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**ORTHOGONAL REPRESENTATIONS OVER FINITE FIELDS AND
THE CHROMATIC NUMBER OF GRAPHS****RENÉ PEETERS****Received January 19, 1994*

We study the relationship between the minimum dimension of an orthogonal representation of a graph over a finite field and the chromatic number of its complement. It turns out that for some classes of matrices defined by a graph the 3-colorability problem is equivalent to deciding whether the class defined by the graph contains a matrix of rank 3 or not. This implies the NP-hardness of determining the minimum rank of a matrix in such a class. Finally we give for any class of matrices defined by a graph that is interesting in this respect a reduction of the 3-colorability problem to the problem of deciding whether or not this class contains a matrix of rank equal to three.

1. Introduction

Let G be a graph with vertex set $V(G) = \{1, 2, \dots, n\}$. The chromatic number of G , which we denote by $\chi(G)$, is the minimum number of colors needed to color the vertices of G , such that the two endpoints of any edge have different colors. To determine the chromatic number of a general graph is a difficult (NP-hard) problem. Even the problem of deciding whether a graph is 3-colorable or not is already NP-complete (cf. [6, 3, 16]). A trivial lower bound on the chromatic number of G is the clique number of G (denoted by $\omega(G)$), that is, the number of vertices in a largest complete subgraph of G . Clearly these $\omega(G)$ vertices must be colored differently in every legal coloring of G . Also the problem of determining the clique number of a general graph is NP-hard (cf. [6, 3, 16]).

In 1979 Lovász [13] introduced for each graph a number $\theta(G)$, for which

$$\omega(G) \leq \theta(\overline{G}) \leq \chi(G).$$

($\overline{G}$ denotes the complementary graph of G .) In [7] it is proved that $\theta(G)$ can be calculated (or, in fact approximated, since $\theta(G)$ doesn't need to be rational) by the ellipsoid method in polynomial time. This implies for instance that for perfect

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graphs (these are graphs for which for all its subgraphs the chromatic number and the clique number are equal) the chromatic number and the clique number can be determined in polynomial time. For perfect graphs even an explicit maximum clique or coclique can be found in polynomial time.

Remark 1.1. The main difficulty in reading this paper and related papers is avoiding to mix up a graph and its complement. For instance, the function called $\theta(G)$ in [14] is called $\theta(\overline{G})$ in [7, 13] and [12]. Because it is more likely to be adopted by other researchers we follow the latter convention.

Let A be an $n \times n$ -matrix (over some field) with all diagonal elements non-zero and with $A_{ij} = 0$ if i and j are adjacent in G . If a matrix satisfies these conditions, we say that the matrix *fits* G ¹. Clearly $\text{rank}(A) \geq \omega(G)$, since A has a diagonal matrix of size $\omega(G)$, with non-zero diagonal entries, as a submatrix. On the other hand, there exists a matrix A that fits G for which $\text{rank}(A) \leq \chi(G)$. Indeed, let G be colored with $\chi(G)$ colors and define the matrix A by

$$A_{ij} := \begin{cases} 1 & \text{if } i \text{ and } j \text{ are in the same color class,} \\ 0 & \text{otherwise,} \end{cases}$$

then A fits G and $\text{rank}(A) = \chi(G)$. So the minimum rank over all matrices fitting G is also a number between the clique number and the chromatic number of G . This number was introduced by Haemers ([8, 9]).

Remark 1.2. In the literature other classes of matrices occur similar to those fitting a graph. One linear class of matrices capturing circuit depth was introduced by Razborov in [18]. Hershkowitz and Schneider [10] study so-called zero patterns and sign patterns which are classes of matrices with some prescribed zero/nonzero and negative/zero/positive pattern respectively.

A question that arises is: What is the complexity of determining the minimum rank of all matrices fitting a graph G ? This is the main problem we concentrate on in this paper. Clearly, this minimum can be determined in polynomial time for perfect graphs. (According to the referee it has been known that the minimal rank problem for a linear class of matrices is NP-hard. However, the notion of a linear class is too broad here.)

In the next section we consider the minimal rank problem for some fixed field and define for that field three reasonable classes of matrices fitting G . We study the chromatic number of all graphs that are fitted by a matrix from such a class with some fixed rank. It turns out (as a sort of curiosity) that for some (small) finite fields the minimum rank over some of these classes is equal to the chromatic number of the graph G if this chromatic number is 3. So for these classes of matrices the

¹ This definition of a matrix fitting a graph is the same as by Haemers in [9], although in that paper the condition that $A_{ij} = 0$ if i and j are adjacent should be read as $A_{ij} = 0$ if i and j are not adjacent to be consistent with the rest of the paper.

3-colorability problem is equivalent to the problem of deciding whether the class defined by the graph contains a matrix of rank 3 or not.

Let for some fixed field F and any graph G , $\mathcal{A}(G)$ be a class of matrices over F , fitting G , such that it contains a matrix of rank $\chi(G)$. Define the following problem:

Name: RANK-3,

Input: A graph G ,

Question: Does $\mathcal{A}(G)$ contain a matrix of rank 3?

Then, since the 3-colorability problem is NP-complete, we obtain as a corollary that for some classes of matrices (over a small finite field) fitting G the problem RANK-3 is NP-complete. This suggests that RANK-3 is NP-hard for all relevant classes of matrices fitting a graph G .

In the last section of this paper we give a reduction from the 3-colorability problem to the RANK-3 problem, where the field is arbitrary and the class of matrices fitting a graph is assumed to contain a matrix of rank at most the chromatic number of the graph. So if this RANK-3 problem is contained in NP (for instance if the field is finite and if it can be checked in time polynomial in $|V(G)|$ whether or not some matrix belongs to $\mathcal{A}(G)$.) it is NP-complete.

2. Some equivalences

In this section we show that for some particular classes of matrices fitting a graph this class contains a matrix of rank 3 if and only if the graph is 3-colorable. We also find nice examples of graphs for which Haemers's rank bound is much better than Lovász's bound.

2.1. Orthogonal representations

Let G be a graph with vertex set $V(G) = \{1, 2, \dots, n\}$ and edge set $E(G)$ and let F be a field. Consider the following classes of matrices:

$$\begin{aligned}\mathcal{A}_3(G, F) &:= \{A \in F^{n \times n} | A \text{ fits } G\} \\ \mathcal{A}_2(G, F) &:= \{A \in \mathcal{A}_3(G, F) | A \text{ symmetric}\} \\ \mathcal{A}_1(G, F) &:= \{A \in \mathcal{A}_2(G, F) | A_{ii} = 1\}.\end{aligned}$$

For any class $\mathcal{A}$ of matrices over F we introduce the following number:

$$R(\mathcal{A}) := \min\{\text{rank}(A) | A \in \mathcal{A}\}.$$

So by the previous section we have that

$$\omega(G) \leq R(\mathcal{A}_3(G, F)) \leq R(\mathcal{A}_2(G, F)) \leq R(\mathcal{A}_1(G, F)) \leq \chi(G).$$

Let V be a vector space of dimension d over the field F provided with a bilinear form $B: V \times V \rightarrow F$ (notation: (V, B)). Vectors will be row vectors. An *orthogonal representation* of G in (V, B) is a system $(v_1, v_2, \dots, v_n)$ of non-isotropic vectors in V (that are vectors v for which $B(v, v) \neq 0$) such that for all $i, j \in V(G)$, $i \neq j$:

$$i \not\sim j \Rightarrow B(v_i, v_j) = 0.$$

This definition is a generalization of the definition given by Lovász [13], who introduced orthogonal representations in the study of the Shannon capacity of a graph. In his definition V is $\mathbb{R}^d$ provided with the standard inner product. It follows by definition that the Gram matrix of $\{v_1, v_2, \dots, v_n\}$ (the $n \times n$ -matrix with entries $B(v_i, v_j)$; notation: $\text{Gram}(v_1, v_2, \dots, v_n)$) is a matrix from $\mathcal{A}_3(\overline{G}, F)$. If furthermore B is symmetric then $\text{Gram}(v_1, v_2, \dots, v_n) \in \mathcal{A}_2(\overline{G}, F)$. On the other hand, if $A \in \mathcal{A}_2(\overline{G}, F)$ and A has rank r , then there is an orthogonal representation of G , $(v_1, \dots, v_n)$ say, in (F^r, B) for some symmetric bilinear form B such that $A = \text{Gram}(v_1, \dots, v_n)$. An orthogonal representation $(v_1, \dots, v_n)$ of G is called *orthonormal* if $B(v_i, v_i) = 1$ for $i = 1, 2, \dots, n$. If for a symmetric bilinear form B , $(v_1, \dots, v_n)$ is an orthonormal representation of G in (V, B) , then $\text{Gram}(v_1, \dots, v_n) \in \mathcal{A}_1(\overline{G}, F)$ and, conversely, if $A \in \mathcal{A}_1(\overline{G}, F)$ and A has rank r , there is an orthonormal representation $(v_1, \dots, v_n)$ of G in F^r such that $A = \text{Gram}(v_1, \dots, v_n)$. So we have:

$$\begin{aligned} R(\mathcal{A}_1(G, F)) &= \min\{d \mid \overline{G} \text{ has an orthonormal representation in } (F^d, B) \\ &\quad \text{for some symmetric bilinear form } B\} \\ R(\mathcal{A}_2(G, F)) &= \min\{d \mid \overline{G} \text{ has an orthogonal representation in } (F^d, B) \\ &\quad \text{for some symmetric bilinear form } B\}. \end{aligned}$$

For later use, define $d_F(G)$ as the minimum d for which there exist an orthonormal representation of G in F^d provided with the standard inner product.

An *orthogonal birepresentation* of G in F^d is a pair (g, h) of mappings $g, h: V(G) \rightarrow F^d$ such that for each vertex v : $g(v)h(v)^T \neq 0$ and for each non-edge vw , $g(v)h(w)^T = g(w)h(v)^T = 0$. This definition is slightly different from the definition in [15], where orthogonal birepresentations are introduced in $\mathbb{R}^d$ only. Clearly, if G has an orthogonal birepresentation in F^d , then $R(\mathcal{A}_3(\overline{G}, F)) \leq d$ and if $R(\mathcal{A}_3(\overline{G}, F)) = r$, then G has an orthogonal birepresentation in F^r . So we have:

$$R(\mathcal{A}_3(G, F)) = \min\{d \mid \overline{G} \text{ has an orthogonal birepresentation in } F^d\}$$

For a graph G , let G^k be the normal product of k copies of G , then $\Theta(G)$, the *Shannon capacity* of G , is defined by

$$\Theta(G) = \sup_k \sqrt[k]{\alpha(G^k)}.$$

Here $\alpha(G)$ denotes the size of the largest coclique (independent set) of G (So $\alpha(G) = \omega(\overline{G})$). This number was first defined by Shannon [20] and has an interpretation in coding theory. The numbers $R(\mathcal{A}_i(\overline{G}, F))$ as well as Lovász number $\theta(G)$ are upper bounds for the Shannon capacity of G (see [8, 9] and [13] respectively). We refer to these bounds by Haemers's and Lovász's bound respectively. It is shown in [8] that Haemers's bound is sometimes better than Lovász's upper bound $\theta(G)$. We will see some more examples of this at the end of this section. For most graphs however, Lovász's bound seems to be smaller than Haemers's bound. The number $\theta(G)$ is defined as the minimum over all orthonormal representations $(v_1, \dots, v_n)$ of G in a Euclidean space of the value

$$\min_c \max_{1 \leq i \leq n} \frac{1}{(cv_i^T)^2}$$

where c ranges over all unit vectors. Since for the pentagon $\alpha(G^2) = 5$ and $\theta(G) = \sqrt{5}$, Lovász solved the problem of determining the Shannon capacity for the pentagon, which was open for over twenty years (cf. [13]). Also for Lovász's bound the inequalities

$$\omega(G) \leq \theta(\overline{G}) \leq \chi(G)$$

hold. In Figure 1 it is shown how the various numbers are related. For this diagram some results from [13] are used. The number $\alpha^*(G)$ is the fractional independence number of G .

2.2. Minimal rank and the chromatic number

It turns out that for some finite fields F , the number $R(\mathcal{A}_i(G, F))$ is equal to the chromatic number of G if this chromatic number is small. First of all we trivially have for any field F :

$$R(\mathcal{A}_i(G, F)) = 1 \Leftrightarrow \chi(G) = 1$$

and since for odd cycles we have that $R(\mathcal{A}_i(G, F)) = 3$, also

$$R(\mathcal{A}_i(G, F)) = 2 \Leftrightarrow \chi(G) = 2$$

for any field F . If for some field F we can prove that also

$$R(\mathcal{A}_i(G, F)) = 3 \Leftrightarrow \chi(G) = 3$$

then we immediately get that for this field F determining the minimal rank of all matrices in $\mathcal{A}_i(G, F)$ is NP-hard. In fact it is enough to prove this equivalence for a class of graphs for which the 3-colorability problem is still NP-complete such as planar graphs or graphs containing a triangle (see for instance [4]). For finite fields it turns out to be relatively easy to verify if for some positive integer k the statement

$$(1) \quad R(\mathcal{A}_i(G, F)) = k \Leftrightarrow \chi(G) = k$$

Fig. 1. Diagram of the order relations between numbers related to the Shannon capacity of a graph

holds. In fact for each $i = 1, 2, 3$, each k and each finite field F the maximal chromatic number of the graphs G for which $R(\mathcal{A}_i(G, F)) = k$ can be determined. Statement (1) holds if and only if this maximum is k . In particular we get that for finite fields the chromatic number of the graphs G for which $R(\mathcal{A}_i(G, F))$ has some specific value is bounded.

Let B be a nondegenerate symmetric bilinear form on the k -dimensional vector space $V = F^k$. We call two vectors of V , x and y say, equivalent if $x = ay$ for some $a \in F \setminus \{0\}$. Define the graph $\Gamma_2(F, k, B)$ as follows. Take as vertex set the equivalence classes of non-isotropic vectors of (V, B) ; two classes being adjacent if they are orthogonal. Let $\Gamma_1(F, k, B)$ be the subgraph of $\Gamma_2(F, k, B)$ induced by the classes of vectors for which $B(v, v)$ is a square for one (and hence each) vector in the class. Furthermore, define the graph $\Gamma_3(F, k)$ as follows. Take as vertex set all ordered pairs of non-orthogonal equivalence classes of vectors. Two pairs, (v_1, w_1) and (v_2, w_2) say, are adjacent if both v_1 and w_2 and v_2 and w_1 are orthogonal. So in fact $\Gamma_3(F, k)$ is defined on the non-incident point-hyperplane pairs in the $(k-1)$ -dimensional projective space over F . Two of these anti-flags, (P_1, H_1) and (P_2, H_2) say, being adjacent if P_1 is incident with H_2 and P_2 is incident with H_1 .

Lemma 2.1. *Let F be a field and k a positive integer, then for $i=1, 2$*

$$\begin{aligned} \max_B \chi(\Gamma_i(F, k, B)) &= \max\{\chi(G) \mid G \text{ s.t. } R(\mathcal{A}_i(G, F)) = k\} \\ \chi(\Gamma_3(F, k)) &= \max\{\chi(G) \mid G \text{ s.t. } R(\mathcal{A}_3(G, F)) = k\} \end{aligned}$$

provided these maxima exist.

Proof. We prove the first statement of the lemma for $i=2$. The proof for the case $i=1$ and the proof of the second statement are similar. Denote the first maximum with M_1 and the second one with M_2 . Notice that by definition for each symmetric bilinear form B the complement of $\Gamma_2(F, k, B)$ has an orthogonal representation in (F^k, B) , so $M_1 \leq M_2$.

Let G be a graph for which $R(\mathcal{A}_2(G, F)) = k$ and $\chi(G) = M_2$. Then $\overline{G}$ has an orthogonal representation, $v_1, \dots, v_n$ say, in (F^k, B) for some symmetric bilinear form B . So $B(v_i, v_j) = 0$ if i and j are adjacent in G . Since $\text{Gram}(v_1, \dots, v_n)$ has rank k , B must be nondegenerate. Let G' be the graph obtained from G by adding the edges between i and j for all pairs of vertices i, j that are not adjacent in G for which $B(v_i, v_j) = 0$. Then $v_1, \dots, v_n$ is also an orthogonal representation for $\overline{G'}$ and $\chi(G') \geq \chi(G)$. If i and j are two vertices of G' for which $v_i = av_j$ for some $a \in F \setminus \{0\}$ then i and j are non-adjacent and have the same set of neighbors. So i and j can always be colored with the same color and the chromatic number of G' does not change if we identify i and j . Let G'' be the graph we get from G' by identifying vertices that are represented by equivalent vectors, then G'' is a subgraph of $\Gamma_2(F, k, B)$, so we get that

$$\begin{aligned} M_2 = \chi(G) &\leq \chi(G') = \chi(G'') \leq \\ &\chi(\Gamma_2(F, k, B)) \leq M_1. \end{aligned}$$

■

The previous lemma is useful if F is a finite field, since then there are only two non-isomorphic nondegenerate symmetric bilinear forms and there is only a finite number of equivalence classes of non-isotropic vectors (We denote bilinear forms by matrices):

Lemma 2.2. Let B be a nondegenerate symmetric bilinear form on $\mathbb{F}_q^k$.

If $\mathbb{F}_q$ has characteristic 2 then B is either symplectic (all vectors are isotropic) or equivalent to the standard inner product(I).

If $\mathbb{F}_q$ has odd characteristic, let $\delta \in \mathbb{F}_q$ be a non-square, then B is equivalent to either the standard inner product(I) or to $\text{diag}(1^{k-1}, \delta)$ (notation: I'). ■

(N.B.: $\text{diag}(1^{k-1}, \delta)$ is the diagonal matrix of size k with one diagonal element equal to δ and the other $k-1$ elements equal to 1.)

For determining the chromatic number of a graph the following two lemmas can be useful.

Lemma 2.3. (See Brooks [1].) Let G be a simple connected graph which is not a complete graph and let $\rho(\geq 3)$ be the largest vertex degree of G , then $\chi(G) \leq \rho$. ■

Lemma 2.4. (See Hoffman [11].) Let G be a graph on n vertices and let $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$ be the eigenvalues of the adjacency matrix of G , then

$$\chi(G) \geq 1 - \frac{\lambda_1}{\lambda_n}. \quad \blacksquare$$

In the following examples we check the chromatic number of the graphs $\Gamma_i(F, k, B)$ for some finite fields F and nondegenerate symmetric bilinear forms B . Many of the graphs we consider are distance-regular. Most of their properties mentioned here (in particular their eigenvalues) can be found in [2].

Example 2.0. If $F = \mathbb{F}_2$ then $\Gamma_1(F, 2m+1, I)$ consists of an isolated vertex plus the complement of the symplectic graph $\mathcal{S}p(2m, 2)$. The graph $\mathcal{S}p(2m, 2)$ is strongly regular with parameters $(2^{2m}-1, 2^{2m-1}, 2^{2m-2})$. In particular, $\Gamma_1(\mathbb{F}_2, 5, I)$ is the line graph of K_6 plus an isolated vertex, so it has $\chi=5$. $\Gamma_1(\mathbb{F}_2, 6, I)$ is a graph on 32 vertices whose chromatic number is equal to 8. The graph $\Gamma_3(\mathbb{F}_2, 3)$ can also be defined as follows: take as vertex set the 28 antiflags (non-incident point-line pairs) of the Fano plane, two antiflags, (P, l) and (Q, m) say, being adjacent if both P and m and Q and l are incident. This graph is the distance 2 graph of the Coxeter graph, so its largest eigenvalue is 6 and its smallest is $-2\sqrt{2}$. By Hoffman's bound we get that $\chi \geq 4$.

Example 2.1. Let $F = \mathbb{F}_q$ with $q = 2^e$ for $e \geq 2$. Then $\Gamma_1(\mathbb{F}_q, 3, I)$ consists of an isolated vertex plus a graph that can also be defined as follows: Let V be a vector space of dimension 2 over $\mathbb{F}_q$ provided with a nondegenerate symplectic form B . Take as vertex set $V \setminus \{0\}$ and let v and w be adjacent whenever $B(v, w) = 1$. This yields a graph of diameter 3 on $q^2 - 1$ vertices which is distance-regular with intersection array $\{q, q-2, 1; 1, 1, q\}$ and which is an antipodal $(q-1)$ -cover of the complete graph K_{q+1} , see [2] Proposition 12.5.3 of which these graphs are a special

case. These graphs have spectrum $q^1, \sqrt{q}^f, -1^q, -\sqrt{q}^f$ with $f = \frac{1}{2}(q+1)(q-2)$, so by Hoffman's bound $\chi \geq 1 + \sqrt{q}$ and so $\Gamma_1(\mathbb{F}_q, 3, I)$ is not 3-colorable for $q \geq 3$. $\Gamma_1(\mathbb{F}_4, 3, I)$ is an isolated vertex plus the line graph of the Petersen graph for which it is easy to check that $\chi=4$.

Example 2.2. If $F = \mathbb{F}_3$, then $\Gamma_1(F, 4, I)$ is isomorphic to $3K_4$, so it has chromatic number 4 and $\Gamma_1(F, 4, I')$ is the complement of the line graph of K_6 and has chromatic number 4 as well. The graph $\Gamma_1(F, 5, I)$ is the collinearity graph of the generalized quadrangle $GQ(4, 2)$, so it is strongly regular with parameters $(45, 12, 3, 3)$. It has chromatic number equal to 5 since the points of $GQ(4, 2)$ can be partitioned into five classes of nine pairwise non-collinear points each. The graph $\Gamma_1(F, 5, I')$ is strongly regular with parameters $(36, 15, 6, 6)$, so by Hoffman's bound $\chi \geq 6$. In fact it is the graph $\mathcal{N}_6^-$, a subgraph of $\mathcal{S}p(6, 2)$. The graphs $\Gamma_2(F, 3, I)$ and $\Gamma_2(F, 3, I')$ both consist of 9 vertices: one triangle plus 3 pairs of adjacent vertices, each pair adjacent to one of the 3 vertices of the triangle, so $\chi = 3$. The graphs $\Gamma_2(F, 4, I)$ and $\Gamma_2(F, 4, I')$ are not 4-colorable.

Example 2.3. Let $F = \mathbb{F}_5$, then $\Gamma_1(F, 3, I)$ is $5K_3$ and $\Gamma_1(F, 3, I')$ is the Petersen graph, so both are 3-colorable. The graph $\Gamma_1(F, 4, I')$ is a distance-regular graph on 65 vertices that is locally the Petersen graph and has intersection array $\{10, 6, 4; 1, 2, 5\}$. From the eigenvalues we get that $\chi \geq 5$.

Example 2.4. Let $F = \mathbb{F}_7$, then $\Gamma_1(F, 3, I)$ is a distance-regular graph on 21 vertices with intersection array $\{4, 2, 2; 1, 1, 2\}$. It is the collinearity graph of the generalized hexagon of order $(2, 1)$ or the line graph of the Heawood graph and can also be defined as follows: Take as vertices the flags (incident point-line pairs) of the Fano plane, two flags being adjacent whenever they contain the same point or the same line. The 3-colorability problem reduces to the question if we can partition the 21 flags of the Fano plane into 3 sets of 7 flags each, such that each point and each line occurs in one of the flags of each set. Since there is a circulant line-point incidence matrix of the Fano plane (for instance with top row (1101000)) such a partition is possible. $\Gamma_1(F, 3, I')$ is the Coxeter graph, the distance-regular graph on 28 vertices with intersection array $\{3, 2, 2, 1; 1, 1, 1, 2\}$. By Brooks' theorem $\chi=3$.

Example 2.5. For $F = \mathbb{F}_9$ the graph $\Gamma_1(F, 3, I)$ is the distance regular graph on 45 vertices with intersection array $\{4, 2, 2, 2; 1, 1, 1, 2\}$. This is the collinearity graph of the generalized octagon of order $(2, 1)$. A different construction of this graph is as follows: Start with the generalized quadrangle of order $(2, 2)$ with point set the 15 pairs from $\{1, 2, 3, 4, 5, 6\}$ and lines the 15 partitions of $\{1, 2, 3, 4, 5, 6\}$ into 3 disjoint pairs, which are the three points on that line. Let the 45 flags of this generalized quadrangle be the vertices of the graph, two flags being adjacent if they contain the same point or the same line. It is a nice exercise to show that the 45 flags can be partitioned into 3 sets of 15 flags each, such that each point and each line (partition) occur in one of the flags of each set, so we have that $\chi=3$. $\Gamma_1(F, 3, I')$ is the Sylvester graph: the distance-regular graph on 36 vertices with intersection array $\{5, 4, 2; 1, 1, 4\}$. We find $\chi=4$ by computer.

Example 2.6. If $F = \mathbb{F}_{11}$ then $\Gamma_1(F, 3, I)$ is a graph on 55 vertices and by computer $\chi = 4$.

By use of Lemma 2.1 the examples yield the following results:

Theorem 2.1. *Let G be any graph, then*

$$\begin{aligned} R(\mathcal{A}_1(G, \mathbb{F}_2)) &= 5 \Leftrightarrow \chi(G) = 5, \\ R(\mathcal{A}_1(G, \mathbb{F}_3)) &= 4 \Leftrightarrow \chi(G) = 4, \\ R(\mathcal{A}_1(G, \mathbb{F}_5)) &= 3 \Leftrightarrow \chi(G) = 3, \\ R(\mathcal{A}_1(G, \mathbb{F}_7)) &= 3 \Leftrightarrow \chi(G) = 3, \\ R(\mathcal{A}_2(G, \mathbb{F}_3)) &= 3 \Leftrightarrow \chi(G) = 3. \end{aligned}$$

■

The above examples also show that there are no more such equivalences for $\mathbb{F}_q$ with $q \leq 11$ or q even. Probably this is all. If we restrict to graphs containing a clique of some fixed size we get some more equivalences:

Theorem 2.2. *Let G be a graph with $\omega(G) = 3$ then*

$$R(\mathcal{A}_1(G, \mathbb{F}_9)) = 3 \Leftrightarrow \chi(G) = 3.$$

Let G be a graph with $\omega(G) = 5$ then

$$R(\mathcal{A}_1(G, \mathbb{F}_3)) = 5 \Leftrightarrow \chi(G) = 5.$$

■

Since the 3-colorability problem for graphs is NP-complete, also if G is restricted to have a triangle (see for instance [4]) we get the following result:

Corollary 2.1. *The rank-3 problem is NP-complete for the classes $\mathcal{A}_i(G, F)$ for $i = 1$ and $F = \mathbb{F}_q$ with $q \in \{2, 3, 5, 7, 9\}$ and for $i = 2$ and $F = \mathbb{F}_3$.*

■

Unfortunately, in this way we can prove for only a few of the defined classes of matrices fitting a graph that the problem of deciding whether they contain a matrix of rank equal to three or not is NP-complete, but the results suggest that the problem is NP-complete for all classes $\mathcal{A}_1(G, F)$ and $\mathcal{A}_2(G, F)$ over finite fields. The following proposition suggests that this problem is also NP-complete for the classes $\mathcal{A}_3(G, F)$, where F is some finite field:

Proposition 2.1. *Let G be any graph, then*

$$\min\{\text{rank}(A + J) \mid A \in \mathcal{A}_3(G, \mathbb{F}_2)\} \leq 2 \Leftrightarrow \chi(G) \leq 3.$$

Proof. Let G be a graph with $\chi(G) \leq 3$ and let G be colored with at most 3 colors. Define the matrix B by

$$B_{ij} = \begin{cases} 0 & \text{if } i \text{ and } j \text{ have the same color} \\ 1 & \text{otherwise.} \end{cases}$$

Then $A := J - B \in \mathcal{A}_3(G, \mathbb{F}_2)$ and B has rank equal to 2.

On the other hand, let for a graph G $A \in \mathcal{A}_3(G, \mathbb{F}_2)$ such that $B := J + A$ has rank at most 2. Then there are at most 3 different nonzero-columns in B (zero-columns of B correspond to isolated vertices of G) and vertices for which the corresponding columns are the same, are non-adjacent in G ; so $\chi(G) \leq 3$. ■

Remark 2.1. Note that by definition for the graphs $\Gamma_{1,2}(F, k, I)$ and $\Gamma_3(F, k)$ the minimal rank of any matrix fitting these graphs is equal to their clique number which is k . So also the Shannon capacity of their complements is k . In particular, the Shannon capacity of the Symplectic graphs $\mathcal{S}p(2m, 2)$ is equal to $2m + 1$ while their Lovász bound is $2^m + 1$ (cf. [9, 19]) and the complementary graphs of $\Gamma_1(\mathbb{F}_q, 3, I)$ for $q = 2^e$ with $e \geq 2$ have Shannon capacity 3 while their Lovász bound is $\sqrt{q} + 1$. These examples illustrate once more that Haemers's bound is sometimes (much) better than Lovász's bound.

Remark 2.2. Let F be any field and G a graph on n vertices. If A and B are matrices over F fitting G and $\overline{G}$ respectively then

$$\text{rank}(A) \cdot \text{rank}(B) \geq n$$

since the matrix C defined by $C_{ij} = A_{ij} \cdot B_{ij}$ is a rank n submatrix of the matrix $A \otimes B$. So if $\mathcal{A}(G)$ is for any graph G a class of matrices over F fitting G then

$$(2) \quad R(\mathcal{A}(G)) \cdot R(\mathcal{A}(\overline{G})) \geq n.$$

Also for Lovász's number we have that

$$\theta(G) \cdot \theta(\overline{G}) \geq n$$

(cf. [13]) and equality holds for instance if G has a vertex-transitive automorphism group (cf. [13]) or if the edge set of G is a union of classes of a symmetric association scheme (cf. [19]) as is the case for distance-regular graphs. If we allow matrices over different fields, relation (2) in general does not longer hold. An infinite class of examples that show this can be found in [9].

3. A reduction from 3C to RANK-3

In this section we prove that for all relevant classes of matrices fitting a graph G it is NP-hard to check whether or not it contains a matrix of rank equal to 3. More precisely, let for some fixed field F and any graph G , $\mathcal{A}(G)$ be a class of matrices over F , fitting G , such that it contains a matrix of rank $\chi(G)$. Define the following problem:

Name: RANK-3.

Input: A graph G .

Question: Does $\mathcal{A}(G)$ contain a matrix of rank 3?

Then

Theorem 3.1. *RANK-3 is NP-hard.*

Proof. Consider the 3-Coloration problem, (3C), which is defined as follows:

Name: 3C.

Input: A graph G .

Question: Is $\chi(G) \leq 3$?

It is proved in [4] (see also [3, 16]) that 3C is an NP-complete problem. We show that 3C is polynomially reducible to RANK-3.

Let $G = (V, E)$ with vertex set $V = \{1, 2, \dots, n\}$ be the input I to the 3C-problem. First we construct a graph $f(I)$ such that $f(I)$ is 3-colorable if and only if G is. Secondly we prove that $f(I)$ is 3-colorable if and only if $\mathcal{A}(f(I))$ contains a rank-3 matrix fitting $f(I)$.

In order to construct the graph $f(I)$, introduce for each unordered pair of vertices from G , i and j with $i < j$ say, four extra vertices a_{ij}, b_{ij}, c_{ij} and d_{ij} and nine extra edges such that these nine edges form the graph H_{ij} as shown in Figure 2. So, apart from a possible edge between i and j , H_{ij} is the induced subgraph of $f(I)$ on the vertices $i, j, a_{ij}, b_{ij}, c_{ij}$ and d_{ij} . So $f(I)$ has $|V| + 2n(n-1)$ vertices and $|E| + \frac{9}{2}n(n-1)$ edges.

Fig. 2. The graph H_{ij}

Notice that there are essential two different valid 3-colorings of H_{ij} , one with color classes $\{\{i, c_{ij}\}, \{a_{ij}, d_{ij}\}, \{b_{ij}, j\}\}$ and one with color classes $\{\{i, j\}, \{a_{ij}, c_{ij}\}, \{b_{ij}, d_{ij}\}\}$, so for one 3-coloring i and j get different colors and for the other coloring i and j are colored the same. It follows that $f(I)$ is 3-colorable if and only if G is since a valid 3-coloring of $f(I)$ induces a 3-coloring of G and a 3-coloring of G can always be completed to a valid 3-coloring for $f(I)$.

It is an exercise to check that for any field there are essentially only two different types of matrices of rank 3 fitting H_{ij} corresponding to the two different 3-colorings of H_{ij} . These two types are shown in Figure 3 where a $*$ denotes a non-zero field element. The first one corresponds to the coloring with color classes $\{\{i, c_{ij}\}, \{a_{ij}, d_{ij}\}, \{b_{ij}, j\}\}$ and the second one to the coloring with color classes $\{\{i, j\}, \{a_{ij}, c_{ij}\}, \{b_{ij}, d_{ij}\}\}$. The row vectors of vertices from the same color class are scalar multiples of each other.

i	$*$	0	0	$*$	0	0
a_{ij}	0	$*$	0	0	$*$	0
b_{ij}	0	0	$*$	0	0	$*$
c_{ij}	$*$	0	0	$*$	0	0
d_{ij}	0	$*$	0	0	$*$	0
j	0	0	$*$	0	0	$*$

i	$*$	0	0	0	0	$*$
a_{ij}	0	$*$	0	$*$	0	0
b_{ij}	0	0	$*$	0	$*$	0
c_{ij}	0	$*$	0	$*$	0	0
d_{ij}	0	0	$*$	0	$*$	0
j	$*$	0	0	0	0	$*$

Fig. 3. The two types of rank-3 matrices fitting H_{ij}

Finally we show that $f(I)$ is 3-colorable if and only if $\mathcal{A}(f(I))$ contains a matrix of rank 3. First of all, by assumption $\mathcal{A}(f(I))$ contains a matrix of rank 3 if $f(I)$ is 3-colorable. Now assume that there exists a rank-3 matrix, M say, fitting $f(I)$. We prove that $f(I)$ is 3-colorable as follows: Denote the 1-dimensional subspace of $\langle M \rangle$ spanned by the row vector of vertex i by V_i . We show that there are only three different spaces V_i inducing a 3-coloring of G that can be completed to a 3-coloring of $f(I)$.

Let for each $i \neq j \in \{1, 2, \dots, n\}$ $H_i(j)$ be the 2-dimensional subspace of $\langle M \rangle$ spanned by the row vectors of the vertices from H_{ij} that are not in V_i , then clearly $\langle M \rangle = V_i \oplus H_i(j)$. For any j different from i the coordinate corresponding to i of all vectors in $H_i(j)$ is zero, while this coordinate is non-zero for each non-zero vector from V_i . So $H_i(j)$ is the 2-dimensional subspace of $\langle M \rangle$ consisting of all vectors for which the coordinate corresponding to vertex i is zero. It follows that $H_i(j)$ is independent of j which defines for any i a 2-dimensional subspace H_i of $\langle M \rangle$.

Since for any two vertices i and j we have a subgraph H_{ij} , the two 1-dimensional subspaces V_i and V_j are the same if and only if $H_i = H_j$. If $V_i \neq V_j$ then $V_i \subset H_j$ and $V_j \subset H_i$, so if $V_i \neq V_j$ then all V_k different from V_i and V_j are in the 1-dimensional subspace $H_i \cap H_j$, so there are only three different V_i 's. ■

Corollary 3.1. *The RANK-3 problem is NP-complete for all the classes $\mathcal{A}_i(G, F)$ ($i=1, 2, 3$) with F a finite field.*

Remark 3.1. In [17] it is proved that RANK-3 is still NP-complete if restricted to planar graphs or to planar unit disc graphs.

Remark 3.2. In [15] the authors remarked that "It seems to be difficult to find the smallest dimension in which a given graph G has an orthonormal representation." (in $\mathbb{R}^d$ provided with the standard inner product). It follows from the above

theorems that if the problem of deciding whether or not a (planar) graph has an orthonormal representation in F^3 is in NP (which is the case if F is finite), it is NP-complete.

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