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Minimality of consistent solutions for strategic games, in particular for potential games[★]

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Summary. Solutions defined on classes of strategic games, satisfying One-Person Rationality (OPR), Non-emptiness (NEM) and Consistency (CONS) are considered. The main question to be answered is whether these conditions characterize the Nash Equilibrium solution *NE* for the given class of games. Depending on the structure of the class of games positive as well as negative answers are obtained. A graph-theoretical framework will be developed to express sufficient conditions for a positive or a negative answer. For the class of (finite) strategic games with at least one Nash equilibrium the answer is positive. For several classes of potential games the answer is negative.

1. Introduction

In Peleg and Tijs [9] the Nash Equilibrium solution defined on *closed* classes of strategic games has been axiomatized by three axioms: One-Person Rationality, Consistency and Converse Consistency. It has been shown that these axioms are logically independent.

In the New Palgrave [2] Aumann writes about consistency the following – in the light of these results intriguing – sentences

Consistency often plays a key role in axiomatizations. Strategic equilibrium is axiomatized by consistency, together with the requirement that in one-person maximization problems, the maximum be chosen.¹

Note that in the quotation there is no mentioning of ‘closed classes of games’ nor ‘converse consistency’. The reader can only guess what class of strategic games is

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¹ We read this as One-Person Rationality.

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meant. In Peleg and Tijs' proof of the logical independence of Converse Consistency they give as a counter example the Strong Nash Equilibrium solution that satisfies One-Person Rationality and Consistency but not Converse Consistency. One may object that many strategic games do not have Strong Nash Equilibria and that, therefore, this is not a convincing counter example. In the example after Theorem 2 we give a class of games Γ and a solution rule φ satisfying One-Person Rationality, Consistency and Non-emptiness and nevertheless it assigns to some games a proper subset of the Nash Equilibrium solution.

It is the purpose of this paper to investigate for what pairs (Γ, φ) Aumann's statement is true. We consider solution concepts φ defined on classes Γ of strategic games for which the following properties hold:

- (1) In each game of Γ the number of players is finite but the number of players is not bounded on the class² Γ .
- (2) The solution φ assigns to each game in Γ a non-empty subset of strategy profiles.
- (3) The class Γ is closed under reduction with respect to elements of the solution φ .
- (4) The solution satisfies consistency.³

In the next section we will repeat some definitions and introduce the concept of minimality with respect to (NEM), (OPR) and (CONS). Roughly speaking minimality means that it is not possible to refine a solution for some (or all) games of the class without losing one of the three properties. Two fundamental theorems give sufficient condition for minimality and non-minimality. In Section 2 we apply the theory of Section 1 to several classes of strategic games and prove or disprove minimality. The classes of cardinal potential games, of ordinal potential games and of generalized ordinal potential games will give us positive as well as negative results.

1. Definitions

Let Γ be a class of strategic games with finite player set. For games $G \in \Gamma$ the finite player set is denoted by $N(G)$, the strategy spaces by $\{A_i\}_{i \in N(G)}$ and the payoff functions by $\{U_i\}_{i \in N(G)}$. The space of strategy profiles is $A(G) := \prod_{i \in N(G)} A_i$.

If $G = \langle N(G), \{A_i\}_{i \in N(G)}, \{U_i\}_{i \in N(G)} \rangle$ is a game in Γ , x is an element of $A(G)$ and S is a proper coalition in $N(G)$ i.e., $S \neq \emptyset, N(G)$, the *reduced game* $G^{S,x}$ is the strategic game with player set S , strategy spaces $\{A_i\}_{i \in S}$ and payoff functions $\hat{U}_i: A_S \rightarrow R$, defined by $\hat{U}_i(y_S) := U_i(y_S, x_{N \setminus S})$ for all strategy S -tuples $y_S \in A_S$.

A *solution* φ with domain Γ is a rule that assigns a subset $\varphi(G)$ of $A(G)$ to each game $G \in \Gamma$. For pairs (Γ, φ) we introduce the following properties:

- (a) Non-Emptiness (NEM): $\varphi(G) \neq \emptyset$ for all games $G \in \Gamma$.
- (b) Closedness under Reduction (CLOS): for every game $G \in \Gamma$, every element $x \in \varphi(G)$ and every proper coalition S of $N(G)$ we have $G^{S,x} \in \Gamma$.

² To avoid trivial counter examples.

³ To make this property meaningful, condition (3) must hold.

- (c) Consistency (CONS): for every game $G \in \Gamma$, for every element $x \in \varphi(G)$ and every proper coalition $S \subset N(G)$ for which ${}^4G^{S,x} \in \Gamma$, we have $x_S \in \varphi(G^{S,x})$.
- (d) One-Person Rationality (OPR): for one person games $G \in \Gamma$ with strategy space A and payoff function U , we have $\varphi(G) = \arg \max U := \{a \in A \mid U(a) \geq U(a') \text{ for all } a' \in A\}$.
- (e) Pareto Optimality (PO): for games $G \in \Gamma$ and elements $x \in \varphi(G)$ there is no element $y \in A(G)$ with $U_i(y) \geq U_i(x)$ for all players $i \in N(G)$ and $(U_i(y))_{i \in N(G)} \neq (U_i(x))_{i \in N(G)}$. The payoff in x is not weakly dominated by the payoff in an other strategy profile.
- (f) Weak Pareto Optimality (WPO): for games $G \in \Gamma$ and elements $x \in \varphi(G)$ there is no element $y \in A(G)$ with $U_i(y) > U_i(x)$ for all players $i \in N(G)$. There is no strict domination of the payoff vector $(U_i(x))_{i \in N(G)}$ by an other payoff vector.
- (g) Converse Consistency (COCONS) has three different forms:
 - (COCONS_A) for all games $G \in \Gamma$, if $x \in A(G)$ is an element with the property $G^{S,x} \in \Gamma$ and $x_S \in \varphi(G^{S,x})$ for all proper coalitions $S \subset N(G)$, then $x \in \varphi(G)$.
 - (COCONS_B) for all games $G \in \Gamma$, if x is a *Pareto Optimal element* of $A(G)$ with the property $G^{S,x} \in \Gamma$ and $x_S \in \varphi(G^{S,x})$ for all proper coalitions $S \subset N(G)$, then $x \in \varphi(G)$.
 - (COCONS_C) the same property as (COCONS_B) with Weak Pareto Optimal instead of Pareto Optimal.⁵

The graph $\text{Graph}(\Gamma, \varphi)$ connected with a pair (Γ, φ)

For a class of strategic games Γ and a solution φ with domain Γ we introduce a directed graph $\text{Graph}(\Gamma, \varphi)$. The vertices of $\text{Graph}(\Gamma, \varphi)$ are pairs (G, x) with $G \in \Gamma$ and $x \in \varphi(G)$. There is an edge from (G, x) to (H, y) if $N(H) \subsetneq N(G)$, $H = G^{N(H),x}$ and $y = x_{N(H)}$.

Note that if an edge is pointing from a vertex (G, x) to a vertex (H, y) the game H has *fewer players than* G . The vertices of $\text{Graph}(\Gamma, \varphi)$ with the same game G form the set $v_G(\Gamma, \varphi)$. If $\varphi(G) \neq \emptyset$ for all $G \in \Gamma$ (NEM), all sets $v_G(\Gamma, \varphi)$ are nonempty and $\{v_G(\Gamma, \varphi)\}_{G \in \Gamma}$ is a partition of the set $v(\Gamma, \varphi)$ of vertices of $\text{Graph}(\Gamma, \varphi)$. As we realized afterwards, the graph $\text{Graph}(\Gamma, \varphi)$ is an example of a social situation in the sense of Greenberg [4].

A solution φ , defined on a class of games Γ and satisfying (NEM), (OPR) and (CONS) is called *minimal* (with respect to (NEM), (OPR) and (CONS)) if for every solution $\bar{\varphi}$ with $\bar{\varphi}(G) \subset \varphi(G)$ for all games $G \in \Gamma$ and satisfying (NEM), (OPR) and (CONS), we have $\bar{\varphi}(G) = \varphi(G)$ for all $G \in \Gamma$. A sufficient condition for minimality is given in the next theorem.

Theorem 1. *A solution φ defined on Γ and satisfying (NEM), (OPR) and (CONS) is minimal with respect to (NEM), (OPR) and (CONS) if the following property holds:*

⁴ This extra condition makes (CONS) independent of (CLOS).

⁵ In Peleg and Tijs [9] these versions of converse consistency are called COCONS, COCONS₄ and COCONS₃.

(AP) (Ancestors Property): for each vertex (G, x) of $\text{Graph}(\Gamma, \varphi)$ there is a game $H \in \Gamma$ with $N(H) \supset N(G)$ and for every element $y \in \varphi(H)$ we have $H^{N(G), y} = G$ and $y_{N(G)} = x$.

Proof. Let $\bar{\varphi}$ be a solution on Γ satisfying (NEM), (OPR) and (CONS) and $\bar{\varphi} \subset \varphi$. Let G be any game of Γ and let x be any element of $\varphi(G)$. Take H as in the theorem. Then $\bar{\varphi}(H)$ contains at least one element y (NEM). Then $x = y_{N(G)} \in \bar{\varphi}(H^{N(G), y}) = \bar{\varphi}(G)$ (by (CONS) of $\bar{\varphi}$). $\square$

In the proof we did not use explicitly that φ satisfies (OPR) but the definition of minimality requires that the solution has the properties (NEM), (OPR) and (CONS) itself. For it would be strange to call a solution minimal with respect to some properties if it does not satisfy one of these properties itself.

This theorem is especially interesting, if φ is the Nash Equilibrium solution NE . If $\bar{\varphi}$ is a solution defined on a class of games Γ and $\bar{\varphi}$ satisfies (OPR) and (CONS), then $\bar{\varphi} \subset NE$ (see Peleg and Tijs [7]). If, moreover, φ satisfies (NEM) and NE has the Ancestors Property on Γ , then $\bar{\varphi} = NE$. This means that for the class Γ (COCONS_A) is implied by (NEM), (OPR) and (CONS). In the next section we shall see some examples of classes of games this theorem can be applied to.

The next theorem gives an opposite result. It gives sufficient conditions that a solution φ is *not* minimal.

A vertex (G, x) of the graph $\text{Graph}(\Gamma, \varphi)$ is a *source* if the vertex has no ancestors in the graph.

Theorem 2. If φ is a solution with domain Γ satisfying (NEM), (OPR) and (CONS) and the graph $\text{Graph}(\Gamma, \varphi)$ has the following property:

(SP) (Source Property): there is a game $G \in \Gamma$ with $|N(G)| > 1$ and the graph $\text{Graph}(\Gamma, \varphi)$ contains a source (G, x) and another vertex $(G, y) \in v_G(\Gamma, \varphi)$, then φ is not minimal with respect to (NEM), (OPR) and (CONS).

Proof. Take $\bar{\varphi}(H) = \varphi(H)$ for $H \neq G$ and $\bar{\varphi}(G) = \varphi(G) \setminus \{x\}$. Then $\bar{\varphi}$ satisfies (NEM), (OPR) and (CONS). $\square$

The next example gives a class Γ of games for which the Nash Equilibrium solution is not minimal with respect to (NEM), (OPR) and (CONS).

Example. (Left-right coordination games)

Let N (the natural numbers) be the universum of players. Each player has two strategies L and R . For a finite subset $M \subset N$ there are four games in the class Γ with player set M , namely $G^{LR}(M)$, $G^L(M)$, $G^R(M)$ and $G^0(M)$. The payoffs in all these games are zero or one. In $G^{LR}(M)$ the payoff is one for all players if and only if the players play the same strategy, that is, only the strategy profiles $L^M := (L, L, \dots, L)$ and $R^M := (R, R, \dots, R)$ give a payoff one. In $G^L(M)$ or $G^R(M)$ the payoff is one if and only if all players play L or R respectively. In $G^0(M)$ the payoff is always zero.

It is clear that L^M and R^M are equilibria in every game $G \in \Gamma$ with $|M| > 1$.

For $G \in \Gamma$ we define $\varphi(G) = \{L^M\} \subset NE(G)$ if G is not a 1-player game. If G is a 1-person game, we define $\varphi(G) := NE(G)$. Then φ satisfies One-Person Rationality.

For $|M| > 1$, $S \subsetneq M$ and $x = L^M$ the reduced game (cf. Peleg and Tijs [9]) $G^{LR}(M)^{S, x}$ is $G^L(S)$, the reduced game $G^L(M)^{S, x} = G^L(S)$, $G^R(M)^{S, x}$ is the game $G^0(S)$

and finally $G^0(M)^{S,x}$ gives $G^0(S)$. As $L_S^M = L^S$ the solution φ satisfies consistency too. For all games with at least two players, the solution φ gives a proper subset of the Nash equilibrium set.

The nodes $(G^{LR}(M), L^M)$ and $(G^{LR}(M), R^M)$ are *sources* in $\text{Graph}(\Gamma, NE)$. Therefore we can apply Theorem 2. But we have already seen that on this class the Nash equilibrium solution NE is not minimal with respect to (NEM), (OPR) and (CONS).

2. Applications, in particular to potential games

Using the theorems of the preceding section we investigate in this section for some classes of games whether (NEM), (OPR) and (CONS) is sufficient for a solution φ to coincide with the solution NE . We shall obtain positive as well as negative results.

(I) Strategic games with at least one Nash equilibrium

Let Γ^{NE} be the class of strategic games with at least one Nash equilibrium. In the following theorem we prove that (NEM) can replace the $(COCONS_A)$ -condition (see Peleg and Tijs [9]) in the characterization of the Nash equilibrium solution on the class Γ^{NE} (cf. Peleg [7] for an analogous result with respect to the core of cooperative games). The class Γ^{NE} is not a well-described class; it does not have an independent description. Notice, however, that every subclass Γ of Γ^{NE} has the same property as long as the game $(N(G) \cup \{0\}, H)$ (constructed in the proof of the next theorem) is an element of the subclass Γ for every game $G \in \Gamma$ and every Nash equilibrium $x \in NE(G)$. Moreover, one finds a similar situation in Aumann [3] where the NTU-Shapley value is characterized on the class of NTU-games *that have a nonempty NTU-Shapley value*.

Theorem 3. *If a solution φ with domain Γ^{NE} satisfies the properties (NEM), (OPR), (CLOS) and (CONS), then $\varphi = NE$.*

Proof. For 1-person games G we have $\varphi(G) = NE(G)$ as both solutions satisfy (OPR). If $\varphi(G) \subset NE(G)$ for games G with less than k players and G_* is a game in Γ^{NE} with k players, we have, for each element $x \in \varphi(G_*)$, that $G_*^{S,x} \in \Gamma^{NE}$ (CLOS) and $x_S \in \varphi(G_*^{S,x})$ (CONS) for every proper coalition $S \subset N(G_*)$. As $G_*^{S,x}$ has less than k players, we find $x_S \in NE(G_*^{S,x})$ for all proper coalitions S and, as NE satisfies $(COCONS_A)$, $x \in NE(G_*)$. Hence, $\varphi \subset NE$.

Let $G = \langle N(G), \{A_i\}_{i \in N(G)}, \{U_i\}_{i \in N(G)} \rangle$ be a game with at least one Nash equilibrium and let x be a Nash equilibrium of G . We will construct⁶ a game $H \in \Gamma^{NE}$ with $N(H) \supset N(G)$ such that H has exactly one Nash equilibrium y and the point y has the properties $y_{N(G)} = x$ and $G = H^{N(G),y}$. If we can do so, $\varphi = NE$ by Theorem 1. The construction is as follows.

In the game H there is one player more, player 0. The strategy space of player 0 contains two strategies: $A_0^* = \{\alpha, \beta\}$ (see footnote⁶). The strategy spaces of the players $i \in N(G)$ are as before: $A_i^* = A_i$.

⁶ In the construction that follows a strategic game and its mixed extension are considered to be different games.

The payoff functions are defined as follows:

$$\begin{cases} U_0^*(a, \alpha) = 2 & \text{if } a = x \\ U_0^*(a, \alpha) = -1 & \text{if } a \neq x \\ U_0^*(a, \beta) = 0 \end{cases} \quad \begin{cases} U_i^*(a, \alpha) = U_i(a) \\ U_i^*(a, \beta) = -1 & \text{if } a_i \neq x_i \text{ for } i \in N(G). \\ U_i^*(a, \beta) = 1 & \text{if } a_i = x_i \end{cases}$$

We prove that (x, α) is the unique equilibrium of H . Clearly, (x, α) is a Nash equilibrium of H and therefore $H \in \Gamma^{NE}$.

Suppose that (a, β) is a Nash equilibrium of H . If $a \neq x$, then for at least one player $i \in N(G)$ $a_i \neq x_i$. Deviation from a_i to x_i increases the payoff of player i from -1 to 1 . So, $a = x$. But then player 0 has a profitable deviation from β to α .

Suppose, (a, α) is a Nash equilibrium. If $a \neq x$, player 0 can profitably deviate to β . If $a = x$, no player has a profitable deviation: Only (x, α) is a Nash equilibrium in H and $H \in \Gamma^{NE}$. Further, $(x, \alpha)_{N(G)} = x$ and $H^{N(G), (x, \alpha)} = G$. $\square$

The same construction can be used for the class $\Gamma^{f, NE}$ of *finite* strategic games with at least one Nash equilibrium. Therefore we also have:

Theorem 3-a. *If a solution φ with domain $\Gamma^{f, NE}$ satisfies (NEM), (OPR) (CLOS) and (CONS), then $\varphi = NE$.*

The construction in the proof of Theorem 3 cannot be used for the class of mixed extensions of finite n -person games or for games with continuous payoff functions on non-discrete strategy spaces, as the payoff functions U_i^* and U_0^* are *not* linear or continuous.

(II) Strong Nash equilibria

According to Aumann [1] a strategy profile x is a *Strong Nash Equilibrium* (SNE) of a strategic game G if, for every coalition $S \subset N$ and every profile $y_S \in A_S$ at least one player $i \in S$ has a payoff $U_i(y_S, x_{N \setminus S})$ that is *not higher* than $U_i(x)$. If Γ^{SNE} is the class of strategic games with at least one strong Nash equilibrium, we can formulate and prove a similar result as in Theorem 3 for Strong Nash equilibria.

Theorem 4. *If φ is a solution with domain Γ^{SNE} and φ satisfies the properties (NEM), (CLOS), (CONS) and (WPO), then $\varphi = SNE$.*

Proof. In Peleg and Tijs [9] it is shown that $\varphi \subset SNE$. Hence, we are left to prove that (Γ^{SNE}, SNE) has the ancestors property (AP). If $G \in \Gamma^{SNE}$ and $x \in SNE(G)$, we can take the same game H as in the proof of Theorem 3. This game has one Nash equilibrium $y = (x, \alpha)$. This equilibrium satisfies $H^{N(G), y} = G$ and (of course) $y_{N(G)} = x$. It is sufficient to prove that $y \in SNE(H)$. If a coalition $S \subset N(H)$ deviates from y_S to z_S and $U_i(z_S, y_{N(H) \setminus S}) > U_i(y)$ for all $i \in S$, the coalition S must contain player 0 (otherwise x is not a Strong Nash Equilibrium). Then player 0 plays β in z_S and his payoff is zero. But in (x, α) his payoff was 2. Hence, (x, α) is strong Nash and $H \in \Gamma^{SNE}$. $\square$

It is easy to prove a similar result for *strictly strong Nash Equilibria*. A strategy profile x is called a *strictly strong Nash Equilibrium* if for every coalition $S \subset N$ and every profile $y_S \in A_S$ there is at least one player $i \in S$ with a *strictly lower* payoff under

$(y_S, x_{N \setminus S})$ than under x or $U_i(x) = U_i(y_S, x_{N \setminus S})$ for all players $i \in S$. In a similar way we can prove

Theorem 4-a. *If φ is a solution with domain Γ^{s-SNE} , the class of strategic games with at least one strictly strong Nash equilibrium, and φ satisfies the properties (NEM), (CLOS), (CONS) and (PO), then $\varphi = s-SNE$.*

(III) Potential games

In Rosenthal [10] *congestion games* are introduced and it is proved that this type of games ‘have a potential function’. In Monderer and Shapley [5] three types of potential games are introduced and finite cardinal potential games (see later for a definition) are proved to be congestion games. Let us recall the definitions. A strategic game $G = \langle N(G), \{A_i\}_{i \in N(G)}, \{U_i\}_{i \in N(G)} \rangle$ is a *cardinal potential game* (CP) if there exists a function $P: A(G) = \prod_{i \in N(G)} A_i \rightarrow R$ such that, for all players $i \in N(G)$, all strategies $x_i, y_i \in A_i$ and all complementary strategy profiles⁷ $x_{-i} \in \prod_{j \neq i} A_j$,

$$P(x_i | x_{-i}) - P(y_i | x_{-i}) = U_i(x_i | x_{-i}) - U_i(y_i | x_{-i}).$$

The function P is called a *cardinal potential function* of the game G . If $x_0 \in A$ is a point of reference and $x \in A$ is an arbitrary point, we can define points $x^{(k)} \in A, k = 0, \dots, n$ by $x_i^{(k)} = x_i$ for $i \leq k$ and $x_i^{(k)} = x_{0,i}$ if $i > k$. Then

$$P(x) - P(x_0) = \sum_{k=1}^n (U_k(x^{(k)}) - U_k(x^{(k-1)}))$$

for every point $x \in A$. This means that P is uniquely determined up to an additive constant. The class of cardinal potential games will be denoted by Γ_{CP} .

A strategic game G is an *ordinal potential game* (OP) if there is a function $P: A \rightarrow R$ with the property that, for all players $i \in N(G)$, all strategies $x_i, y_i \in A_i$ and all complementary profiles $x_{-i} \in A_{-i}$,

$$P(x_i | x_{-i}) > P(y_i | x_{-i}) \text{ if and only if } U_i(x_i | x_{-i}) > U_i(y_i | x_{-i}).$$

The function P is called an *ordinal potential function* of G and the class of ordinal potential games is denoted by Γ_{OP} . The ordinal potential functions of an ordinal potential game are, in general, not uniquely determined up to an additive constant.

A strategic game G is a *generalized ordinal potential game* (GP) if there exists a function $P: A \rightarrow R$ such that, for all players $i \in N(G)$, strategies $x_i, y_i \in A_i$ and all complementary profiles $x_{-i} \in A_{-i}$

$$P(x_i | x_{-i}) > P(y_i | x_{-i}) \text{ if } U_i(x_i | x_{-i}) > U_i(y_i | x_{-i}).$$

The function P is called a *generalized potential function* and the class of generalized ordinal potential games is denoted by Γ_{GP} .

A cardinal potential function is also an ordinal potential function and an ordinal potential function is a generalized potential function. Therefore, $\Gamma_{CP} \subset \Gamma_{OP} \subset \Gamma_{GP}$.

⁷ The notation may be clear. We write $(x_i | x_{-i})$ for a strategy profile $x = (x_1, \dots, x_n)$ if we want to emphasize the strategy component of player i .

If P is an ordinal or generalized ordinal potential function for a game G and $f: R \rightarrow R$ is a strictly increasing function, then $f \circ P$ is also an ordinal or generalized ordinal potential function for G . Consequently, every (generalized) ordinal potential game has a bounded potential function (take e.g. $f(x) = \arctan(x)$.)

Notice that a cardinal potential game G may have besides cardinal potential functions (uniquely determined up to an additive constant) ordinal potential functions and generalized potential functions that are not cardinal potential functions. If $G \in \Gamma_\omega$, we denote the set of potential functions of the same type by $\mathcal{P}_\omega(G)$. So we have $\mathcal{P}_{CP}(G) \subset \mathcal{P}_{OP}(G) \subset \mathcal{P}_{GP}(G)$ for every game $G \in \Gamma_{CP}$. If we define $\mathcal{P}_\omega(G) = 0$ if the strategic game G is **not** a potential game of type ω , the sets $\mathcal{P}_\omega(G)$ are defined for every strategic game G and $\omega \in \{CP, OP, GP\}$.

Now we define the solution rules PM_ω as follows:

$x \in PM_\omega(G)$ if there is a potential function of type ω that takes its maximum in the point x .

The set $PM_\omega(G)$ can be empty for two reasons: firstly, the game G may have no potential functions of type ω or secondly, no function in $\mathcal{P}_\omega(G)$ takes a maximum value on A . For finite games only the first reason may cause emptiness of $PM_\omega(G)$.

Therefore, we define a subclass $\Gamma_\omega^* \subset \Gamma_\omega$ as the set of games $G \in \Gamma_\omega$ for which $PM_\omega(G) \neq \emptyset$. Now $PM_\omega (\omega \in \{CP, OP, GP\})$ are solution rules defined on Γ_ω^* satisfying (NEM) (by definition). In fact the solutions PM_ω are also satisfying (OPR) and (CONS) but $PM_\omega \subsetneq NE$ for $\omega \in \{CP, OP\}$. For $\omega = GP$ we have $PM_\omega = NE$.

Theorem 5. *The solution rules PM_ω defined on Γ_ω^* , satisfy (NEM), (OPR), (CLOS) and (CONS). For $\omega = CP$ or OP we have $PM_\omega \subsetneq NE$ and for $\omega = GP$ $PM_\omega = NE$.*

Proof. If $x \in PM_\omega(G)$ for some game $G \in \Gamma_\omega^*$, there is a potential function $P \in \mathcal{P}_\omega(G)$ that takes its maximum in x . Then x is a Nash equilibrium. For, if $U_i(y_i | x_{-i}) > U_i(x)$ for some player $i \in N$, we have $P(y_i | x_{-i}) > P(x)$. For one-person games G of Γ_ω^* , this means that $PM_\omega(G)$ is a subset of the points where the payoff function takes its maximum. On the other side the payoff function is a cardinal potential for G and therefore, $PM_\omega(G)$ consists of the points where the payoff function of G takes its maximum (OPR).

If $G \in \Gamma_\omega^*$, $x \in PM_\omega(G)$ and $S \subset N$ is a proper subcoalition of N , the game $G^{S,x}$ is a potential function of type ω with (at least) the potential function $\bar{P}(y_S) := P(y_S, x_{N \setminus S})$ for $P \in \mathcal{P}_\omega(G)$. If $P \in \mathcal{P}_\omega(G)$ takes its maximum in x , then $\bar{P}$ is a potential function for $G^{S,x}$ that takes its maximum in x_S . So, Γ_ω^* is closed with respect to PM_ω and PM_ω satisfies consistency (CONS).

The following counter example shows that $PM_\omega \neq NE$ if $\omega = CP$ or OP .

Let $A_1 := \{T, B\}$ and $A_2 := \{L, M, R\}$ be the strategy spaces of a two-person game and let P be an ordinal potential function given by the matrix

$$\begin{array}{c} \begin{array}{ccc} & L & M & R \\ \begin{array}{c} T \\ B \end{array} & \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 3 \end{pmatrix} \end{array}$$

Then (T, L) is a Nash equilibrium for every ordinal potential game with potential function P but there is no ordinal potential function P' that takes its maximum in

(T, L) . Indeed, if $P'(T, L) = a$, we also have $P'(T, M) = a$ (as $P(T, L) = P(T, M)$ implies $U_2(T, L) = U_2(T, M)$). A similar reasoning gives $P'(T, L) = P'(T, M) < P'(B, M) < P'(B, R)$. Therefore, (T, L) is not an element of $PM_\omega(G)$ for $\omega = CP$ or OP .

Finally, we prove that, if $G \in \Gamma_{GP}$ and $x \in NE(G)$, there is a generalized potential function that takes its maximum in x . Let P be a potential function in $\mathcal{P}_{GP}$. Then $P_1 := \arctan \circ P$ is a bounded element of $\mathcal{P}_{GP}$. Define P_2 by $P_2(y) = P_1(y)$ for $y \neq x$ and $P_2(x) := P_1(x) + \pi$. Then P_2 takes its maximum in x and $P_2 \in \mathcal{P}_{GP}(G)$. We prove the last statement. There is no player $i \in N$ who has an alternative strategy $y_i \neq x_i$ with $U_i(y_i | x_{-i}) > U_i(x_i | x_{-i})$. So increasing the value of P_1 in the point x keeps the function being a generalized potential. There may be alternatives y_i with $U_i(y_i | x_{-i}) = U_i(x_i | x_{-i})$ but a *generalized* potential is allowed to have $P_2(y_i | x_{-i}) < P_2(x)$ nevertheless. $\square$

The last statement of the Theorem ($PM_{GP} = NE$) can be generalized as follows:

Theorem 6. *If φ is a solution rule defined on Γ_{GP}^* and φ satisfies (NEM), (OPR) and (CONS), then $\varphi = NE$.*

Proof. (OPR) and (CONS) imply that $\varphi(G) \subset NE(G)$ for all games $G \in \Gamma_{GP}$. If we can prove that NE has the ancestors property on Γ_{GP}^* , the theorem follows from Theorem 1.

Let G be a strategic game in Γ_{GP} and let $\hat{x}$ be a Nash equilibrium of G . Let P be a *bounded* generalized potential for G . We construct a new game $H \in \Gamma_{GP}^*$ satisfying the ancestors property. Take $\Delta \gg 0$. Let $N \cup \{0\}$ be the player set of H and give the new player (player 0) two strategies α and β . The payoff functions in game H are as follows:

$$\begin{aligned} U_i^*(x, \alpha) &:= U_i(x) \text{ for all } x \in A \text{ and all } i \in N. \\ U_i^*(x, \beta) &:= (k + 1)\Delta \text{ where } k := \#\{i \in N \mid x_i = \hat{x}_i\}. \\ U_0^*(x, \alpha) &:= 0 \text{ if } x \neq \hat{x} \text{ and } U_0^*(\hat{x}, \alpha) := 2(n + 1)\Delta \\ U_0^*(x, \beta) &:= (k + 1)\Delta \text{ for all } x \in A. \end{aligned}$$

This game has one equilibrium namely $(\hat{x}, \alpha)$.

Indeed, strategy profiles (x, α) , $x \neq \hat{x}$ are not equilibria as player 0 can deviate profitably. Strategy profiles (x, β) , $x \neq \hat{x}$ fail to be equilibria as every player $i \in N$ with $x_i \neq \hat{x}_i$ can gain if he deviates to $\hat{x}_i$. Finally, $(\hat{x}, \beta)$ is not an equilibrium as player 0 can win by deviation.

It is easy to see that $H^{N, (\hat{x}, \alpha)} = G$ and $(\hat{x}, \alpha)_N = \hat{x}$. We show that $H \in \Gamma_{GP}^*$.

A generalized ordinal potential function P^* for H can be defined by:

$$\begin{aligned} P^*(x, \alpha) &:= P(x) \text{ if } x \neq \hat{x} \text{ and } P^*(\hat{x}, \alpha) := P(\hat{x}) + 2(n + 1)\Delta. \\ P^*(x, \beta) &:= P(x) + (k + 1)\Delta \text{ for all } x \in A. \end{aligned}$$

It is clear that deviations of player 0 from α to β have the right sign (positive if $x \neq \hat{x}$ and negative if $x = \hat{x}$). Deviations for the players $i \in N$ have also the good sign, if player 0 plays α , as P is a generalized potential function for G . If player $i \in N$ deviates from (x, β) to (y_i, x_{-i}, β) , his payoff increases if and only if $x_i \neq y_i = \hat{x}_i$. In that case the value of P^* increases with $\Delta - P(x) + P(\hat{x}_i | x_{-i})$ and this is positive if Δ is large enough (P is bounded). $\square$

For cardinal potential games we can characterize the solution rule PM by four properties. One of these properties is (OPR). We need also one new property, namely invariance (INV) and two new versions of old properties: Pareto Optimality for Normalized Potential Games (PO for NPG) and Converse Consistency for Normalized Potential Games (COCONS_B for NPG). First we define: two strategic games G and G' with the same player set and strategy spaces are called *equivalent* (notation; $G \simeq G'$) if there are functions $\delta_i: A_{-i} \rightarrow R$ with $U'_i(a) = U_i(a) + \delta_i(a_{-i})$ for all strategy N -tuples $a \in A$ and all players $i \in N$. Equivalent strategic games have the same *best response multifunctions* and therefore the same Nash equilibria. For cardinal potential games two games are equivalent iff they have the same potential functions. Every potential game is equivalent with a *normalized potential game* i.e. a potential game with $U_i = P$ for all $i \in N$ (take $\delta_i(a_{-i}) = U_i(a_i | a_{-i}) - P(a_i | a_{-i})$).

Now we can formulate the new properties:

Invariance (INV): two equivalent potential games have the same solution: if $G \simeq G'$ then $\varphi(G) = \varphi(G')$.

Pareto Optimality for Normalized Potential Games (PO for NPG): $\varphi(G) \subset PO(G)$ if G is a *normalized* potential game.

Converse Consistency for Normalized Potential Games (COCONS_B for NPG): if G is a normalized potential game, $x \in PO(G)$ and $x_S \in \varphi(G^{S,x})$ for all proper subcoalitions $S \subset N$, then $x \in \varphi(G)$.

Theorem 7. If φ is a solution with domain Γ_{CP}^* and φ has the properties (OPR), (INV), (PO for NPG) and (COCONS_B for NPG), then $\varphi = PM$.

Proof: As φ and PM satisfy (INV), it is enough to prove that $\varphi(G) = PM(G)$ for all normalized potential games. Note that, for *normalized* potential games G a strategy profile x is in $PM(G)$ if and only if x is Pareto Optimal (= Weakly Pareto Optimal, as all payoff vectors in a normalized potential game have equal coordinates). Therefore, (PO for NPG) gives $\varphi(G) \subset PM(G)$ if G is normalized.

We prove that $\varphi = PM$ by induction.

(OPR) implies that $\varphi(G) = PM(G)$ for 1-person games G . Suppose that $\varphi(G) = PM(G)$ has been proven for normalized potential games G with less than k players. If G is a normalized potential game with k players and x is an element of $PM(G)$, then x_S is an element of $PM(G^{S,x})$ for every proper coalition $S \subset N$ ((CONS) for PM) and $G^{S,x}$ is a *normalized* potential game. From the induction hypothesis we find $x_S \in \varphi(G^{S,x})$ for all proper coalitions S . Further, x is Pareto Optimal in A and therefore $x \in \varphi(G)$ by (COCONS_B). $\square$

The solution PM does *not* satisfy (PO) or (COCONS_B) for all cardinal potential games as the following example shows:

Let N be $\{1, 2\}$, let A_1 be $\{T, B\}$ and $A_2 = \{L, R\}$. The potential function and the payoff matrices are

$$\begin{array}{cc} & \begin{array}{cc} L & R \end{array} \\ \begin{array}{c} T \\ B \end{array} & \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \end{array} \quad \begin{array}{cc} & \begin{array}{cc} L & R \end{array} \\ \begin{array}{c} T \\ B \end{array} & \begin{pmatrix} (1,1) & (a,0) \\ (0,b-2) & (a+2,b) \end{pmatrix} \end{array}$$

If $a > 0$ and $b > 2$, the point (B, R) is the unique Pareto optimal point, for $a < -1$ and $b < 1$ the strategy 2-tuple (T, L) is the unique Pareto optimal point. So, Pareto

optimality is not invariant under equivalence. Note that PM is always (B, R) . For $a < -1$ and $b < 1$ the strategy profile (T, L) satisfies the conditions in the definition of $(COCONS_B)$ but it is not an element of $PM(G)$.

(IV) Special classes of potential games

In this section we investigate two subclasses of (cardinal, ordinal and generalized ordinal) potential games. The first subclass to be considered is the class Γ^f of mixed extensions of finite games. Let Γ_ω^f be the class of potential games of type ω that are mixed extensions of finite games. We prove that there exists a solution defined on Γ_ω^f satisfying (NEM), (OPR) and (CONS) that does not coincide with NE . Let $\overline{PNE}$ be the solution rule that assigns to a game $G \in \Gamma_\omega^f$ the set of pure Nash equilibria if $|N(G)| > 1$. It assigns to one-person games the set $\arg \max U$ as usual. The solution $\overline{PNE}$ is the solution rule we are looking for.

Theorem 8. For $\omega \in \{CP, OP, GP\}$ the solution rule $\overline{PNE}$ defined on Γ_ω^f satisfies (NEM), (OPR) and (CONS) but there is a game $G \in \Gamma_\omega^f$ with $\overline{PNE}(G) \neq NE(G)$

Proof: The rule $\overline{PNE}$ satisfies (OPR) by definition. The proof of consistency is straightforward. So we have to prove that $\overline{PNE}(G) \neq \emptyset$ for all games in Γ_{GP}^f and that there is a game $G \in \Gamma_{CP}^f$ with $\overline{PNE}(G) \neq NE(G)$.

Let G be the mixed extension of a finite game that has a generalized ordinal potential function P . Let $\hat{e}$ be a pure strategy profile where P takes its maximum value on the (finite) set of pure strategy profiles. If $\hat{e}$ is not a Nash equilibrium, at least one player, say player i , has a pure strategy e_i with $U_i(e_i | \hat{e}_{-i}) > U_i(\hat{e})$. Then $P(e_i | \hat{e}_{-i}) > P(\hat{e})$. This is impossible and therefore, $\overline{PNE}(G) \neq \emptyset$. Take for $0 < x < 1$ the bimatrix game with payoff matrices

$$\begin{array}{cc} & \begin{matrix} L & R \end{matrix} \\ \begin{matrix} T \\ B \end{matrix} & \begin{pmatrix} (1, 1) & (0, x) \\ (x, 0) & (x, x) \end{pmatrix} \end{array} \quad \text{with cardinal potential function} \quad \begin{array}{cc} & \begin{matrix} L & R \end{matrix} \\ \begin{matrix} T \\ B \end{matrix} & \begin{pmatrix} 1 & x \\ x & 2x \end{pmatrix} \end{array}$$

The cardinal potential bimatrix game has two pure equilibria and one completely mixed equilibrium $((x, 1-x); (x, 1-x))$: $\overline{PNE}(G) \neq NE(G)$. $\square$

The second class Γ^c of games consists of the strategic games $G = \langle N, \{A_i\}_{i \in N(G)}, \{U_i\}_{i \in N(G)} \rangle$ with

- (a) the strategy sets A_i are compact convex subsets of an Euclidian space,
- (b) the payoff functions $U_i: A(G) \rightarrow R$ are continuous and (quasi-)concave in the i -th coordinate. The subclasses of potential games of type ω with those properties will be denoted by Γ_ω^c . Notice that $\Gamma_{CP}^c \subset \Gamma_{CP}^*$ as the potential function is a continuous function and takes therefore its maximum somewhere in the compact space of strategy profile. Therefore the solution PM_{CP} satisfies (NEM) on Γ_{CP}^c . It also satisfies (CONS) and (OPR) and therefore, the Nash Equilibrium solution is on Γ_{CP}^c not minimal with respect to (NEM), (OPR) and (CONS). For the class Γ_{OP}^c we can do the same as soon as we know that PM_{OP} satisfies (NEM). This is certainly

true if every game $G \in \Gamma^c$ has a *continuous* ordinal potential function. Unfortunately, this is not known.

For the class Γ_{GP}^c we can prove that PM_{GP} satisfies (NEM). The idea is simple. Take an element $G \in \Gamma_{GP}^c$ and a *bounded* generalized ordinal potential function P for G . Change the values of P on the (non-empty) set $NE(G)$ such that all points in $NE(G)$ have the same value larger than an upper bound of P . The changed function is still a generalized ordinal potential function for G and $NE(G) \subset PM(G)$.

This means that $\Gamma_{GP}^c \subset \Gamma_{GP}^*$. From Theorem 5 we infer that $PM = NE$ on Γ_{GP}^c . It is unclear whether NE is minimal on Γ_{GP}^c with respect to (NEM), (OPR) and (CONS).

3. Summary

In this paper we investigated for what classes of strategic games the properties (NEM), (OPR) and (CONS) characterize the Nash Equilibrium solution.

For Γ^{NE} (the class of games with at least one Nash equilibrium) and Γ_{GP}^* (the class of generalized ordinal potential games with at least one Nash equilibrium)⁸ we found a positive result. For Γ_{CP}^* , Γ_{OP}^* and $\Gamma_{\omega}^f (\omega \in \{CP, OP, GP\})$ we found negative results. In the first two cases PM (the union of the $\arg \max$ of the potential functions) provided us with a counter example, in the last classes $\overline{PNE}$ was a counter example.

We also found that the (strictly) strong Nash equilibrium solution can be characterized without $(COCONS_C)$ (or $(COCONS_B)$) on the largest family of games where the (strictly) strong Nash equilibrium solution satisfies (NEM).

In the first section we formulated in Theorem 1 an important criterion for minimality of a solution on a class of strategic games. Inspired by this result and the proof of Theorem 3 Norde et al. [6] proved that also on the classe Γ^f and Γ^c the NE-solution is characterized by (NEM), (OPR) and (CONS). Peleg and Sudhölter [8] proved an analogous result for games with continuous quasi-convex payoff functions.

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⁸ If $G \in \Gamma_{GP} \cap \Gamma^{NE}$, we can construct – by a minor change in the construction in the proof of Theorem 5 – a generalized ordinal potential function that takes its maximum on $NE(G)$. Therefore, $\Gamma_{GP}^* = \Gamma_{GP} \cap \Gamma^{NE}$.

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