Tilburg University

Axiomatic characterizations of solutions for Bayesian games

van Heumen, R.; Peleg, B.; Tijs, S.H.; Borm, P.E.M.

Published in:
Theory and Decision

Publication date:
1996

[Link to publication in Tilburg University Research Portal](#)

Citation for published version (APA):
van Heumen, R., Peleg, B., Tijs, S. H., & Borm, P. E. M. (1996). Axiomatic characterizations of solutions for Bayesian games. *Theory and Decision*, 40(2), 103-129.

General rights

Copyright and moral rights for the publications made accessible in the public portal are retained by the authors and/or other copyright owners and it is a condition of accessing publications that users recognise and abide by the legal requirements associated with these rights.

- Users may download and print one copy of any publication from the public portal for the purpose of private study or research.
- You may not further distribute the material or use it for any profit-making activity or commercial gain
- You may freely distribute the URL identifying the publication in the public portal

Take down policy

If you believe that this document breaches copyright please contact us providing details, and we will remove access to the work immediately and investigate your claim.

ROBERT VAN HEUMEN, BEZALEL PELEG, STEF TIJS AND
PETER BORM

AXIOMATIC CHARACTERIZATIONS OF SOLUTIONS FOR BAYESIAN GAMES

ABSTRACT. Bayesian equilibria are characterized by means of consistency and one-person rationality in combination with non-emptiness or converse consistency. Moreover, strong and coalition-proof Bayesian equilibria of extended Bayesian games are introduced and it is seen that these notions can be characterized by means of consistency, one-person rationality, a version of Pareto optimality and a modification of converse consistency. It is shown that, in case of the strong Bayesian equilibrium correspondence, converse consistency can be replaced by non-emptiness. As examples we treat Bayesian potential games and Bayesian congestion games.

KEY WORDS: Bayesian games, consistency, converse consistency, potentials, congestion games.

1. INTRODUCTION

In Peleg and Tijs (1992), axiomatic characterizations are given for the Nash equilibrium correspondence on closed families of strategic games using consistency, converse consistency and one-person rationality. Also some refinements of the Nash correspondence are characterized and an indication is given that some of the results can be extended to Bayesian games.

In a subsequent paper Peleg, Potters and Tijs (1993) study the question under which conditions converse consistency can be replaced by the non-emptiness property. In this connection see also Norde, Potters, Reijnierse and Vermeulen (1993).

The purpose of this paper is to make a systematic study of axiomatizations for solutions of extended Bayesian games. Bayesian games were introduced by Harsanyi (1967), and extended Bayesian games by Einy and Peleg (1991). In Section 2 we give the necessary definitions. In Section 3 it is shown that the Bayesian equilibrium correspondence can be axiomatized by consistency and one-person rationality in combination with converse consistency or non-emptiness.

Section 4 introduces strong and coalition-proof Bayesian equilibria and both concepts are axiomatized by consistency, one person rationality and modifications of Pareto optimality and converse consistency. There is also a discussion on the definition of strong Bayesian equilibria and it is shown that, in order to characterize strong Bayesian equilibria, converse consistency can be replaced by non-emptiness. Finally, a modification of the strong Bayesian equilibrium correspondence is given. Section 5 extends the notion of potential game of Monderer and Shapley (1993) to Bayesian games and considers the existence of pure Bayesian equilibria. Also a congestion situation in the spirit of Rosenthal (1973) is considered, which gives rise to a Bayesian potential game.

2. EXTENDED BAYESIAN GAMES

In this section we formally describe the class of extended Bayesian games. This generalized form of ordinary Bayesian games enables us to define reduced Bayesian games.

2.1. DEFINITION (Einy and Peleg, 1991). An *extended Bayesian game (EBG)* is a system

$$G = \langle N, N^+, (A_i)_{i \in N}, (T_i)_{i \in N^+}, (p_i)_{i \in N}, (u_i)_{i \in N} \rangle$$

where

- (i) N is the (finite) set of *players*,
- (ii) N^+ is a finite set with $N^+ \supseteq N$ and $N^+ \setminus N$ is the set of *outside players*.
- (iii) for every $i \in N$, A_i is the set of *actions* of player i ,
- (iv) for every $i \in N^+$, T_i is the finite set of possible *types* of player i ,
- (v) for every $i \in N$, p_i is a probability distribution on $T := \prod_{k \in N^+} T_k$ which represents the *prior* of player i ,
- (vi) for every $i \in N$, $u_i : A \times T \rightarrow \mathbb{R}$ is the *utility-function* of player i , where $A := \prod_{k \in N} A_k$.

Note that, in case $N^+ = N$, we have an ordinary Bayesian game. If $N^+ \neq N$ then, intuitively, one may consider the outside players as those players who have already chosen their strategies (in a larger

Bayesian game). So an extended Bayesian game can be considered as a reduction of an ordinary Bayesian game.

From now on, if we mention an arbitrary Bayesian game G without further specification, we assume that G is the game $\langle N, N^+, (A_i)_{i \in N}, (T_i)_{i \in N^+}, (p_i)_{i \in N}, (u_i)_{i \in N} \rangle$. Sometimes we will refer to N_G as the player set belonging to the game G , to avoid confusion.

Let G be an EBG, and $i \in N$. A *strategy* of player i is a function $x_i : T_i \rightarrow A_i$. By X_i we denote the set of strategies of player i . If $S \subseteq N$, $S \neq \emptyset$, then $X_S := \prod_{k \in S} X_k$ and $X := X_N$.

Let Γ be a set of EBG's. A *solution* on Γ is a function ϕ that assigns to each game $G \in \Gamma$ a subset $\phi(G)$ of the space X of strategy profiles.

2.2. DEFINITION. Let G be an EBG and $S \subseteq N$, $S \neq \emptyset$, $x \in X$. The *reduced* Bayesian game of G with respect to S and x is given by

$$G^{S,x} = \langle S, N^+, (A_i)_{i \in S}, (T_i)_{i \in N^+}, (p_i)_{i \in S}, (u_i^x)_{i \in S} \rangle$$

where, for every $i \in S$, $a_S \in A_S$ and $t \in T$

$$u_i^x(a_S, t) := u_i(((x_k(t_k))_{k \in N \setminus S}, a_S), t).$$

(Note that $A_S = \prod_{k \in S} A_k$, $a_S = (a_k)_{k \in S}$ etc.)

A class Γ of EBG's is *closed* if for every $G \in \Gamma$ and for every $S \subseteq N$, $S \neq \emptyset$ and $x \in X$ it holds that $G^{S,x} \in \Gamma$. It is easy to see that the class of all EBG's is closed.

Now we consider an example of a Bayesian game.

2.3. Example

Let a two-person Bayesian game be given by $G = \langle \{1, 2\}, \{1, 2\}, A_1, A_2, T_1, T_2, p_1, p_2, u_1, u_2 \rangle$ where $A_1 = \{T, B\}$, $A_2 = \{L, R\}$, $T_1 = \{\alpha, \beta\}$, $T_2 = \{\gamma, \delta\}$ and $p_1(\alpha, \gamma) = p_1(\alpha, \delta) = p_1(\beta, \gamma) = p_1(\beta, \delta) = \frac{1}{4}$, $p_2(\alpha, \gamma) = p_2(\beta, \delta) = \frac{1}{2}$ and $p_2(\alpha, \delta) = p_2(\beta, \gamma) = 0$. The payoff-functions are denoted in Table I. So if player 1 is of type α and player 2 is of type γ , they play the upper left game. We denote a strategy of a player by a pair of actions. So

$$X_1 = \{TT, TB, BT, BB\}, \quad X_2 = \{LL, LR, RL, RR\},$$

TABLE I

		γ		δ	
		L	R	L	R
α	T	0, 0	1, 2	T	1, 0
	B	2, 1	0, 0	B	1, 0
β	T	1, 0	1, 0	T	1, 1
	B	0, 1	0, 1	B	3, 2

where for example TB is the strategy of player 1 in which he plays T if he is of type α and B if he is of type β .

Now this Bayesian game is an example of a so-called Bayesian potential game (for the definition we refer to Section 5), which means that for every pair of types the corresponding bimatrix game is an ordinary potential game, where a corresponding potential is described by Table II. We denote the given (potential) function by q , so for example,

$$q((T, R), (\alpha, \gamma)) = 2, \quad q((B, L), (\beta, \gamma)) = -1.$$

If just one player deviates, then the difference in the payoff for that player is indicated by the difference in the potential function. For example

$$\begin{aligned} u_1((T, L), (\beta, \gamma)) - u_1((B, L), (\beta, \gamma)) \\ = q((T, L), (\beta, \gamma)) - q((B, L), (\beta, \gamma)) \\ = 0 - (-1) = 1. \end{aligned}$$

We will elaborate on potential games in Section 5.

2.4. Remark

In the definition of extended Bayesian games, p_i is a probability distribution on T , for every $i \in N$. In the following definition we

TABLE II

		γ				δ		
			L	R			L	R
α	T		0	2		T	0	0
	B		2	1		B	0	0
β	T		0	0		T	0	1
	B		-1	-1		B	2	0

use, for every $i \in N$ and $t_i \in T_i$, the related probability distribution $p_i(\cdot|t_i)$ on $T^{-i} := \prod_{k \neq i} T_k$, defined by

$$p_i(t^{-i}|t_i) := \frac{p_i(t)}{\sum_{s^{-i} \in T^{-i}} p_i(t_i, s^{-i})}$$

for every $t^{-i} \in T^{-i}$. Note that $t = (t_i, t^{-i})$.

Of course this definition is meaningful only in the case that $\sum_{s^{-i}} p_i(t_i, s^{-i}) \neq 0$, for every $t_i \in T_i$, which means that every player puts positive probability on the occurrence of each of his types. In the sequel we shall assume that this is indeed the case.

2.5. DEFINITION. Let G be an EBG and $x \in X$. x is a *Bayesian equilibrium (BE)* of G if for all $i \in N$, $t_i \in T_i$ and $a_i \in A_i$:

$$\begin{aligned} & \sum_{t^{-i}} p_i(t^{-i}|t_i) u_i((x_j(t_j))_{j \in N}, t) \\ & \geq \sum_{t^{-i}} p_i(t^{-i}|t_i) u_i((x_j(t_j))_{j \in N \setminus \{i\}}, a_i, t). \end{aligned}$$

We denote $BE(G) := \{x \in X | x \text{ is a BE of } G\}$.

To shorten notation we define

$$U_i(x|t_i) := \sum_{t^{-i}} p_i(t^{-i}|t_i) u_i((x_j(t_j))_{j \in N}, t)$$

for every $x \in X$, $t_i \in T_i$. Then, for every $x \in X$:

$$x \in BE(G) \text{ iff for all } i \in N, t_i \in T_i, y_i \in X_i : \\ U_i(x|t_i) \geq U_i(x^{-i}, y_i|t_i).$$

In Section 5 we show that each Bayesian potential game with consistent priors has at least one Bayesian equilibrium.

In Example 2.3, the strategy tuple (TB, RL) is a Bayesian equilibrium, but the players do not have consistent priors.

3. AXIOMATIZATIONS OF THE BAYESIAN EQUILIBRIUM CORRESPONDENCE

In this section we give two different characterizations of the Bayesian equilibrium correspondence. The first one is based on consistency and converse consistency (cf. Peleg and Tijs, 1992), the second one on consistency and non-emptiness and uses in its proof the ancestor property (cf. Peleg, Potters and Tijs, 1993).

For a detailed discussion on the interpretation of consistency and converse consistency we refer to Peleg and Tijs (1992). Here we restrict ourselves to the general ideas. Roughly speaking a solution concept is called consistent if renegotiation of the (sub-) solution by subcoalitions on the basis of the same solution concept applied to an intuitively appealing reduced game, will lead to the same (sub-) outcome. The main idea behind converse consistency is the following. Given a game on player set N and a solution concept: if it is the case that an outcome fits with the prescribed solution for the corresponding reduced games on all appropriate subcoalitions of N , this outcome should be prescribed by the solution concept for the original game on N .

3.1. DEFINITION. Let Γ be a closed set of EBG's and ϕ a solution on Γ .

(i) ϕ satisfies *one-person rationality (OPR)* on Γ if

$$\phi(G) = \{x_i \in X_i | U_i(x_i|t_i) \geq U_i(y_i|t_i) \\ \text{for every } t_i \in T_i \text{ and } y_i \in X_i\}$$

for every one-player Bayesian game G with $N_G = \{i\}$ in Γ .

(ii) ϕ satisfies *consistency* (*CONS*) on Γ if for every game $G \in \Gamma$, for every coalition $S \subsetneq N$, $S \neq \emptyset$ and for every $x \in \phi(G)$ it holds that $x_S \in \phi(G^{S,x})$. With $\tilde{\phi}(G) := \{x \in X \mid \text{for every } S \subsetneq N, S \neq \emptyset : x_S \in \phi(G^{S,x})\}$, we have that ϕ satisfies *CONS* iff $\phi(G) \subseteq \tilde{\phi}(G)$ for every $G \in \Gamma$.

We will show that the Bayesian equilibrium solution satisfies *OPR* and *CONS*. However, *OPR* and *CONS* do not axiomatize *BE*. In Section 4 we will see that the strong Bayesian equilibrium solution (*SBE*) also satisfies *OPR* and *CONS*. To characterize *BE* we use the following property.

3.2. DEFINITION. We say that a solution ϕ satisfies *converse consistency* (*COCONS*) on a closed set Γ of EBG's if

$$\tilde{\phi}(G) \subseteq \phi(G) \quad \text{for every } G \in \Gamma \text{ with } |N| \geq 2.$$

For a detailed discussion of consistency and converse consistency we refer to Peleg and Tijs (1992).

3.3. LEMMA. Let Γ be a closed set of EBGs. Then *BE* satisfies *OPR*, *CONS* and *COCONS* on Γ .

Proof. (i) By definition *BE* satisfies *OPR*.

(ii) Let $G \in \Gamma$, $|N| \geq 2$ and $x \in BE(G)$. Let $S \subsetneq N$, $i \in S$ and $t_i \in T_i$. Then for every $y_i \in X_i$:

$$\begin{aligned} U_i^x(x_S | t_i) &= U_i(x | t_i) \geq U_i(x^{-i}, y_i | t_i) \\ &= U_i^x(x_{S \setminus \{i\}}, y_i | t_i) \end{aligned}$$

where

$$U_i^x(x_S | t_i) := \sum_{t^{-i}} p_i(t^{-i} | t_i) u_i^x((x_j(t_j))_{j \in S}, t).$$

Hence $x_S \in BE(G^{S,x})$, so $BE(G) \subseteq \tilde{BE}(G)$.

(iii) Let $G \in \Gamma$, $|N| \geq 2$ and $x \in X$ be such that $x_S \in BE(G^{S,x})$ for all $S \subsetneq N$, $S \neq \emptyset$. Take $i \in N$ and $t_i \in T_i$. Then $x_i \in BE(G^{\{i\},x})$ so for every $y_i \in X_i$:

$$\begin{aligned} U_i^x(x_i | t_i) &\geq U_i^x(y_i | t_i) \quad \text{and hence} \\ U_i(x | t_i) &\geq U_i(x^{-i}, y_i | t_i). \end{aligned}$$

So $x \in BE(G)$ and $\tilde{BE}(G) \subseteq BE(G)$. ■

In fact, *OPR*, *CONS* and *COCONS* characterize *BE*, as the next theorem shows.

3.4. THEOREM. *Let ϕ be a solution on a closed set Γ of EBGs. Then ϕ satisfies *OPR*, *CONS* and *COCONS* iff $\phi(G) = BE(G)$ for every $G \in \Gamma$.*

Proof. We give a proof of the ‘only if’-part by induction on the number of players. Suppose ϕ satisfies *OPR*, *CONS* and *COCONS*.

- Let G be a one person game in Γ . Then $\phi(G) = BE(G)$ by *OPR* of ϕ and *BE*.
- Let $k \in \{2, 3, 4, \dots\}$ be such that, for every $G \in \Gamma$ with less than k players, we have that $BE(G) = \phi(G)$, and let G be a k -person game in Γ . Then we have

$$\begin{array}{ccccccc} \phi(G) & \subseteq & \tilde{\phi}(G) & = & \widetilde{BE}(G) & \subseteq & BE(G) \\ & & \text{CONS of } \phi & & \text{(induction hyp.)} & & \text{(COCONS of BE)} \end{array}$$

and

$$\begin{array}{ccccccc} BE(G) & \subseteq & \widetilde{BE}(G) & = & \tilde{\phi}(G) & \subseteq & \phi(G). \\ & & \text{CONS of BE} & & \text{(induction hyp.)} & & \text{(COCONS of } \phi) \end{array}$$

So $\phi(G) = BE(G)$. ■

For the second characterization we introduce, for every set Γ of EBGs and every solution ϕ on Γ , a directed graph $Graph(\Gamma, \phi)$. The vertices of this graph are pairs (G, x) where $G \in \Gamma$ and $x \in \phi(G)$. There is an edge from (G, x) to (H, y) if $N_H \subsetneq N_G$, $H = G^{N_H, x}$ and $y = x_{N_H}$. In this case we call (G, x) an *ancestor* of (H, y) .

3.5. DEFINITION. Let Γ be a closed class of EBG’s and ϕ a solution on Γ . The graph $Graph(\Gamma, \phi)$ satisfies the *ancestor property (AP)* if for every vertex (H, y) there is a $G \in \Gamma$ such that $\phi(G) \neq \emptyset$ and (G, x) is an ancestor of (H, y) for every $x \in \phi(G)$.

3.6. DEFINITION. (i) ϕ satisfies *non-emptiness (NEM)* on Γ if $\phi(G) \neq \emptyset$ for every $G \in \Gamma$.

(ii) ϕ is *minimal w.r.t NEM, OPR and CONS* if ϕ satisfies these properties and for every solution $\bar{\phi}$ with $\bar{\phi} \subseteq \phi$ on Γ which satisfies *NEM, OPR and CONS*, we have that $\bar{\phi} = \phi$.

These definitions are due to Peleg, Potters and Tijs (1993).

3.7. LEMMA. *For every closed class Γ of EBG's and every solution ϕ on Γ satisfying NEM, OPR and CONS: if $\text{Graph}(\Gamma, \phi)$ satisfies AP, then ϕ is minimal w.r.t. NEM, OPR and CONS.*

Proof. Let ϕ and $\bar{\phi}$ be solutions on Γ that satisfy *NEM, OPR and CONS* such that $\bar{\phi} \subset \phi$. Assume, moreover, that $\text{Graph}(\Gamma, \phi)$ satisfies AP. Let H be any game of Γ and let $y \in \phi(H)$. It suffices to prove that $y \in \bar{\phi}(H)$.

By definition of AP there is a game $G \in \Gamma$ such that $\phi(G) \neq \bar{\phi}$ and

$$N_H \subsetneq N_G, \quad H = G^{N_H, x} \quad \text{and} \quad y = x_{N_H}$$

for all $x \in \phi(G)$.

By *NEM* we have that $\bar{\phi}(G)$ contains at least one element z . Since $\bar{\phi}(G) \subset \phi(G)$ it follows that $z \in \phi(G)$ and, consequently,

$$y = z_{N_H} \in \phi(G^{N_H, z}) = \bar{\phi}(H)$$

by *CONS* of $\bar{\phi}$. ■

This lemma has an interesting application if we take $\phi = BE$ and $\Gamma = \Gamma^{BE}$ (the class of all EBGs which have at least one *BE*). We already know (see the proof of Theorem 3.4) that a solution $\bar{\phi}$ which satisfies *OPR* and *CONS* on Γ^{BE} is contained in *BE*. If $\bar{\phi}$ also satisfies *NEM* and if we can prove that $\text{Graph}(\Gamma^{BE}, BE)$ has the ancestor property, then $\bar{\phi} = BE$, so *BE* is characterized on Γ^{BE} by *NEM, OPR and CONS*.

3.8. THEOREM. *Let ϕ be a solution on Γ^{BE} . Then ϕ satisfies NEM, OPR and CONS iff $\phi(G) = BE(G)$ for every $G \in \Gamma^{BE}$.*

Proof. We prove the 'only if' part.

By Lemma 3.7, it suffices to prove that $\text{Graph}(\Gamma^{BE}, BE)$ satisfies AP. Let $G \in \Gamma^{BE}$ and $x \in BE(G)$. We construct a game $H \in \Gamma^{BE}$ with $N_H \supsetneq N_G$ such that H has exactly one Bayesian equilibrium

y with the property that $y_{N_G} = x$ and $H^{N_G, y} = G$. Define $H := \langle N_G \cup \{0\}, N_G^+ \cup \{0\}, (A_i^*)_{i \in N_G \cup \{0\}}, (T_i^*)_{i \in N_G^+ \cup \{0\}}, (p_i^*)_{i \in N_G \cup \{0\}}, (u_i^*)_{i \in N_G \cup \{0\}} \rangle$, where

- $A_0^* := \{\alpha, \beta\}, \quad A_i^* := A_i \quad (i \in N_G),$
- $T_0^* := \{t_0\}, \quad T_i^* := T_i \quad (i \in N_G^+),$
- for every $i \in N_G$:

$$p_i^*(t, t_0) := p_i(t) \quad \text{for every } t \in T$$

and:

$$\begin{aligned} p_0^*(s, t_0) &:= 1 \quad \text{for a fixed } s \in T \\ p_0^*(t, t_0) &:= 0 \quad \text{for every } t \in T \setminus \{s\}, \end{aligned}$$

- for every $a \in A$ and every $i \in N_G, t \in T$:

$$\begin{aligned} u_i^*((a, \alpha), (t, t_0)) &:= u_i(a, t) \\ u_i^*((a, \beta), (t, t_0)) &:= -1 \quad \text{if } a_i \neq x_i(t_i) \\ u_i^*((a, \beta), (t, t_0)) &:= 1 \quad \text{if } a_i = x_i(t_i) \end{aligned}$$

and:

$$\begin{aligned} u_0^*((a, \alpha), (t, t_0)) &:= 2 \quad \text{if } a = (x_1(t_1), \dots, x_n(t_n)) \\ u_0^*((a, \alpha), (t, t_0)) &:= -1 \quad \text{if } a \neq (x_1(t_1), \dots, x_n(t_n)) \\ u_0^*((a, \beta), (t, t_0)) &:= 0. \end{aligned}$$

Since player 0 has only one type, we can identify a strategy of player 0 with an action.

Claim: (x, α) is the unique BE in H .

Proof of the claim. Let $(y, \beta) \in X \times A_0^*$ and $y \neq x$. Choose $i \in N_G, t_i \in T_i$ such that $y_i(t_i) \neq x_i(t_i)$. Then

$$\begin{aligned} &U_i^*((y, \beta)|t_i) \\ &= \sum_{t^{-i} \in T^{-i}} p_i^*(t^{-i}, t_0|t_i) u_i^*((y_j(t_j))_{j \in N_G}, \beta), (t, t_0)) \\ &= \sum_{t^{-i} \in T^{-i}} p_i^*(t^{-i}, t_0|t_i) \cdot -1 < \sum_{t^{-i} \in T^{-i}} p_i^*(t^{-i}, t_0|t_i) \\ &= \sum_{t^{-i} \in T^{-i}} p_i^*(t^{-i}, t_0|t_i) u_i^*((y_j(t_j))_{j \in N_G \setminus \{i\}}, x_i(t_i), \beta), (t, t_0)) \\ &= U_i^*((y^{-i}, x_i, \beta)|t_i). \end{aligned}$$

So player i can profitably deviate, which shows that $(y, \beta) \notin BE(H)$.

Also $(x, \beta) \notin BE(H)$, because

$$\begin{aligned} U_0^*((x, \beta)|t_0) &= \sum_{t \in T} p_0^*(t|t_0) u_0^*((x_i(t_i))_{i \in N_G}, \beta), (t, t_0)) \\ &< \sum_{t \in T} p_0^*(t|t_0) u_0^*((x_i(t_i))_{i \in N_G}, \alpha), (t, t_0)) \\ &= U_0^*((x, \alpha)|t_0). \end{aligned}$$

Now let $(y, \alpha) \in X \times A_0^*$ and $y \neq x$. Then

$$\begin{aligned} U_0^*((y, \alpha)|t_0) &= \sum_{t \in T} p_0^*(t|t_0) u_0^*((y_i(t_i))_{i \in N_G}, \alpha), (t, t_0)) \\ &= \sum_{t \in T} p_0^*(t|t_0) \cdot -1 < \sum_{t \in T} p_0^*(t|t_0) \cdot 0 \\ &= \sum_{t \in T} p_0^*(t|t_0) u_0^*((y_i(t_i))_{i \in N_G}, \beta), (t, t_0)) \\ &= U_0^*((y, \beta)|t_0). \end{aligned}$$

So player 0 can profitably deviate, and $(y, \alpha) \notin BE(H)$.

Clearly $(x, \alpha) \in BE(H)$, which finishes the proof. ■

4. STRONG AND COALITION PROOF BAYESIAN EQUILIBRIA

In this section we define and characterize strong Bayesian equilibria and coalition-proof Bayesian equilibria which are generalizations of strong Nash equilibria (see Aumann, 1959) and coalition-proof Nash equilibria (see Bernheim, Peleg and Whinston, 1987) for games in strategic form. We also discuss the definitions and provide some modifications.

4.1. DEFINITION. Let G be an EBG and $x \in BE(G)$. x is a *strong Bayesian equilibrium (SBE)* if there is no coalition $S \subset N$, $S \neq \emptyset$, which has an *improvement upon* x , which means that there is no $y_S \in X_S$ such that, for all $i \in S$, $t_i \in T_i$:

$$U_i(x|t_i) < U_i(x_{N \setminus S}, y_S|t_i).$$

By $SBE(G)$ we denote the set of strong Bayesian equilibria of G .

TABLE III

	α		β	
	L	R	L	R
T	0	1	2	0

This definition is due to Ichiishi and Idzik (1992). Ichiishi and Idzik investigate Bayesian societies, which are more general than Bayesian games and allow binding agreements. However, their definition, when applied to Bayesian games, is essentially the same as our Definition 4.1.

We explicitly defined *SBE* as a refinement of *BE*. In order to get *SBE*(G) as a subset of *BE*(G), it is not sufficient to define *SBE*(G) as the set of all strategy combinations which cannot be improved upon, as the next example shows.

4.2. Example. Let $N = N^+ = \{1, 2\}$, $A_1 = \{T\}$, $A_2 = \{L, R\}$, $|T_1| = 1$, $T_2 = \{\alpha, \beta\}$ and the payoff functions u_1 and u_2 as denoted in Table III. Note that $u_1 = u_2$. The priors are arbitrary.

The strategy combination (T, LL) is no *BE*, because $U_2(T, LL|\alpha) < U_2(T, RL|\alpha)$. However, (T, LL) cannot be improved upon: if there would be an improvement it can only be (T, RL) , but in that case only type α of player 2 profits, while type β gets the same amount.

Two questions may be raised with respect to the validity of the definition of *SBE*.

(i) Let $x \in X$ and let y_S be an improvement of a coalition S upon x . Can S choose to play y_S without changing the beliefs of its members? Would the beliefs change then the payoffs to the types of the players in S would also change. Therefore, some members of S might no longer prefer $(y_S, x_{N \setminus S})$ to x after y_S is chosen.

The answer is simple. Because all the members of S know x and y_S and all the types of all members of S prefer $(y_S, x_{N \setminus S})$ to x , S can choose to play y_S without a change in the beliefs. In order to be

TABLE IV

	L	R
T	2, 2	0, 3
B	3, 0	1, 1

completely precise we supply the following simple model for choice (by the members of S) between x_S and y_S .

Let $G^* = \langle S, N^+, (A_i^*)_{i \in S}, (T_i)_{i \in N^+}, (p_i)_{i \in S}, (u_i^*)_{i \in S} \rangle$ be the following EBG: $A_i^* = \{y_S, x_S\}$ for all $i \in S$ and $u_i^* : A^* \times T \rightarrow \mathbb{R}$ for all $i \in S$ is given by

$$u_i^*((a_j^*)_{j \in S}, t) = \begin{cases} U_i(x_{N \setminus S}, y_S | t_i) & \text{if } a_j^* = y_S \text{ for all } j \in S \\ U_i(x | t_i) & \text{otherwise.} \end{cases}$$

We shall say that y_S is chosen by S if each $i \in S$ plays in G^* the strategy y_i^* , where $y_i^*(t_i) := y_S$ for every $t_i \in T_i$. As the reader may easily verify, y_S^* is dominant in G^* and the posterior probability attributed by a player $i \in S$ to an n -tuple t of types does not change when y_S is chosen by S (in the game G^*).

(ii) The second question is more subtle. When will S indeed choose y_S over x_S ? The obvious answer is that y_S will be chosen by S because it is an improvement upon x . However, we shall show, by means of two examples, that this is true only if the players are short-sighted. This kind of criticism of the *SBE* does not directly stem from the incomplete information environment; it even applies to *SNE* for games with complete information. Indeed, our two examples are games with complete information.

4.3. Example. We consider the following (numerical) version of the prisoner's dilemma (Table IV).

This game has no strong Nash equilibrium. The unique Nash equilibrium (B, R) can be improved upon by (T, L) . However, (T, L) is not a *NE* and therefore it cannot be implemented. In this sense the rejection of (B, R) in favor of (T, L) is short-sighted.

TABLE V

		L	R		
T		0, 0, 0	0, 0, 0	T	
B		0, 0, 0	0, 0, 0	B	
		M_1			
			L	R	
T			2, 2, -1	-1, -1, -1	T
B			-1, -1, -1	1, 1, 1	B
			M_2		

The reader might think that players who want to implement improvements that are also *NEs*, are not short-sighted. The following example shows that this is not necessarily true.

4.4. Example

Consider the three-person game with complete information shown in Table V.

Here $x = (T, L, M_1)$ is not an *SNE* because $y = (B, R, M_2)$ is an improvement. Now y is a strict *NE* which is Pareto optimal. However, from the point of view of player 3, playing y is short-sighted behaviour. Indeed, if players 1 and 2 know that 3 will play M_2 , then they will play (T, L) .

The basic assumption underlying the definitions is the following. During the interim phase of a Bayesian game (i.e. when the players know their types but have not yet implemented their strategies) transmission of information between the players is not allowed.

Thus, in our model the players may communicate in order to coordinate the choice of strategies, but they are not allowed to reveal any part of private information to each other. Under these conditions *SBEs* may be the only stable points of G if the players are sufficiently short-sighted. As discussed, the above assumption does not eliminate short-sighted behaviour.

The foregoing assumption is common in applications of BGs to economics. Transmission of information or signaling is usually done by the actual use of strategies (or local strategies in extensive form games).

As an illustration we consider the following simple game.

TABLE VI

		γ		δ	
		L	R	L	R
α	T	-1, 2	-1, 3	T	2, 5
	B	1, -1	1, 1	B	-1, -1

4.5. *Example.* Let $N = N^+ = \{1, 2\}$, $A_1 = \{T, B\}$, $A_2 = \{L, R\}$, $T_1 = \{\alpha\}$, $T_2 = \{\gamma, \delta\}$, $p_1(\alpha, \gamma) = 0.1$, $p_1(\alpha, \delta) = 0.9$, $p_2(\alpha, \gamma) = p_2(\alpha, \delta) = 0.5$ and u_1 and u_2 are given by the matrices shown in Table VI.

Let $x = (B, LR)$. Then $y = (T, RL)$ is an improvement upon x by N . However, if player 2 is of type γ and his type is somehow revealed to player 1 (before y is implemented), then y is no longer an improvement upon x for N . By the foregoing assumption player 2 is not allowed to tell his type to player 1 and vice versa. By the foregoing discussion y can be implemented without revelation of information.

4.6. **DEFINITION.** A solution ϕ on a set Γ of EBGs satisfies *weak Pareto-optimality (WPO)* if, for all $G \in \Gamma$, no strategy combination in $\phi(G)$ can be improved upon by N (see also Definition 4.1).

The next lemma shows that *SBE* satisfies *OPR*, *WPO* and (on closed sets) *CONS*.

4.7. **LEMMA.** Let Γ be a closed set of EBGs. Then *SBE* satisfies *OPR*, *WPO* and *CONS* on Γ .

Proof. (i) Let $G = \langle \{i\}, \{i\}^+, A_i, (T_j)_{j \in \{i\}^+}, p_i, u_i \rangle$ be a one-person game in Γ . To prove *OPR* it suffices to show that $SBE(G) = BE(G)$, because *BE* satisfies *OPR*.

Let $x_i \in BE(G)$. Then for every $t_i \in T_i$, $y_i \in X_i$:

$$U_i(x_i|t_i) \geq U_i(y_i|t_i)$$

so $\{i\}$ does not have an improvement upon x_i , hence $x_i \in SBE(G)$.

(ii) Let $G \in \Gamma$, $x \in SBE(G)$ and $S \subset N$, $S \neq \emptyset$. We prove that $x_S \in SBE(G^{S,x})$. We know that $x_S \in BE(G^{S,x})$ because $x \in BE(G)$ and BE satisfies *CONS*. Suppose $R \subset S$ is a coalition which has an improvement y_R upon x_S in $G^{S,x}$. Then y_R is also an improvement upon x in G , which contradicts the fact that $x \in SBE(G)$. Hence there is no coalition which has an improvement upon x . So SBE satisfies *CONS*. $\blacksquare$

Since, in general, the sets of Bayesian equilibria and strong Bayesian equilibria do not coincide, we know that SBE will not satisfy *COCONS*. However, we can formulate a weaker version of converse consistency that is satisfied by SBE .

4.8. DEFINITION. Let ϕ be a solution on a closed set of EBGs. Then ϕ satisfies *COCONS-S* if for every $G \in \Gamma$ with $|N| \geq 2$ and every $x \in X$: if $x \in \tilde{\phi}(G)$ and x cannot be improved upon by N , then $x \in \phi(G)$.

4.9. LEMMA. Let Γ be a closed set of EBGs. Then SBE satisfies *COCONS-S* on Γ .

Proof. Let $G \in \Gamma$, $|N| \geq 2$ and $x \in \widetilde{SBE}(G)$ such that x cannot be improved upon by N . Then in particular $x \in \widetilde{BE}(G)$ and consequently $x \in BE(G)$ since BE satisfies *COCONS*.

Suppose $S \subsetneq N$ is a coalition which has an improvement y_S upon x . Then y_S is also an improvement upon x_S in $G^{S,x}$, which contradicts the fact that $x \in \widetilde{SBE}(G)$. Because x cannot be improved by N either, there is no coalition which has an improvement upon x and we may conclude that $x \in SBE(G)$. $\blacksquare$

4.10. THEOREM. Let Γ be a closed set of EBGs. There is a unique solution on Γ that satisfies *OPR*, *WPO*, *CONS* and *COCONS-S*, and it is SBE .

Proof. From the previous lemmas we know that SBE satisfies *OPR*, *WPO*, *CONS* and *COCONS-S*. Now let ϕ be a solution on Γ

that satisfies the foregoing four axioms. We prove by induction on the number of players that $\phi(G) = SBE(G)$ for every $G \in \Gamma$.

- If G is a one-person game, then by *OPR*, $\phi(G) = SBE(G)$.
- Now assume $k \in \{2, 3, \dots\}$ and that $\phi(G) = SBE(G)$ for every $G \in \Gamma$ with less than k players. Let $G \in \Gamma$ be a k -person game and let $x \in \phi(G)$. By *CONS* of ϕ , $x \in \tilde{\phi}(G)$ and by induction, $x \in \tilde{SBE}(G)$. Hence by *WPO* of ϕ and *COCONS-S* of *SBE*, $x \in SBE(G)$. Thus $\phi(G) \subset SBE(G)$. Similarly, one can prove that $SBE(G) \subset \phi(G)$. ■

Now we provide a characterization of *SBE* using non-emptiness in the spirit of the ancestor property of Definition 3.5. Let Γ^{SBE} denote the set of EBGs which have at least one *SBE*.

4.11. THEOREM. *Let ϕ be a solution on Γ^{SBE} . Then ϕ satisfies NEM, OPR, CONS and WPO iff $\phi = SBE$.*

Proof. We know that a solution which satisfies *OPR*, *CONS* and *WPO*, is a refinement of *SBE* (see the proof of Theorem 4.10). We are left to prove that $Graph(\Gamma^{SBE}, SBE)$ has the ancestor property.

If $G \in \Gamma^{SBE}$ and $x \in SBE(G)$, we can consider the same H as in the proof of Theorem 3.8. It immediately follows that the unique Bayesian equilibrium (x, α) is also an *SBE*. ■

Now we define coalition-proof Bayesian equilibria.

4.12. DEFINITION. Let G be an EBG.

- (i) Let $x \in BE(G)$ and $S \subset N$, $S \neq \emptyset$. We define an *internally consistent improvement (ICI)* of S upon x in G by induction on $|S|$.
 - If $S = \{i\}$, then $y_i \in X_i$ is an *ICI* of S upon x if it is an improvement (see Definition 4.1).
 - If $|S| > 1$, then $y_S \in X_S$ is an *ICI* of S upon x if y_S is an improvement upon x and no coalition $T \subsetneq S$, $T \neq \emptyset$ has an *ICI* upon $(y_S, x_{N \setminus S})$.
- (ii) Let $x \in BE(G)$. x is a *coalition-proof Bayesian equilibrium (CPBE)* if no coalition has an *ICI* upon x .

For two-person games we have the following characterization.

4.13. LEMMA. *Let G be a two-person EBG. Then, for every $x \in X$: $x \in CPBE(G)$ iff $x \in BE(G)$ and the grand coalition N has no improvement y upon x , such that $y \in BE(G)$.*

Proof. Let $x \in CPBE(G)$. By definition $x \in BE(G)$. Therefore $\{1\}$ and $\{2\}$ have no improvements upon x . Suppose N has an improvement $y \in BE(G)$ upon x . Then y is also internally consistent, because $y \in BE(G)$. This contradicts the fact that $x \in CPBE(G)$. For the converse case, it suffices to note that if N has an *ICI* y upon x , then $y \in BE(G)$. ■

Using the following modified forms of Pareto optimality and converse consistency, we are able to provide an axiomatic characterization for the coalition-proof Bayesian equilibrium correspondence.

4.14. DEFINITION. Let ϕ be a solution on a closed set Γ of EBGs.

- (i) ϕ satisfies *relative Pareto-optimality (RPO)* if for every $G \in \Gamma$: if $x \in \phi(G)$ then there is no $y \in \tilde{\phi}(G)$ which is an improvement of N upon x .
- (ii) ϕ satisfies *COCONS-CP* if for every $G \in \Gamma$: if $x \in \tilde{\phi}(G)$ and there is no $y \in \tilde{\phi}(G)$ which is an improvement of N upon x , then $x \in \phi(G)$.

It is not difficult to prove that *CPBE* satisfies *OPR*, *RPO* and *CONS*. In the next lemma we prove that *CPBE* also satisfies *COCONS-CP*.

4.15. LEMMA. *Let Γ be a closed set of EBGs. Then *CPBE* satisfies *COCONS-CP* on Γ .*

Proof. Let $G \in \Gamma$ and $x \in X$. Suppose $x \in \widetilde{CPBE}(G)$ and suppose $x \notin CPBE(G)$. We show that there is an $y \in CPBE(G)$ which is an improvement of N upon x . Choose a coalition $S \subset N$ which has an *ICI* y_S upon x . Then y_S is also an *ICI* of S upon x_S in $G^{S,x}$.

If $S \neq N$ then we have a contradiction because $x \in \widetilde{CPBE}(G)$, so $S = N$. Clearly $y = y_N$ is an improvement upon x , moreover $y \in \widetilde{CPBE}(G)$ since $y_R \in CPBE(G^{R,y})$ for all $R \subsetneq N$. For, if $T \subsetneq R$

has an *ICI* z_T upon y_R in $G^{R,y}$ then z_T is also an *ICI* of T upon y in G , which contradicts the fact that y is an *ICI* upon x in G . ■

For the proof of the following theorem we refer to the analogue in Peleg and Tijs (1992) for coalition-proof Nash equilibria.

4.16. THEOREM. *Let Γ be a closed set of EBGs. Then there is a unique solution on Γ that satisfies OPR, RPO, CONS and COCONS-CP, and it is the CPBE.*

We conclude this section with a modification of the definition of strong Bayesian equilibrium.

4.17. DEFINITION. Let G be an EBG and $x \in X$. x is a *strictly strong Bayesian equilibrium (SSBE)* if there is no coalition $S \subset N$, $S \neq \emptyset$ which has a *weak improvement upon x* , which means that there is no $y_S \in X_S$ such that, for all $i \in S$, $t_i \in T_i$: $U_i(y_S, x_{N \setminus S} | t_i) \geq U_i(x | t_i)$ and there is at least one $i \in S$ such that for all $t_i \in T_i$: $U_i(y_S, x_{N \setminus S} | t_i) > U_i(x | t_i)$.

In the definition of *SSBE* we look at a coalition in which at least one player gains *in every type*. One can also imagine a concept in which every player in a certain coalition gains *in at least one type*. It will not be difficult to characterize these concepts by *OPR*, *CONS* and slightly modified versions of Pareto optimality and converse consistency.

Finally, we want to mention that some of the Theorems of Section 3 and 4 can be strengthened by replacing ‘closedness’ by ‘ ϕ -closedness’, where ϕ is the solution in question. We call a set Γ of EBGs *ϕ -closed* if, for every $G \in \Gamma$, every $S \subset N$, $S \neq \emptyset$ and every $x \in \phi(G)$ it holds that $G^{S,x} \in \Gamma$.

5. BAYESIAN POTENTIAL GAMES

In this section we introduce a specific closed class of EBG’s, namely the class of Bayesian potential games. Bayesian potential games are

generalizations of (strategic) potential games, introduced by Monderer and Shapley (1993). It turns out that, under a special condition on the priors, each Bayesian potential game has a pure Bayesian equilibrium.

5.1. DEFINITION. Let G be an EBG. G is a *Bayesian potential game (BPG)* if there exists a function $q : A \times T \rightarrow \mathbb{R}$ such that, for every $i \in N$, $a \in A$, $b_i \in A_i$ and $t \in T$

$$u_i(a, t) - u_i((a^{-i}, b_i), t) = q(a, t) - q((a^{-i}, b_i), t).$$

Such a function q is called a *potential* for G .

One can easily verify that the class of BPG's is closed. More precisely: if G is a BPG with potential q , $S \subset N$ and $x \in X$, then $q^x : A_S \times T \rightarrow \mathbb{R}$ defined by

$$q^x(a_S, t) := q(((x_i(t_i))_{i \in N \setminus S}, a_S), t) \quad (a_S \in A_S, t \in T)$$

is a potential for $G^{S,x}$.

Moreover, if G is a BPG with potential q then, in order to determine the set of Bayesian equilibria of G we can replace each player's utility function by q .

5.2. DEFINITION. Let G be an EBG.

- (i) We say that G has *consistent priors* if each player has the same prior p on T . If G has consistent priors, we write

$$G = \langle N, N^+, (A_i)_{i \in N}, (T_i)_{i \in N^+}, p, (u_i)_{i \in N} \rangle.$$

- (ii) We define the *ex ante game* $\hat{G}$ as the strategic game $\langle X_1, \dots, X_n, \hat{u}_1, \dots, \hat{u}_n \rangle$ where, for every $i \in N$, X_i is the set of pure strategies of player i , and for every $x \in X$, $i \in N$:

$$\hat{u}_i(x) := \sum_{t \in T} p_i(t) u_i((x_j(t_j))_{j \in N}, t).$$

Note that the *ex ante game* $\hat{G}$ is a potential game if G is a Bayesian potential game with consistent priors.

Fig. 1.

We mention the following important relation between an EBG and the corresponding ex ante game. For the proof we refer to Harsanyi (1967), Part II, Theorem I.

5.3. THEOREM. *Let G be an EBG with consistent priors. Then, for every $x \in X$, x is a Bayesian equilibrium of G if and only if x is a Nash equilibrium of the ex ante game $\hat{G}$.*

5.4. COROLLARY. *Let G be a BPG with consistent priors, such that every type has positive probability. Then $BE(G) \neq \emptyset$.*

Proof. If G is a BPG, then $\hat{G}$ is a potential game, so $\hat{G}$ has a NE (see Monderer and Shapley, 1993, Corollary 2.3). Hence G has a BE. ■

5.5. Example. (A congestion situation: cf. Rosenthal, 1973; Monderer and Shapley, 1993.) We consider a situation, corresponding to the network in Figure 1.

This network gives rise to a two-person Bayesian potential game with consistent priors. Suppose player 2 lives in A and has to go to C , either directly using road AC or via the detour ABC . Suppose player 1 lives in C and has to go with probability $\frac{1}{2}$ to A (using the road CA or CBA) and with probability $\frac{1}{2}$ to B (using CB or CAB).

These probabilities are common knowledge to both players. Suppose that if one player uses a road AC , BC or AB he has to pay 2 units and if two players use the same road, then both of them have to pay 8 units. Suppose further that the reward for player 2 going to C is 200 and that the reward for player 1 is 100 (or 50) if he goes to A (or B). This situation corresponds to the following Bayesian

TABLE VII

		type C		type C	
		AC	ABC	AC	ABC
type A	CA	92, 192	98, 196	CA	290
	CBA	96, 198	84, 184	CBA	294
	CB	-2, 198	-8, 190	CB	196
	CAB	-10, 192	-10, 190	CAB	188
		AC	ABC	AC	ABC
type B	CA	-8, 192	-2, 196	CA	190
	CBA	-4, 198	-16, 184	CBA	194
	CB	48, 198	42, 190	CB	246
	CAB	40, 192	40, 190	CAB	238
		AC	ABC	AC	ABC
		190	194	194	180
		246	238	238	236
		238	236		
		the Bayesian game		a potential	

potential game:

$$N = N^+ = \{1, 2\}, \quad A_1 = \{CA, CBA, CB, CAB\}, \\ A_2 = \{AC, ABC\}, \quad T_1 = \{A, B\}, \quad T_2 = \{C\},$$

where, e.g., A is interpreted as 'player 1 has to go to A '. Further $p_i(A, C) = p_i(B, C) = \frac{1}{2}$ for $i \in \{1, 2\}$ and u_1, u_2 and a potential are given in Table VII.

So, for example $u_1((CBA, ABC), (A, C)) = 84$ and $u_2((CBA, ABC), (A, C)) = 184$ which we obtain as follows. Player 1 and 2 obtain a reward of 100 and 200, respectively, but both have costs 16 because both use the roads CB and BA . The Bayesian game in Table VII gives rise to a 16×2 -*ex ante* bimatrix game. In Table VIII we only give the relevant 4×2 -bimatrix, leaving out 12 dominated rows, and also a 'knotted' 4×2 -potential game.

Note that the unique pure Nash equilibrium $((CBA, CB), AC)$ corresponds to the following behaviour in the network: player 2 goes straight to his goal C using AC and player 1 goes straight to his goal

TABLE VIII

	<i>AC</i>	<i>ABC</i>		<i>AC</i>	<i>ABC</i>
(CA, CB)	70, 195	70, 193	(CA, CB)	268	266
(CA, CAB)	66, 192	69, 193	(CA, CAB)	264	265
(CBA, CB)	72, 198*	63, 187	(CBA, CB)	270*	259
(CBA, CAB)	68, 195	62, 187	(CBA, CAB)	266	258

the 'knotted' ex ante game	a 'knotted' potential
----------------------------	-----------------------

B , using CB , if he is of type B ; otherwise player 1 goes to A making the detour CBA . This Nash equilibrium corresponds in the original game to the Bayesian equilibrium (x_1, x_2) where $x_1(A) = CBA$, $x_1(B) = CB$ and $x_2(C) = AC$.

For a systematic study of congestion situations we refer to Tijs *et al.* (1994).

The following example shows that the *ex ante* game of a BPG which does not have consistent priors need not be a potential game. This shows that one cannot follow the same line of reasoning as in Corollary 5.4 to prove that BPGs with inconsistent priors have a *BE*.

5.6. Example. Let $G = \langle \{1, 2\}, \{1, 2\}, A_1, A_2, T_1, T_2, p_1, p_2, u \rangle$ be the BPG defined by $A_1 := \{T, B\}$, $A_2 := \{L, R\}$, $T_1 := \{\alpha, \beta\}$, $T_2 := \{\gamma, \delta\}$ and p_1, p_2 and a potential q given by the matrices in Table IX.

The corresponding *ex ante* game is given in Table X.

This game is not a potential game, because we have the following cycle of improvements:

$$(TT, LL) \rightarrow (TT, RR) \rightarrow (BB, RR) \rightarrow (BB, LL) \\ \rightarrow (TT, LL).$$

So if q were a potential for this game, we would have that

TABLE IX

		$\gamma \quad \delta$				$\gamma \quad \delta$	
$p_1 :$	α	$\frac{1}{2}$	0	$p_2 :$	α	0	$\frac{1}{2}$
	β	0	$\frac{1}{2}$		β	$\frac{1}{2}$	0
$q :$		$\boxed{\gamma}$				$\boxed{\delta}$	
		$L \quad R$				$L \quad R$	
$\boxed{\alpha}$	T	1	0	T	0	0	
	B	0	0		B	1	0
$\boxed{\beta}$	T	0	1	T	0	0	
	B	0	0		B	0	1

$$q(TT, LL) < q(TT, RR) < q(BB, RR) < q(BB, LL) < q(TT, LL)$$

which is a contradiction.

Note that the BPG in Example 5.6 does have a *BE*, for example (TB, LL) . The following example shows that a *three-person* BPG with inconsistent priors need not have Bayesian equilibria.

5.7. Example. Let $N = N^+ = \{1, 2, 3\}$, $A_1 = \{T, B\}$, $A_2 = \{L, R\}$, $A_3 = \{M\}$ and $T_1 = \{\gamma\}$, $T_2 = \{\delta\}$, $T_3 = \{\alpha, \beta\}$. Also $p_1(\gamma, \delta, \alpha) = p_2(\delta, \gamma, \beta) = 1$, $p_1(\gamma, \delta, \beta) = p_2(\gamma, \delta, \alpha) = 0$, $p_3(\gamma, \delta, \alpha) = p_3(\gamma, \delta, \beta) = \frac{1}{2}$.

A potential is defined in Table XI. One can easily verify that this game has no Bayesian equilibria in pure strategies.

We do not know whether *two-person* BPGs with inconsistent priors always have Bayesian equilibria. We have found the following partial solution.

TABLE X

	LL	RR	LR	RL
TT	$\frac{1}{2}, 0$	$0, \frac{1}{2}$	$\frac{1}{2}, 0$	$0, \frac{1}{2}$
BB	$0, \frac{1}{2}$	$\frac{1}{2}, 0$	$\frac{1}{2}, 0$	$0, \frac{1}{2}$
TB	$\frac{1}{2}, 0$	$\frac{1}{2}, 0$	$1, 0$	$0, 0$
BT	$0, \frac{1}{2}$	$0, \frac{1}{2}$	$0, 0$	$0, 1$

TABLE XI

	α			β	
	L	R		L	R
T	1	0	T	0	1
B	0	1	B	1	0

5.8. THEOREM. *Every two-player BPG with complete information on one side and such that every type has positive probability, has a pure BE.*

Proof. Let G be a two-player BPG with $T_1 = \{\delta\}$. So player 2 has complete information. We prove that player 2's prior p_2 can be replaced by p_1 , without changing the set of BEs. Then the priors are consistent, so we know by Corollary 5.4 that $BE(G) \neq \emptyset$.

For every $a \in A$, $t \in T_2$:

$$\begin{aligned}
 U_2(a|t) &= \sum_{s \in T_1} p_2(s|t) u_2(a, (s, t)) \\
 &= p_2(\delta|t) u_2(a, (\delta, t)) \\
 &= \frac{p_2(\delta, t)}{\sum_{s \in T_1} p_2(s, t)} u_2(a, (\delta, t)) = u_2(a, (\delta, t)).
 \end{aligned}$$

So U_2 does not depend on p_2 , which means that, to calculate $BE(G)$, we can replace p_2 by p_1 . ■

ACKNOWLEDGEMENTS

This research was supported by the Dutch Organization for Scientific Research (NWO).

REFERENCES

- Aumann, R.J.: 1959, 'Acceptable Points in General Cooperative n -Person Games', *Contributions to the Theory of Games IV*, A.W. Tucker and R.D. Luce (Eds.), Princeton, N.Y.: Princeton University Press, pp. 287–324.
- Bernheim, B.D., Peleg, B. and Whinston, M.D.: 1987, 'Coalition-Proof Nash Equilibria I. Concepts', *Journal of Economic Theory* **42**, 1–12.
- Einy, E. and Peleg, B.: 1991, 'Coalition-Proof Communication Equilibria', Mimeo, CentER of Rationality and Interactive Decision Theory, The Hebrew University of Jerusalem.
- Facchini, G., Megen, F. van, Tijs, S.H. and P. Borm: 1995, 'Congestion Models and Weighted Bayesian Potential Games', Mimeo, Tilburg University, The Netherlands.
- Harsanyi, J.C.: 1967, 'Games with Incomplete Information Played by 'Bayesian' Players' I–III, *Management Science* **14**(3), 159–182, 320–334, 486–502.
- Ichiishi, T. and Idzik, A.: 1992, 'Bayesian Cooperative Choice of Strategies', Discussion paper, Ohio State University, U.S.A.
- Monderer, D. and Shapley, L.S.: 1993, 'Potential Games', Discussion paper, Department of Economics, UCLA, Los Angeles, California.
- Myerson, R.B.: 1991, *Game Theory*, Cambridge, Massachusetts: Harvard University Press.
- Norde, H., Potters, J., Reijnierse, H. and Vermeulen, D.: 1993, 'Equilibrium Selection and Consistency', Discussion paper #9341, Nijmegen University, The Netherlands. To appear in *Games and Economic Behavior*.
- Peleg, B., Potters, J. and Tijs, S.H.: 1994, 'Minimality of Consistent Solutions for Strategic Games, in Particular for Potential Games', Discussion paper #9409, Nijmegen University, The Netherlands. To appear in *Economic Theory*.
- Peleg, B. and Tijs, S.H.: 1992, 'The Consistency Principle for Games in Strategic Form', Discussion paper #19, CentER of Rationality and Interactive Decision Theory, The Hebrew University of Jerusalem. To appear in *International Journal of Game Theory*.

Rosenthal, R.W.: 1973, 'A Class of Games Possessing Pure-Strategy Nash Equilibria', *International Journal of Game Theory* **2**, 65–67.

Robert van Heumen

*Department of Econometrics,
Tilburg University,
P.O. Box 90153,
5000 LE Tilburg,
The Netherlands*

Bezalel Peleg

*Center for Economic Research,
Tilburg University, and
Institute of Mathematics,
The Hebrew University of Jerusalem,
Jerusalem, Israel.*

Stef Tijs and Peter Borm

*Department of Econometrics,
Tilburg University,
Tilburg, The Netherlands.*