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UNILATERAL SUPPORT EQUILIBRIA

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Abstract

The concept of Berge equilibria is based on supportive behavior among the players: each player is supported by the group of all other players. In this paper, we extend this concept by maintaining the idea of supportive behavior among the players, but eliminating the underlying coordination issues. We suggest to consider individual support rather than group support. The main idea is to introduce support relations, modeled by derangements. In a derangement, every player supports exactly one other player and every player is supported by exactly one other player. Subsequently, we define a new equilibrium concept, called a unilateral support equilibrium, which is unilaterally supportive with respect to every possible derangement.

We show that a unilateral support equilibrium can be characterized in terms of pay-off functions so that every player is supported by every other player individually. Moreover, it is shown that every Berge equilibrium is also a unilateral support equilibrium and we provide an example in which there is no Berge equilibrium, while the set of unilateral support equilibria is non-empty. Finally, the relation between the set of unilateral support equilibria and the set of Nash equilibria is explored.

Keywords: mutual support equilibria, Berge equilibria, unilateral support equilibria

JEL classification: C72

1 Introduction

Non-cooperative game theory deals with situations of interaction and conflict between individuals (or players). Within the field of non-cooperative game theory, special attention goes to providing equilibrium concepts to somehow solve these conflicts. Such a concept should answer the question how to determine a good strategy if you do not know the strategies of the other players beforehand. The standard equilibrium concept is the one defined by Nash (1951). His idea, resulting in a Nash equilibrium, is based on the fact that no player should have an incentive to unilaterally deviate from the equilibrium strategy. For every player, it should hold that playing the Nash equilibrium strategy is the best thing (that is, resulting in the highest pay-off) this player could do if all the other players stick to their equilibrium strategies. In other words, each player's pay-off is maximized given the Nash equilibrium strategy combination of the other players.

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In this paper, we further explore the idea of Berge (1957), in which players are not maximizing their own pay-offs, but instead maximize the other players' pay-offs instead. More precisely, in a Berge equilibrium, every player's pay-off is maximized by the group of all other players, that is, every player is supported by all other players together. Therefore, a Berge equilibrium is also called a mutual support equilibrium (cf. Colman, Körner, Musy, and Tazdaït, 2011).

The history of the concept of a Berge equilibrium is remarkable. It has started with the publication of a book, written by Berge (1957). However, this book had only minor impact at that time. According to the view of Courtois, Nessah, and Tazdaït (2015), this was due to several reasons: the book was published in French. Furthermore, the book was written from a mathematical point of view and in particular, from a graph theoretical and topological perspective. To quote Shubik (1961):

“The argument is presented in a highly abstract manner and no consideration is given to applications to economics. (...) the book will be of little direct interest to most economists.”

Despite the negative review of Shubik (1961), the book of Berge was translated into Russian. It was Zhukovskii (1985) who reconsidered the concept and named it a Berge equilibrium. At first, the focus was to find existence theorems for a very general setting of non-cooperative games, but later on, the research also included results for more specific classes of games (in particular for mixed extensions of finite games) and related problems. We mention the papers of Radjef (1988), Abalo and Kostreva (2004), Nessah, Larbani, and Tazdaït (2007) and Larbani and Nessah (2008) for existence theorems, and Colman et al. (2011), Musy, Pottier, and Tazdaït (2012), Corley and Kwain (2014) and Corley (2015), for research on mixed extensions of finite games. This paper builds on the ideas and concepts in the spirit of the last four papers.

In particular, this paper focuses on the essence of the concept of a Berge equilibrium, which is the supportive behavior. In a Berge equilibrium, every player is supported by the group of all other players together and in that sense is reflecting the idea of mutual support. According to both Larbani and Nessah (2008) and Corley (2015), this can be seen as the idea of ‘one for all, and all for one’. Indeed, every player supports (as part of a larger group) every other player and all other players support every single player. To quote Colman et al. (2011):

“A Berge equilibrium can be viewed as an implication of the altruistic social value orientation of interdependence theory, just as Nash equilibrium is an implication of the individualistic orientation.”

An interesting research area is to study the combination of both equilibrium concepts. This was first explored by Abalo and Kostreva (1996), who studied the existence of Berge equilibria which are also Nash equilibria. Courtois, Nessah, and Tazdaït (2017) continued this line of research. Furthermore, both Colman et al. (2011) and Corley (2015) tried to find a link between the set of Berge equilibria and the set of Nash equilibria. They proved that any Berge equilibrium is also a Nash equilibrium of a game where the pay-offs are interchanged in a suitable way.

However, in order to fully understand the combination of both a Nash equilibrium and a Berge equilibrium, one has to have a better idea of the supportive behavior underlying the set of Berge equilibria. This paper aims to fill that gap and provides more insight into the idea of supportive behavior. More precisely, a new equilibrium concept for non-cooperative games is introduced, which closely relates to a link between the set of Berge equilibria and the set of Nash equilibria as studied by Colman et al. (2011) and Corley (2015).

One of the key factors of a Berge equilibrium is that the group of all players except for one single player has to coordinate their actions in order to support the single player in the best way possible. To avoid this rather complex coordination issue, we suggest to consider individual support rather than group support. The main idea is to introduce support relations among the individual players, which can be modeled by using a special type of bijections, called derangements. The interpretation of such a derangement is that every player supports exactly one other player and every player is supported by exactly one other player. Subsequently, we define a new equilibrium concept, called a unilateral support equilibrium, which is unilaterally supportive with respect to every possible derangement.

It is shown that it is sufficient to only consider cyclic derangements, a special type of derangements: every strategy combination that is unilaterally supportive with respect to every possible cyclic derangement is a unilateral support equilibrium. Moreover, we show that a unilateral support equilibrium can be characterized in terms of pay-off functions so that every player is supported by every other player individually, whereas in a Berge equilibrium, every player is supported by the group of all other players. It is proven that every Berge equilibrium is also a unilateral support equilibrium. In that sense, this new equilibrium concept extends the concept of Berge equilibria. For any two-person game, the set of unilateral support equilibria coincides with the set of Berge equilibria. Moreover, we provide an example in which there is no Berge equilibrium, while the set of unilateral support equilibria is non-empty.

Furthermore, we explore the relation between the set of unilateral support equilibria and the set of Nash equilibria. We show that the intersection between the set of unilateral support equilibria and the set of Nash equilibria coincides with the intersection of the sets of unilaterally competitive strategy combinations with respect to every possible bijection. Moreover, we reformulate this statement in terms of coordination games. In a coordination game, every player faces the same pay-off function, which results in the fact that the set of unilateral support equilibria coincides with the set of Nash equilibria. All together, more insight into the diversity of supportive behavior and in the link between unilateral support equilibria, Berge equilibria and Nash equilibria is acquired.

This paper is structured in the following way. Section 2 contains the preliminaries on non-cooperative game theory and the two equilibrium concepts, Nash equilibria and Berge equilibria. Section 3 introduces and analyzes the set of unilateral support equilibria. In particular, it contains a characterization of unilateral support equilibria in terms of pay-off functions and the example in which there is a unilateral support equilibrium, while there are no Berge equilibria. Section 4 provides some further results, mainly with respect to unilateral support equilibria in relation with Nash equilibria.

2 Preliminaries

A *non-cooperative game* is a triple $G = (N, (X_i)_{i \in N}, (\pi_i)_{i \in N})$, where N is a non-empty, finite set of players, with $|N| \geq 2$, X_i the set of strategies for player $i \in N$ and $\pi_i : X \rightarrow \mathbb{R}$ the pay-off function of player $i \in N$. Here, X is defined as the product of all sets of strategies, $X = \prod_{i \in N} X_i$, and is called the *set of strategy combinations*. A non-cooperative game $G = (N, (X_i)_{i \in N}, (\pi_i)_{i \in N})$ is called *finite* if $|X_i| < \infty$ for every $i \in N$. Some important notations regarding the (sets of) strategy combinations: a strategy combination $x \in X$ is sometimes written as $x = (x_{-i}, x_i) = (x_i, x_{-i})$ for a certain $i \in N$, where $x_{-i} = (x_j)_{j \in N \setminus \{i\}} \in X_{-i}$, with $X_{-i} = \prod_{j \in N \setminus \{i\}} X_j$ denoting the set of strategy combinations of the players in $N \setminus \{i\}$.

For a non-cooperative game $G = (N, (X_i)_{i \in N}, (\pi_i)_{i \in N})$, a strategy combination $x^* \in X$ is called a *Nash equilibrium* if for every $i \in N$ it holds that $\pi_i(x_{-i}^*, x_i) \leq \pi_i(x_{-i}^*, x_i^*)$ for every $x_i \in X_i$ (Nash, 1950). The set of Nash equilibria for G is denoted by $NE(G)$. A Nash equilibrium is thus a strategy combination in which every player maximizes his own pay-off if the other players stick to their equilibrium strategies. This can be translated into the notion of a best reply strategy: for a non-cooperative game $G = (N, (X_i)_{i \in N}, (\pi_i)_{i \in N})$, a player $i \in N$ and a strategy combination $x_{-i} \in X_{-i}$, define the *set of best reply strategies against* x_{-i} , $BR_i(x_{-i})$, as follows:

$$BR_i(x_{-i}) = \{x_i \in X_i \mid \pi_i(x_{-i}, x'_i) \leq \pi_i(x_{-i}, x_i) \text{ for every } x'_i \in X_i\}.$$

Clearly, $x^* \in NE(G)$ if and only if $x_i^* \in BR_i(x_{-i}^*)$ for every $i \in N$.

For a non-cooperative game $G = (N, (X_i)_{i \in N}, (\pi_i)_{i \in N})$, a strategy combination $x^* \in X$ is called a *Berge equilibrium* if for every $i \in N$ it holds that $\pi_i(x_i^*, x_{-i}) \leq \pi_i(x_i^*, x_{-i}^*)$ for every $x_{-i} \in X_{-i}$ (Berge, 1957). The set of Berge equilibria for G is denoted by $BE(G)$. A Berge equilibrium is thus a strategy combination in which every group of all players except for one single player supports this single player in the best way possible. In other words, every player is supported by the group of all other players. This can be translated into the notion of a best support strategy combination (cf. Musy et al., 2012): for a non-cooperative game $G = (N, (X_i)_{i \in N}, (\pi_i)_{i \in N})$, a player $i \in N$ and a strategy $x_i \in X_i$, define the *set of best support strategy combinations against* x_i , $BS_i(x_i)$, as follows:

$$BS_i(x_i) = \{x_{-i} \in X_{-i} \mid \pi_i(x_i, x'_{-i}) \leq \pi_i(x_i, x_{-i}) \text{ for every } x'_{-i} \in X_{-i}\}.$$

Clearly, $x^* \in BE(G)$ if and only if $x_{-i}^* \in BS_i(x_i^*)$ for every $i \in N$.

For Nash equilibria, existence can be proven under several (weak) conditions, see for example the existence theorem in Rosen (1965). That theorem implies existence of Nash equilibria in several interesting classes of games, e.g. the classes of bimatrix or trimatrix games. Although there are several existence theorems known for Berge equilibria (Larbani and Nessah (2008), among others), these all include more complicated conditions than the ones for Nash equilibria. In fact, Corley (2015) provided an example of a trimatrix game in which there does not exist a Berge equilibrium.

Example 2.1 Consider the following game, proposed by Corley (2015):

$$G = \begin{array}{cc} & \begin{array}{cc} f_1 & f_2 \end{array} \\ \begin{array}{c} e_1 \\ e_2 \end{array} & \left[\begin{array}{cc|cc} (1, 1, 0) & (0, 0, 0) & (0, 0, 1) & (0, 0, 0) \\ (0, 0, 0) & (0, 0, 1) & (0, 0, 0) & (1, 1, 0) \end{array} \right] \\ & \begin{array}{cc} g_1 & g_2 \end{array} \end{array}.$$

Here, G is represented as a trimatrix game, defined as a three-person game in which the strategies for player 1 consist of probability distributions among the two pure strategies e_1 and e_2 , the strategies for player 2 consist of probability distributions among f_1 and f_2 and the strategies for player 3 distributions among g_1 and g_2 . The pay-off functions are defined as the expected pay-offs with respect to the pure strategies as given in the table above. Here, e.g., when (e_2, f_2, g_1) is played, players 1, 2 and 3 receive a pay-off of 0, 0 and 1, respectively.

It can be shown that $BE(G) = \emptyset$ by using the sets of best support strategy combinations, which are presented in Figure 1. For example, Figure 1a shows the set of best support strategy combinations for player 1, which consists of two parts. For the bottom part, if player 1 chooses

a strategy $\lambda e_1 + (1 - \lambda)e_2 \in X_1$ with $\lambda \in [\frac{1}{2}, 1]$, then the best thing players 2 and 3 can do to support player 1 is the strategy combination $(f_1, g_1) \in X_{-1}$. For the upper part, if player 1 chooses a strategy $\lambda e_1 + (1 - \lambda)e_2 \in X_1$ with $\lambda \in [0, \frac{1}{2}]$, then the only best support strategy combination is given by $(f_2, g_2) \in X_{-1}$.

From these three figures it can be readily seen that the intersection between the three sets of best support strategy combinations is empty, leading to the conclusion that $BE(G) = \emptyset$. Note that, e.g., $(e_1, f_1, g_2) \in NE(G)$, so $NE(G) \neq \emptyset$. $\triangle$

Figure 1 – The three sets of best support strategy combinations corresponding to $BE(G)$

In order to define and denote the support relations, it is convenient to number the players, i.e. $N = \{1, 2, \dots, n\}$ for a finite player set N . A *bijection* is a map that is both surjective (onto) and injective (one-to-one). For a finite player set $N = \{1, 2, \dots, n\}$, we denote a bijection $\sigma : N \rightarrow N$ by $\sigma = (\sigma(1), \sigma(2), \dots, \sigma(n))$. The set of all such bijections is denoted by $\Pi(N)$. The identity bijection is denoted by Id , i.e. $Id = (1, 2, \dots, n)$. In the context of supportive behavior, a bijection $\sigma \in \Pi(N)$, for a player set N , should be interpreted as follows: player $i \in N$ supports player $\sigma(i) \in N$.

The set of all *derangements* is given by

$$D(N) = \{\delta \in \Pi(N) \mid \delta(i) \neq i \text{ for every } i \in N\}.$$

In a derangement, no player supports himself. Finally, we introduce the set $C(N)$ of *cyclic derangements*, given by

$$C(N) = \{\gamma \in D(N) \mid \text{there exists a number } \alpha \in \{1, 2, \dots, n-1\} \text{ such that:} \\ \gamma(i) = i + \alpha \pmod{n} \text{ for every } i \in N\}^1.$$

In a cyclic derangement, every player supports the player that is a fixed number of shifts away from himself. The number $\alpha \in \{1, \dots, n-1\}$ represents the number of shifts. E.g., a shift of $\alpha = 1$ results in the cyclic derangement $(2, 3, 4, \dots, n, 1)$, while a shift of $\alpha = n-1$ results in $(n, 1, 2, \dots, n-2, n-1)$.

¹Here, we use the assumption that $0 \pmod{n} = n$.

3 Unilateral support equilibria

Berge equilibria are based on mutually supportive behavior, which means that every player is supported by the group of all other players together. This mutually supportive behavior however could create coordination issues. This section introduces a new equilibrium concept, the concept of unilateral support equilibria, that retains supportive behavior, but eliminates these coordination issues.

Before introducing the set of unilateral support equilibria, first the definition of a unilaterally supportive strategy combination with respect to a bijection is given. Afterwards, this is generalized to the definition of a unilateral support equilibrium, where the dependence on a certain bijection is removed.

Definition 3.1 Let $G = (N, (X_i)_{i \in N}, (\pi_i)_{i \in N})$ be a non-cooperative game. Moreover, let $\sigma \in \Pi(N)$ be a bijection on the set of players. A strategy combination $x^* \in X$ is called *unilaterally supportive with respect to σ* if for every $i \in N$ it holds that

$$\pi_{\sigma(i)}(x_{-i}^*, x_i) \leq \pi_{\sigma(i)}(x_{-i}^*, x_i^*),$$

for every $x_i \in X_i$. The set of such strategy combinations is denoted by $USE_\sigma(G)$.

Hence, a strategy combination is unilaterally supportive with respect to a bijection $\sigma \in \Pi(N)$ if every player $i \in N$ supports player $\sigma(i) \in N$ in the best way possible. If for a given non-cooperative game $G = (N, (X_i)_{i \in N}, (\pi_i)_{i \in N})$ and a bijection $\sigma \in \Pi(N)$ it holds that $\sigma(i) = i$ for a certain player $i \in N$, then this means that player i supports himself. In that case, player i is going to maximize his pay-off with respect to the unilaterally supportive strategy combination of the other players. In fact, also if $\sigma(i) \neq i$, player i is still going to maximize a pay-off with respect to the unilaterally supportive strategy combination of the other players, but now player i maximizes player $\sigma(i)$'s pay-off. This results in the proposition below, which shows that every set of unilaterally supportive strategy combinations with respect to a bijection σ coincides with the set of Nash equilibria of the game with twisted pay-off functions, in which player i 's pay-off function is replaced by the pay-off function of player $\sigma(i)$. More formally, for a non-cooperative game $G = (N, (X_i)_{i \in N}, (\pi_i)_{i \in N})$ and a bijection $\sigma \in \Pi(N)$, the game with twisted pay-off functions is given by $G_\sigma = (N, (X_i)_{i \in N}, (\pi_{\sigma(i)})_{i \in N})$. The proof of the proposition is straightforward and therefore omitted.

Proposition 3.2 Let $G = (N, (X_i)_{i \in N}, (\pi_i)_{i \in N})$ be a non-cooperative game and let $\sigma \in \Pi(N)$ be a bijection. Then it holds that

$$USE_\sigma(G) = NE(G_\sigma).$$

Remark Note that if one applies Proposition 3.2 to the identity bijection, it follows that $USE_{Id}(G) = NE(G)$.

Definition 3.1 is applicable to every bijection on the set of players. However, only derangements really reflect the idea of supportive behavior. If a player is mapped to himself, then this player does not support another player. The set of unilaterally supportive strategy combinations with respect to a derangement has the disadvantage that it is not anonymous in the sense that it relies on the predetermined support relations given by the derangement. For this reason, we will consider the set of unilaterally supportive strategy combinations with respect to all derangements.

Definition 3.3 Let $G = (N, (X_i)_{i \in N}, (\pi_i)_{i \in N})$ be a non-cooperative game. Then the set of *unilateral support equilibria* is defined as

$$USE(G) = \bigcap_{\delta \in D(N)} USE_\delta(G).$$

The following theorem provides a characterization of the set of unilateral support equilibria in terms of the pay-off functions. This contrasts the definition, which is formulated as the intersection of sets of unilaterally supportive strategy combinations. This theorem clearly highlights the underlying feature of unilaterally supportive behavior.

Theorem 3.4 Let $G = (N, (X_i)_{i \in N}, (\pi_i)_{i \in N})$ be a non-cooperative game and let $x^* \in X$ a strategy combination. Then $x^* \in USE(G)$ if and only if for every $i \in N$ and every $j \in N \setminus \{i\}$ it holds that

$$\pi_i(x_{-j}^*, x_j) \leq \pi_i(x_{-j}^*, x_j^*) \quad \text{for every } x_j \in X_j.$$

Proof: Set $N = \{1, 2, \dots, n\}$. Let $x^* \in USE(G)$ and let $i \in N, j \in N \setminus \{i\}$ and $x_j \in X_j$. Now, define a bijection $\sigma \in \Pi(N)$ as follows: $\sigma(k) = k + (i - j) \pmod{n}$ for every $k \in N$. Since $i \neq j$, it follows that $\sigma \in D(N)$. Moreover, $\sigma(j) = i$. Then it holds that

$$\pi_i(x_{-j}^*, x_j) = \pi_{\sigma(j)}(x_{-j}^*, x_j) \leq \pi_{\sigma(j)}(x_{-j}^*, x_j^*) = \pi_i(x_{-j}^*, x_j^*),$$

where the inequality follows from the fact that $x^* \in USE_\sigma(G)$.

For the reverse implication, let x^* be such that for every $i \in N$ and every $j \in N \setminus \{i\}$ it holds that $\pi_i(x_{-j}^*, x_j) \leq \pi_i(x_{-j}^*, x_j^*)$ for every $x_j \in X_j$. Suppose for the sake of contradiction that $x^* \notin USE(G)$. Then there is a derangement $\delta \in D(N)$ such that $x^* \notin USE_\delta(G)$. Accordingly, there is a player $i \in N$ and a strategy $x_i \in X_i$ such that

$$\pi_{\delta(i)}(x_{-i}^*, x_i) > \pi_{\delta(i)}(x_{-i}^*, x_i^*).$$

However, this contradicts our assumption. □

Theorem 3.4 shows that in a unilateral support equilibrium, every player is supported by every other player individually. In a Berge equilibrium, every player is supported by the group of all other players together. Every Berge equilibrium is in fact a unilateral support equilibrium. This result is based on the fact that a Berge equilibrium is unilaterally supportive with respect to any derangement.

Theorem 3.5 Let $G = (N, (X_i)_{i \in N}, (\pi_i)_{i \in N})$ be a non-cooperative game. Then it holds that $BE(G) \subseteq USE(G)$.

Proof: Let $x^* \in BE(G)$. Moreover, let $\delta \in D(N)$, $i \in N$ and $x_i \in X_i$. Then,²

$$\begin{aligned} \pi_{\delta(i)}(x_{-i}^*, x_i) &= \pi_{\delta(i)}(x_{\delta(i)}^*, x_{-i, -\delta(i)}^*, x_i) \\ &\leq \pi_{\delta(i)}(x_{\delta(i)}^*, x_{-i, -\delta(i)}^*, x_i^*) = \pi_{\delta(i)}(x_{-i}^*, x_i^*), \end{aligned}$$

where the inequality follows from the fact that $x^* \in BE(G)$ (since $(x_{-i, -\delta(i)}^*, x_i) \in X_{-\delta(i)}$). Hence, $x^* \in USE_\delta(G)$ for every $\delta \in D(N)$ and consequently, $x^* \in USE(G)$. □

We conclude this section with two examples regarding the existence of unilateral support equilibria. Example 3.6 shows that there are games in which there is a unilateral support equilibrium, while there is no Berge equilibrium. Example 3.7 provides a game in which no unilateral support equilibria exist.

²Here, $x_{-i, -\delta(i)}^*$ is the notation for the strategy combination induced by x^* for the players in $N \setminus \{i, \delta(i)\}$.

Example 3.6 Reconsider the three-person game G in trimatrix form as discussed in Example 2.1:

$$G = \begin{array}{cc} & \begin{array}{cc} f_1 & f_2 \end{array} \\ \begin{array}{c} e_1 \\ e_2 \end{array} & \left[\begin{array}{cc|cc} (1, 1, 0) & (0, 0, 0) & (0, 0, 1) & (0, 0, 0) \\ (0, 0, 0) & (0, 0, 1) & (0, 0, 0) & (1, 1, 0) \end{array} \right] \\ & \begin{array}{cc} g_1 & g_2 \end{array} \end{array}.$$

As noted before, $BE(G) = \emptyset$. It holds that $(e_1, f_1, g_1) \in USE(G)$. Using the characterization provided in Theorem 3.4, this can be seen from the following six inequalities:

$$\begin{aligned} \pi_1(e_1, x_2, g_1) &\leq 1 = \pi_1(e_1, f_1, g_1) && \text{for every } x_2 \in X_2; \\ \pi_1(e_1, f_1, x_3) &\leq 1 = \pi_1(e_1, f_1, g_1) && \text{for every } x_3 \in X_3; \\ \pi_2(x_1, f_1, g_1) &\leq 1 = \pi_2(e_1, f_1, g_1) && \text{for every } x_1 \in X_1; \\ \pi_2(e_1, f_1, x_3) &\leq 1 = \pi_2(e_1, f_1, g_1) && \text{for every } x_3 \in X_3; \\ \pi_3(x_1, f_1, g_1) &= 0 = \pi_3(e_1, f_1, g_1) && \text{for every } x_1 \in X_1; \\ \pi_3(e_1, x_2, g_1) &= 0 = \pi_3(e_1, f_1, g_1) && \text{for every } x_2 \in X_2. \end{aligned}$$

△

Example 3.7 Consider the following three-person game G in trimatrix form:

$$G = \begin{array}{cc} & \begin{array}{cc} f_1 & f_2 \end{array} \\ \begin{array}{c} e_1 \\ e_2 \end{array} & \left[\begin{array}{cc|cc} (0, 1, 1) & (0, 0, 0) & (1, 0, 0) & (1, 0, 0) \\ (0, 0, 0) & (1, 0, 0) & (0, 0, 0) & (0, 1, 1) \end{array} \right] \\ & \begin{array}{cc} g_1 & g_2 \end{array} \end{array}.$$

Clearly, for a player set consisting of three players, i.e. $N = \{1, 2, 3\}$, there are only two derangements, $(2, 3, 1)$ and $(3, 1, 2)$ respectively. To show that $USE(G) = \emptyset$, we have to show that $USE_{(2,3,1)}(G) \cap USE_{(3,1,2)}(G) = \emptyset$. First, consider $\delta = (2, 3, 1)$. Using Proposition 3.2, in a unilaterally supportive strategy combination with respect to δ , player 1 is maximizing the pay-off of player 2, since player 1 supports player 2 according to δ . In other words, player 1 is playing a best reply strategy with regard to the pay-off function of player 2. Figure 2a shows this set of best reply strategies. Note that player 1 is indifferent between his pure strategies e_1 and e_2 for every strategy combination $(\lambda f_1 + (1 - \lambda)f_2, (1 - \lambda)g_1 + \lambda g_2) \in X_{-1}$ with $\lambda \in [0, 1]$.

Figure 2b shows the set of best reply strategies for player 2, who supports player 3. Player 2 is indifferent (with regard to the pay-off function of player 3) between his pure strategies f_1 and f_2 for every strategy combination $(\lambda e_1 + (1 - \lambda)e_2, (1 - \lambda)g_1 + \lambda g_2) \in X_{-2}$ with $\lambda \in [0, 1]$.

The set of best reply strategies for player 3 is shown in Figure 2c. He is indifferent (with regard to the pay-off function of player 1) between his pure strategies g_1 and g_2 for every strategy combination $(\lambda e_1 + (1 - \lambda)e_2, \frac{1-2\lambda}{1-\lambda}f_1 + \frac{\lambda}{1-\lambda}f_2) \in X_{-3}$ with $\lambda \in [0, \frac{1}{2}]$.

By intersecting the three sets of best reply strategies as shown in Figures 2a, 2b and 2c, we obtain the set of unilaterally supportive strategy combinations with respect to the derangement

$(2, 3, 1)$:

$$USE_{(2,3,1)}(G) = \{(e_1, f_1, g_2), (e_2, f_2, g_1), (\alpha e_1 + (1 - \alpha)e_2, \alpha f_1 + (1 - \alpha)f_2, (1 - \alpha)g_1 + \alpha g_2)\},$$

with $\alpha := \frac{3}{2} - \frac{\sqrt{5}}{2}$.

Figure 2 – The three sets of best reply strategies corresponding to $USE_{(2,3,1)}(G)$

Figure 3 – The three sets of best reply strategies corresponding to $USE_{(3,1,2)}(G)$

Figure 3 summarizes a similar analysis when one considers $\delta = (3, 1, 2)$. One finds that player 1 is indifferent for every strategy combination $(\lambda f_1 + (1 - \lambda)f_2, (1 - \lambda)g_1 + \lambda g_2) \in X_{-1}$ with $\lambda \in [0, 1]$. Player 2 is indifferent for every strategy combination $(e_1, \lambda g_1 + (1 - \lambda)g_2) \in X_{-2}$ with $\lambda \in [0, 1]$ and every strategy combination $(\lambda e_1 + (1 - \lambda)e_2, g_2) \in X_{-2}$ with $\lambda \in [0, 1]$. Player 3 is indifferent for every strategy combination $(\lambda e_1 + (1 - \lambda)e_2, (1 - \lambda)f_1 + \lambda f_2) \in X_{-3}$ with $\lambda \in [0, 1]$.

Consequently,

$$USE_{(3,1,2)}(G) = \text{Conv} \{(e_1, f_1, g_1), (e_1, f_2, g_1)\} \cup \text{Conv} \{(e_2, f_1, g_2), (e_2, f_2, g_2)\}.$$

Hence,

$$USE(G) = USE_{(2,3,1)}(G) \cap USE_{(3,1,2)}(G) = \emptyset.$$

△

4 Further results

We first show that in order to compute the set of unilateral support equilibria, one can restrict to the intersection of the sets of unilaterally supportive strategy combinations with respect to cyclic derangements only.

Theorem 4.1 *Let $G = (N, (X_i)_{i \in N}, (\pi_i)_{i \in N})$ be a non-cooperative game. Then it holds that*

$$USE(G) = \bigcap_{\gamma \in C(N)} USE_{\gamma}(G).$$

Proof: Obviously, $C(N) \subseteq D(N)$. Hence,

$$USE(G) = \bigcap_{\delta \in D(N)} USE_{\delta}(G) \subseteq \bigcap_{\gamma \in C(N)} USE_{\gamma}(G).$$

To prove that $\bigcap_{\gamma \in C(N)} USE_{\gamma}(G) \subseteq USE(G)$, let $x^* \in \bigcap_{\gamma \in C(N)} USE_{\gamma}(G)$. Using Theorem 3.4, it suffices to show that for every $i \in N$ and every $j \in N \setminus \{i\}$ it holds that

$$\pi_i(x_{-j}^*, x_j) \leq \pi_i(x_{-j}^*, x_j^*) \quad \text{for every } x_j \in X_j.$$

Set $N = \{1, 2, \dots, n\}$. Let $i \in N, j \in N \setminus \{i\}$ and $x_j \in X_j$. Define $\sigma \in \Pi(N)$ in the following way: $\sigma(k) = k + (i - j) \pmod{n}$ for every $k \in N$. Clearly, $\sigma \in C(N)$ and $\sigma(j) = i$. Then,

$$\pi_i(x_{-j}^*, x_j) = \pi_{\sigma(j)}(x_{-j}^*, x_j) \leq \pi_{\sigma(j)}(x_{-j}^*, x_j^*) = \pi_i(x_{-j}^*, x_j^*),$$

where the inequality follows from the fact that $x^* \in USE_{\sigma}(G)$. □

n	$ \Pi(N) $	$ D(N) $	$ C(N) $
2	2	1	1
3	6	2	2
4	24	9	3
5	120	44	4
6	720	265	5
7	5040	1854	6
$\vdots$	$\vdots$	$\vdots$	$\vdots$
n	$n!$	$n! \cdot \sum_{k=0}^n \frac{(-1)^k}{k!}$	$n - 1$

Table 1 – The number of bijections, derangements and cyclic derangements with $N = \{1, \dots, n\}$

Table 1 gives an overview of the number of bijections, the number of derangements and the number of cyclic derangements for a given player set. It shows that Theorem 4.1 leads to a drastic reduction of the number of sets of unilaterally supportive strategy combinations that has to be computed. Moreover, since there are $n - 1$ cyclic derangements for a player set with n players, every cyclic derangement is responsible for exactly one support relation for every player. In other words, every player supports another player due to exactly one cyclic derangement. Therefore, every set of unilaterally supportive strategy combinations with respect to a cyclic derangement is necessary in a unilateral support equilibrium. This is not the case in Definition 3.3, in which supporting a player is due to possibly more derangements.

For $N = \{1, 2\}$, there is only one (cyclic) derangement, $(2, 1)$. Consequently, the set of unilateral support equilibria is equal to the set of unilaterally supportive strategy combinations with respect to $(2, 1)$. Recall that the set of unilaterally supportive strategy combination with respect to $(2, 1)$ coincides with the set of Nash equilibria of the game with twisted pay-off functions, in which the players' pay-off functions are interchanged, according to Proposition 3.2.

Moreover, in a Berge equilibrium, every player is supported by the group of all other players together. For a two-person game, this group consists of only one other player, which implies that the set of Berge equilibria consists of those strategy combinations in which every player is supported by the other player in the best way possible. Hence, the set of Berge equilibria coincides with the set of unilateral support equilibria. This proves the following proposition, which has the consequence, when $|N| = 2$, that we can adopt existence theorems regarding Nash equilibria to the setting of unilateral support equilibria.

Proposition 4.2 *Let $G = (N, (X_i)_{i \in N}, (\pi_i)_{i \in N})$ be a non-cooperative game with $|N| = 2$. Then it holds that*

$$USE(G) = BE(G) = NE(G_{(2,1)}).$$

In a unilateral support equilibrium, every player is supported by every other player individually. In a Nash equilibrium, which is a strategy combination that is unilaterally supportive with respect to the identity, every player supports himself in the best way possible. Together this implies that if a strategy combination is both a unilateral support equilibrium and a Nash equilibrium, every player is supported by every player, including himself. The following theorem shows that the intersection between the set of unilateral support equilibria and the set of Nash equilibria is equal to the intersection over all bijections of the sets of unilaterally supportive strategy combinations with respect to these bijections.

Theorem 4.3 *Let $G = (N, (X_i)_{i \in N}, (\pi_i)_{i \in N})$ be a non-cooperative game. Then it holds that*

$$USE(G) \cap NE(G) = \bigcap_{\sigma \in \Pi(N)} USE_{\sigma}(G).$$

Proof: First, if $N = \{1, 2\}$, we have that $\Pi(N) = \{(1, 2), (2, 1)\}$. Hence, $USE(G) \cap NE(G) = USE_{(2,1)}(G) \cap USE_{(1,2)}(G) = \bigcap_{\sigma \in \Pi(N)} USE_{\sigma}(G)$.

Let $|N| \geq 3$. Obviously, $\bigcap_{\sigma \in \Pi(N)} USE_{\sigma}(G) \subseteq USE(G) \cap NE(G)$.

For the reverse inclusion, let $x^* \in USE(G) \cap NE(G)$. Then, $\pi_i(x_{-j}^*, x_j) \leq \pi_i(x_{-j}^*, x_j^*)$ for every $i \in N, j \in N \setminus \{i\}$ and every $x_j \in X_j$, according to Theorem 3.4. Moreover, $\pi_i(x_{-i}^*, x_i) \leq \pi_i(x_{-i}^*, x_i^*)$ for every $i \in N$ and every $x_i \in X_i$, according to the definition of a Nash equilibrium. Hence,

$$\pi_i(x_{-j}^*, x_j) \leq \pi_i(x_{-j}^*, x_j^*) \quad \text{for every } i \in N, j \in N \text{ and } x_j \in X_j. \quad (1)$$

It suffices to prove that $x^* \in USE_{\sigma}(G)$ for every $\sigma \in \Pi(N) \setminus D(N)$ with $\sigma \neq Id$. Let $\sigma \in \Pi(N) \setminus D(N), \sigma \neq Id$. We have to prove that for every $i \in N$ and every $x_i \in X_i$ it holds that $\pi_{\sigma(i)}(x_{-\sigma(i)}^*, x_i) \leq \pi_{\sigma(i)}(x_{-\sigma(i)}^*, x_i^*)$. This, however, is a direct consequence of Equation (1), with $\sigma(i)$ and i instead of i and j respectively. $\square$

Finally, we formulate two statements about the class of coordination games, in which every player faces the same pay-off function. In the light of supportive behavior, this means that if a player supports another player, he also supports himself. Interestingly, this fact can be used to describe the intersection between the set of unilateral support equilibria and the set of Nash equilibria for general games.

Definition 4.4 Let $G = (N, (X_i)_{i \in N}, (\pi_i)_{i \in N})$ be a non-cooperative game. Then G is a *coordination game* if $\pi_i = \pi_j$ for every $i, j \in N$.

For a coordination game, the set of unilateral support equilibria coincides with the set of Nash equilibria, as shown by the proposition below.

Proposition 4.5 Let $G = (N, (X_i)_{i \in N}, (\pi_i)_{i \in N})$ be a coordination game. Then, for every $\sigma \in \Pi(N)$, it holds that

$$USE_\sigma(G) = NE(G).$$

Consequently, $USE(G) = NE(G)$.

Proof: Let $\sigma \in \Pi(N)$. Proposition 3.2 implies that $USE_\sigma(G) = NE(G_\sigma)$, where G_σ is the game with twisted pay-off functions. Since G is a coordination game, $G_\sigma = G$. Consequently, $USE_\sigma(G) = NE(G_\sigma) = NE(G)$. $\square$

The second statement is about the intersection between the set of unilateral support equilibria and the set of Nash equilibria for general games. We reformulate Theorem 4.3 in terms of coordination games. For a given non-cooperative game $G = (N, (X_i)_{i \in N}, (\pi_i)_{i \in N})$ and a player $k \in N$, we can define the coordination game $G_k = (N, (X_i)_{i \in N}, (\pi_k)_{i \in N})$, in which every player faces the pay-off function of player k .

Theorem 4.6 Let $G = (N, (X_i)_{i \in N}, (\pi_i)_{i \in N})$ be a non-cooperative game. Then it holds that

$$USE(G) \cap NE(G) = \bigcap_{k \in N} NE(G_k).$$

Proof: First, similar to the proof of Theorem 4.3, note that for a strategy combination $x^* \in X$ it holds that $x^* \in USE(G) \cap NE(G)$ if and only if

$$\pi_i(x_{-j}^*, x_j) \leq \pi_i(x_{-j}^*, x_j^*) \quad \text{for every } i \in N, j \in N \text{ and } x_j \in X_j. \quad (2)$$

For the first inclusion, let $x^* \in USE(G) \cap NE(G)$. We have to prove that $x^* \in NE(G_k)$ for every $k \in N$, or equivalently, that $\pi_k(x_{-i}^*, x_i) \leq \pi_k(x_{-i}^*, x_i^*)$ for every $k \in N, i \in N$ and $x_i \in X_i$. This is a direct consequence of Equation (2).

For the reverse inclusion, let $x^* \in X$ be a strategy combination such that $x^* \in \bigcap_{k \in N} NE(G_k)$. It suffices to prove that $\pi_i(x_{-j}^*, x_j) \leq \pi_i(x_{-j}^*, x_j^*)$ for every $i \in N, j \in N$ and $x_j \in X_j$. This follows immediately from the fact that $x^* \in NE(G_i)$ for every $i \in N$. $\square$

Theorem 4.6 shows that every strategy combination that is both a unilateral support equilibrium and a Nash equilibrium is also a Nash equilibrium of the game in which every player faces the pay-off function of a single player and vice versa. This is a result of the fact that it is possible to rearrange the support relations in a suitable way.

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