \in \mathbb{R}$. Suppose there is an $i \in \mathbb{N}$ such that

$$\sup_{j' \in \mathbb{N} : i \in B_1^{\varepsilon/2}(j')} P(i, j') \geq \sup_{j' \in \mathbb{N}} P(i, j') - \varepsilon/2.$$

If $\sup_{j' \in \mathbb{N}} P(i, j') = +\infty$ then $\sup_{j' \in \mathbb{N} : i \in B_1^{\varepsilon/2}(j')} P(i, j') = +\infty$ and therefore $\sup_{j' \in \mathbb{N} : i \in B_1^{\varepsilon/2}(j')} u_2(i, j') = +\infty$. Now choose j such that $i \in B_1^{\varepsilon/2}(j)$ and $u_2(i, j) \geq k$. Then $(i, j) \in E^{(\varepsilon, k)}(\Gamma)$. If $\sup_{j' \in \mathbb{N}} P(i, j') \in \mathbb{R}$ then choose j such

that $i \in B_1^{\varepsilon/2}(j)$ and $P(i, j) \geq \sup_{j' \in \mathbb{N}} P(i, j') - \varepsilon$. Then $(i, j) \in E^{(\varepsilon, k)}(\Gamma)$. If there is a $j \in \mathbb{N}$ such that

$$\sup_{i' \in \mathbb{N}: j \in B_2^{\varepsilon/2}(i')} P(i', j) \geq \sup_{i' \in \mathbb{N}} P(i', j) - \varepsilon/2,$$

the proof is analogous.

b) Suppose P is not determined. Then there is an $\varepsilon_0 > 0$ such that

$$\sup_{j \in \mathbb{N}: i \in B_1^{\varepsilon_0}(j)} P(i, j) < \sup_{j \in \mathbb{N}} P(i, j) - \varepsilon_0$$

for every $i \in \mathbb{N}$ and

$$\sup_{i \in \mathbb{N}: j \in B_2^{\varepsilon_0}(i)} P(i, j) < \sup_{i \in \mathbb{N}} P(i, j) - \varepsilon_0$$

for every $j \in \mathbb{N}$. Now define the dummy payoff functions $d_1, d_2 : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{R}$ in the following way:

$$d_1(i, j) := -\max\{P(1, j), \dots, P(j, j), \sup_{i' \in \mathbb{N}: j \in B_2^{\varepsilon_0}(i')} P(i', j)\},$$

$$d_2(i, j) := -\max\{P(i, 1), \dots, P(i, i), \sup_{j' \in \mathbb{N}: i \in B_1^{\varepsilon_0}(j')} P(i, j')\}.$$

Note that d_1 is constant in i and that d_2 is constant in j . Let Γ be the $\infty \times \infty$ -bimatrix game with payoff functions $u_1 := P + d_1$, $u_2 := P + d_2$. Since $u_1(i, j) \leq 0$ for $i \leq j$ and $u_2(i, j) \leq 0$ for $i \geq j$ this game has no k -equilibria for $k > 0$. If $\varepsilon \in (0, \varepsilon_0]$ and strategy i of player 1 is an ε -best (and therefore ε_0 -best) response to strategy j of player 2 then j is not an ε_0 -best (and therefore not an ε -best) response to i . So the game Γ has no ε -equilibria for every $\varepsilon \in (0, \varepsilon_0]$ either. Moreover, if $\varepsilon \in (0, \varepsilon_0]$ and strategy i of player 1 is an ε -best (and therefore ε_0 -best) response to strategy j of player 2, then, since $u_2(i, j) \leq 0$, j is not a k -guaranteeing response to i . Since this last statement remains true if we interchange the roles of players 1 and 2 we have $E^{(\varepsilon, k)}(\Gamma) = \emptyset$ for every $\varepsilon \in (0, \varepsilon_0]$, $k > 0$. Therefore Γ is not weakly determined. ■

5 Concluding remarks

There is a parallel between (weak) determinateness in mixed strategies for general games and (weak) determinateness in pure strategies for potential games. In fact, finite games are determined in mixed strategies, whereas finite potential games are determined in pure strategies. For games, in which one player has an infinite (countable) action space, weak determinateness is known to be true if the number of players is two. The existence of approximate equilibria for such games with more than two players is still an open problem. Potential games, in which one player has an infinite (countable) action space, are always weakly determined. For games with two (or more) players having a

large action space the example of Wald in the introduction and the Wald-like potential game in section 4 are games which are not weakly determined in mixed and pure strategies respectively. So, roughly speaking, (weak) determinateness (in mixed strategies for general games and in pure strategies for potential games) is true for games in which at most one player has an infinite (countable) action space.

References

- [1] Facchini G, van Megen F, Borm P, Tijs S (1995) Congestion models and weighted Bayesian potential games. Research Memorandum FEW 689, Tilburg University, The Netherlands
- [2] Lucchetti R, Patrone F, Tijs SH (1986) Determinateness of two-person games. *Bollettino U.M.I.* 6:907–924
- [3] Monderer D, Shapley LS (1996) Potential games. *Games and Economic Behavior* 14:124–143
- [4] Nash JF (1951) Non-cooperative games. *Annals of Mathematics* 54:286–295
- [5] von Neumann J (1928) Zur Theorie der Gesellschaftsspiele. *Mathematische Annalen* 100:295–320
- [6] Norde H, Potters J (1997) On the determinateness of $m \times \infty$ -bimatrix games. *Mathematics of Operations Research* 22(3):631–638
- [7] Rosenthal RW (1973) A class of games possessing pure-strategy Nash equilibria. *International Journal of Game Theory* 2:65–67
- [8] Slade ME (1994) What does an oligopoly maximize? *The Journal of Industrial Economics* 42:45–61
- [9] Tijs SH (1981) Nash-equilibria for non-cooperative n -person games in normal form. *SIAM Review* 23:225–237
- [10] Wald A (1945) Generalization of a theorem by von Neumann concerning zero sum two-person games. *Annals of Mathematics* 46:281–286