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**THE DEGREE AND COST ADJUSTED FOLK SOLUTION
FOR MINIMUM COST SPANNING TREE GAMES**

By

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The degree and cost adjusted folk solution for minimum cost spanning tree games

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Abstract

In this paper two cost sharing solutions for minimum cost spanning tree problems are introduced, the degree adjusted folk solution and the cost adjusted folk solution. These solutions overcome the problem of the classical reductionist folk solution as they have considerable strict ranking power, without breaking established axioms. As such they provide affirmative answers to open questions, put forward in Bogomolnaia and Moulin (2010) and Bogomolnaia et al. (2010).

Keywords: cost sharing, minimum cost spanning tree games, networks.

JEL-code: C71.

1 Introduction

The problem of finding a minimum cost spanning tree (mcst) is a classical combinatorial optimization problem. Finitely many agents want to be connected to a source, directly or indirectly via other agents. The problem is to organize this at minimal cost. Two well-known algorithms for this problem are the greedy algorithms of Kruskal (1956) and Prim (1957), that compute an optimal tree network in nearly quadratic and quadratic time respectively (in the number of agents).

Once an optimal tree has been computed the question is how to divide the optimal cost in a fair way among the agents. This issue still receives a lot of attention in the (game theoretic) cost sharing literature. Claus and Kleitman (1973) were the first to discuss this division problem, soon followed by the famous paper of Bird (1976), which proposes an easy to compute core element of the strict stand alone cooperative game. Since then other solutions have been proposed. Granot and Huberman (1984) consider the nucleolus, Kar (2002) the Shapley value, and Dutta and Kar (2004) an adapted version of the Bird solution. None of these solutions however simultaneously obey attractive properties like *continuity*, *core selection*, *cost monotonicity* and *polynomial complexity*. A solution which does is the folk solution. This solution has several appearances in different papers. It is the Equal Remaining Obligation solution (Feltkamp et al. (1994)), the Potters-value (Branzei et al. (2004)), the average of a number of population monotonic solutions introduced in Norde et al. (2004), and it was given its name in Bogomolnaia and Moulin (2010). Several characterizations of this solution

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have been provided, for example in Bergantinos and Vidal-Puga (2007), Bergantinos and Kar (2010), Bergantinos and Vidal-Puga (2011) and Branzei et al. (2004). A central role in some of these characterizations is the axiom of *population monotonicity* (Sprumont (1990)) that guarantees dynamic stability: if a new agent joins then the other agents are not harmed.

The folk solution is a ‘reductionist’ solution in the sense that it depends on the cost of edges occurring in optimal trees only and disregards the cost of other edges. The property of reductionism seems self-evident: why bothering about other costs if you are dividing the cost of edges in an optimal tree? Bergantinos and Vidal-Puga (2007) raised this question whether it is worthwhile to study nonreductionist solutions as well. In particular, they stated that so far in the literature no solution has been studied that is nonreductionist, has appealing properties, and for which no reductionist solution can be found with the same properties.

The necessity to study nonreductionist solutions was first pointed out by Bogomolnaia and Moulin (2010). Although the folk solution satisfies the axiom of *ranking*, i.e. agents with weakly smaller connection costs are allocated a weakly smaller cost share, it fails to discriminate strictly between agents in situations where this seems to be more than fair. Consider for example the two problems in Figure 1. For both problems the folk solution

Figure 1: Two cost situations where the folk solution produces an ‘unfair’ outcome.

(and any other anonymous reductionist solution), yields the symmetric allocation $(1, 1, 1)$ as outcome. In both cases however, a strict smaller cost share for agent 1 seems to be more appropriate. In the first cost situation one may argue that agent 1 should be rewarded for having free connections with all other agents (all agents having the same connection cost with the source), whereas in the second cost situation this should happen because of the strict lower connection cost of agent 1 with the source (all other connection costs being zero). Bogomolnaia and Moulin (2010) formulate two strict ranking axioms for solutions, *agent-based strict ranking* and *source-based strict ranking*¹, describing such discriminative power of solutions, and present classes of solutions obeying one or both of these strict ranking axioms. Also the cycle complete solution presented in Trudeau (2012) has some discriminative power. Nevertheless, this solution still has some reductionist flavor and fails to reward agent 1 strictly in the cost situation in Figure 2 (where all depicted edges have zero cost and all other edges cost 1) for the fact that he has free connections with all other agents, as it divides the unit cost equally among all agents.

Bogomolnaia et al. (2010) study capacity synthesis problems. Here the problem is to construct a minimal cost graph such that any pair of agents is connected by a path of which any edge has at least some minimum level of capacity. Under some simplifying assumptions the capacity synthesis problem boils down to the problem of finding a *maximum* cost spanning tree, albeit that there is no ‘source’ in this problem. This observation enables a more or less direct

¹In Bogomolnaia and Moulin (2010) these axioms are called Strict Ranking₁ and Strict Ranking₂.

Figure 2: A cost situation where the cycle complete solution produces an ‘unfair’ outcome.

translation of solutions and axioms from the mcst literature to the context of capacity synthesis problems. Bogomolnaia et al. (2010) do so and introduce the BHM solution satisfying several versions of the ranking axiom.

None of the solutions with strict ranking potential mentioned above satisfy all ‘established’ axioms (provided they do not imply reductionism of course). Bogomolnaia and Moulin (2010) conclude their paper with an open problem: do solutions exist which are nonnegative, continuous, cost and population monotonic and which meet ranking and agent-based strict ranking? In this paper we present the *degree adjusted folk solution*, a piecewise linear solution which provides an affirmative answer to this question. Moreover, abandoning piecewise linearity, we combine the ideas underlying the degree adjusted folk solution and the ideas presented in Bogomolnaia and Moulin (2010) in order to construct the *cost adjusted folk solution* which also satisfies the appealing properties *strict cost monotonicity* and *source-based strict ranking* (in Bogomolnaia and Moulin (2010) it is argued that piecewise linearity rules out these axioms). As this solution has a direct counterpart in the context of capacity synthesis problems we also respond in this way to an open question put forward in Bogomolnaia et al. (2010), stating “Can we define a solution with a closed form expression, sharing some of the distinctive normative properties of the BHM solution?”

This paper is organized as follows. In section 2 we present some preliminaries on mcst problems and present the axioms which are mentioned in the open problem of Bogomolnaia and Moulin (2010). The degree adjusted folk solution is presented in section 3, where we show that this solution solves the aforementioned open problem. In section 4 we present the cost adjusted folk solution. We conclude in section 5.

2 Preliminaries

In this paper we consider mcst problems with different agent sets. We assume that these finite agent sets are subsets of $\mathbb{N} = \{1, 2, 3, \dots\}$ and use 0 to denote the source. For a finite set $N \subseteq \mathbb{N}$ we write $N_0 = N \cup \{0\}$ and we let E_{N_0} denote the collection of subsets of N_0 of size 2 (the *edge set on N_0*). Such a subset will be denoted as ij instead of $\{i, j\}$. The collection of nonnegative cost vectors on E_{N_0} will be denoted by $\mathcal{C}^N$, i.e. $\mathcal{C}^N = \mathbb{R}_+^{E_{N_0}}$. Moreover, $\mathcal{RC}^N$ will denote the collection of cost vectors $c \in \mathcal{C}^N$ such that $c_e < c_{0k}$ for all $e \in E_{N_0}$ with $e \subseteq N$ and all $k \in N$. So $\mathcal{RC}^N$ corresponds to the problems where the cost of an edge between two agents is always strictly smaller than the cost of an edge between some agent and the source. If $c \in \mathcal{C}^N$ is such that $c_e \in \{0, 1\}$ for every $e \in E_{N_0}$ then c is called an *elementary cost vector*. The collection of elementary cost vectors on E_{N_0} is denoted by $\mathcal{E}^N$. An *mcst problem* is a pair (N, c) where $N \subseteq \mathbb{N}$ is a finite set and $c \in \mathcal{C}^N$. The collection of all mcst problems is denoted

by $\mathcal{M}$. The collection of mcst problems with an elementary cost vector is denoted by $\mathcal{E}$.

Let $(N, c) \in \mathcal{M}$. A *spanning tree* is a $T \subseteq E_{N_0}$ such that (N_0, T) is a connected graph without cycles. The cost of such a tree is $\sum_{e \in T} c_e$. A *minimum cost spanning tree* is a spanning tree with minimal cost. The cost of an mcst will be denoted by $C(N, c)$. Using the terminology of Bogomolnaia and Moulin (2010) the associated *strict stand alone game* is the TU cost game $(N, \hat{c})$ defined by $\hat{c}(S) = C(S, c_S)$ for every $S \subseteq N, S \neq \emptyset$, where $c_S \in \mathcal{C}^S$ is the restriction of c to E_{S_0} . The *stand alone game* is the TU cost game $(N, \tilde{c})$ defined by $\tilde{c}(S) = \min\{\hat{c}(T) : S \subseteq T \subseteq N\}$ for every $S \subseteq N, S \neq \emptyset$. In the strict stand alone game coalition S can only use edges in E_{S_0} , whereas in the stand alone game agents in S are allowed to use edges in $E_{N_0} \setminus E_{S_0}$ in order to construct an optimal tree.

A *solution* is a map φ with domain $\mathcal{M}$ that assigns to every $(N, c) \in \mathcal{M}$ a vector $\varphi(N, c) \in \mathbb{R}_+^N$ (nonnegativity) such that $\sum_{i \in N} \varphi_i(N, c) = C(N, c)$ (cost efficiency). This vector represents a division of the cost of building an mcst among the agents, where each agent has to pay a nonnegative amount. A *partial solution* is a map with domain $\mathcal{E}$ satisfying nonnegativity and cost efficiency as well.

In the recent literature on mcst problems many desirable properties for solutions have been proposed. Let us first list the properties mentioned in the open problem stated by Bogomolnaia and Moulin (2010). A solution φ satisfies

- *Continuity (Co)* if for all finite $N \subseteq \mathbb{N}$ the map $\varphi(N, \cdot)$ is a continuous map from $\mathcal{C}^N$ to $\mathbb{R}_+^N$;
- *Cost Monotonicity (CM)* if for all finite $N \subseteq \mathbb{N}$, all $c, c' \in \mathcal{C}^N$, all $i \in N$, and all $k \in N_0 \setminus \{i\}$ with $c_{ik} < c'_{ik}$ and $c_e = c'_e$ for all $e \in E_{N_0} \setminus \{ik\}$, we have $\varphi_i(N, c) \leq \varphi_i(N, c')$;
- *Population Monotonicity (PM)* if for all finite $N \subseteq \mathbb{N}$, all $i \in N$, all $k \in \mathbb{N} \setminus N$, all $c \in \mathcal{C}^N$, and all $c' \in \mathcal{C}^{N \cup \{k\}}$ such that $c'_N = c$, we have $\varphi_i(N \cup \{k\}, c') \leq \varphi_i(N, c)$;
- *Ranking (Ra)* if for all finite $N \subseteq \mathbb{N}$, all $c \in \mathcal{C}^N$, and all $i, j \in N$ such that $c_{ik} \leq c_{jk}$ for every $k \in N_0 \setminus \{i, j\}$, we have $\varphi_i(N, c) \leq \varphi_j(N, c)$;
- *Agent-based Strict Ranking (ASR)* if for all finite $N \subseteq \mathbb{N}$ with $|N| \geq 3$, all $c \in \mathcal{RC}^N$, and all $i, j \in N$ such that $c_{ik} < c_{jk}$ for all $k \in N \setminus \{i, j\}$ and $c_{0i} \leq c_{0j}$, we have $\varphi_i(N, c) < \varphi_j(N, c)$.

The interpretation of these axioms is obvious: *Co* implies that small changes in the costs do not lead to large changes in the division of cost, *CM* states that decreasing the cost of one edge should not harm one of the agents involved, *PM* states that new entrants should be always welcome as they can not increase the costs of agents already present and *Ra* implies that agents with smaller connection costs should not pay more than agents with larger connection costs. *ASR* is a sophistication of *Ra* as it requires agents with strictly smaller connection costs to other agents and weakly smaller connection costs to the source to be strictly better off.

3 The degree adjusted folk solution

Many papers use a canonical decomposition of cost vectors into elementary cost vectors in order to obtain solutions on $\mathcal{M}$. In order to describe this decomposition method we need the concept of carrier of an elementary cost vector: for an elementary cost vector $b \in \mathcal{E}^N$ we define $Ca(b) = \{e \in E_{N_0} : b_e = 1\}$, the *carrier* of b , and $G^b = (N_0, E_{N_0} \setminus Ca(b))$, the *graph corresponding to b* . So the edges in G^b are precisely the edges having zero cost according to

b. Now for a nonzero cost vector $c \in \mathcal{C}^N$ let $\alpha_1 < \alpha_2 < \dots < \alpha_l$ be the distinct positive values of c and, for all $k \in \{1, \dots, l\}$, let $b_k \in \mathcal{E}^N$ be such that $Ca(b_k) = \{e \in E_{N_0} : c_e \geq \alpha_k\}$. Then

$$c = \sum_{k=1}^l \beta_k \cdot b_k, \quad (1)$$

where $\beta_1 = \alpha_1$ and, for all $k \in \{2, \dots, l\}$ (if any), $\beta_k = \alpha_k - \alpha_{k-1}$. Note that decomposition (1) can be computed in an algorithmic way in polynomial time in $|N|$.

An at first sight inefficient way of writing decomposition (1) is the following

$$c = \sum_{b \in \mathcal{E}^N} \gamma(c, b) \cdot b, \quad (2)$$

where

$$\gamma(c, b) = \begin{cases} \max\left\{\min_{e \in Ca(b)} c_e - \max_{e \notin Ca(b)} c_e, 0\right\} & \text{if } Ca(b) \neq E_{N_0} \\ \min_{e \in Ca(b)} c_e & \text{if } Ca(b) = E_{N_0}. \end{cases}$$

The advantage of closed form formula (2) however is that it is immediate that the nonnegative coefficients $\gamma(\cdot, b)$ are continuous on $\mathcal{C}^N$ for every $b \in \mathcal{E}^N$.

In Norde et al. (2004) it was shown that decomposition (2) carries over to the corresponding strict stand alone TU games:

$$\hat{c} = \sum_{b \in \mathcal{E}^N} \gamma(c, b) \cdot \hat{b}.$$

This observation has a double consequence. First it enables us to extend any partial solution L on $\mathcal{E}$ to a ‘full’ solution φ^L on $\mathcal{M}$ via the formula

$$\varphi^L(N, c) = \sum_{b \in \mathcal{E}^N} \gamma(c, b) \cdot L(N, b), \quad (3)$$

which is moreover continuous and piecewise linear². Second, nice properties of L may carry over to nice properties of φ^L by using linearity arguments. This second statement has been exploited explicitly in Proposition 6 of Bogomolnaia and Moulin (2010).

Therefore let us consider partial solutions. For every $(N, b) \in \mathcal{E}$ one easily verifies that $C(N, b)$ equals the number of connected components in G^b minus one. It is a straightforward exercise to check that all nonnegative core elements³ of the corresponding (strict) stand alone games $(N, \hat{b})$ and $(N, \tilde{b})$ are obtained by assigning zero cost to all agents in the component containing 0, and by dividing the unit cost of all other components in some (nonnegative) way among the agents in this component. Two obvious ways of doing this is by equal division and by allocation proportional to the degree of the agents in graph G^b (the higher the degree, the smaller the cost share). This leads to the partial solutions ED and deg . In order to introduce these partial solutions formally we need some more notation. For every $i \in N$ let $C_i(N, b)$ denote the component in G^b containing agent i and let $n_i(N, b) = |C_i(N, b)|$ denote the cardinality of this component. Moreover, let $d_i(N, b) = 1 + |\{j \in C_i(N, b) \setminus \{i\} : b_{ij} = 1\}|$,

²For any bijection $\pi : \{1, 2, \dots, |E_{N_0}|\} \rightarrow E_{N_0}$ the map $\varphi^L(N, \cdot)$ is linear on the domain $\{c \in \mathcal{C}^N : c_{\pi(1)} \leq c_{\pi(2)} \leq \dots \leq c_{\pi(|E_{N_0}|)}\}$.

³A *core element* of TU cost game (N, c') is a vector $x \in \mathbb{R}^N$ such that $\sum_{i \in S} x_i \leq c'(S)$ for every $S \subseteq N$, $S \neq \emptyset$, and $\sum_{i \in N} x_i = c'(N)$.

i.e. $d_i(N, b)$ equals the cardinality of $C_i(N, b)$ minus the degree of i in G^b . Now, for every $i \in N$ we define

$$ED_i(N, b) = \begin{cases} \frac{1}{n_i(N, b)} & \text{if } 0 \notin C_i(N, b) \\ 0 & \text{if } 0 \in C_i(N, b), \end{cases}$$

and

$$deg_i(N, b) = \begin{cases} \frac{d_i(N, b)}{\sum_{j \in C_i(N, b)} d_j(N, b)} & \text{if } 0 \notin C_i(N, b) \\ 0 & \text{if } 0 \in C_i(N, b). \end{cases}$$

Now $\mathcal{F} = \varphi^{ED}$ is the well-known folk solution that satisfies *Co*, *CM*, *PM* and *Ra*, but fails to satisfy *ASR*, as argued in Bogomolnaia and Moulin (2010). The solution $\mathcal{D} = \varphi^{deg}$ is not so famous. We refer to it as the *degree solution*.⁴ It is not hard to see that it satisfies *Co*, *Ra* and *ASR*, but not *CM* and *PM*.

Example 1 Let $(N, b) \in \mathcal{E}$ be such that $N = \{1, 2, 3, 4, 5, 6, 7, 8\}$ and G^b is as depicted below.

Note that $d_1(N, b) = 2$, $d_i(N, b) = 7$ for $i \in \{2, 3, 4, 5, 6, 8\}$, and $d_7(N, b) = 6$. So $\mathcal{D}_1(N, b) = 2/50$. Now let $b' \in \mathcal{E}^N$ be the elementary cost vector obtained from b by increasing the cost of edge 17 from 0 to 1. That is, $G^{b'}$ is obtained from G^b by deleting edge 17. One easily verifies that $\mathcal{D}_1(N, b') = 1/26 < 2/50 = \mathcal{D}_1(N, b)$, which illustrates that $\mathcal{D}$ does not satisfy *CM*.

Now consider $S = N \setminus \{8\}$ and the corresponding elementary cost vector $b_S \in \mathcal{E}^S$. Note that G^{b_S} is obtained from G^b by deleting node 8 and edge 78. Now $\mathcal{D}_1(S, b_S) = 1/37 < 2/50 = \mathcal{D}_1(N, b)$, which illustrates that $\mathcal{D}$ does not satisfy *PM*. $\triangleleft$

Next, we define the degree adjusted folk solution.

Definition 2 The partial solution K is defined in the following way: for every $(N, b) \in \mathcal{E}$ and every $i \in N$ we have

$$K_i(N, b) = \left(1 - \frac{1}{2(n_i(N, b) + 1)^2}\right) \cdot ED_i(N, b) + \frac{1}{2(n_i(N, b) + 1)^2} \cdot deg_i(N, b).$$

The resulting solution $\mathcal{DF} = \varphi^K$ is called the *degree adjusted folk solution*.

Partial solution K divides the unit cost of any component not containing source 0 by taking a convex combination of the allocations according to *ED* and *deg*, where the weights depend upon the size of the component (the larger the component the larger the weight of *ED*). Note that $K_i(N, b) = 0$ if $0 \in C_i(N, b)$ and $K_i(N, b) = 1$ if $n_i(N, b) = 1$ (i.e. if $C_i(N, b) = \{i\}$).

⁴The degree solution $\mathcal{D}$ has some flavor of the analog of the BHM solution (Bogomolnaia et al. (2010)) in the context of minimum cost spanning tree problems: if $b \in \mathcal{E}^N$ and $i \in N$ is such that $C_i(N, b)$ is a tree with $0 \notin C_i(N, b)$ then the latter solution assigns cost $1 - \frac{1}{2}(n_i(N, b) - d_i(N, b))$ to agent i .

Example 3 Let $(N, b) \in \mathcal{E}$ be such that $N = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ and G^b is as depicted.

Here

$$\begin{aligned}\mathcal{F}(N, b) &= \frac{1}{12}(0, 0, 3, 3, 3, 3, 4, 4, 4), \\ \mathcal{D}(N, b) &= \frac{1}{40}(0, 0, 10, 10, 5, 15, 8, 16, 16), \\ \mathcal{DF}(N, b) &= \frac{1}{2400}(0, 0, 600, 600, 594, 606, 790, 805, 805).\end{aligned}$$

Note that the fact that G^b has components of different size implies that $\mathcal{DF}(N, b)$ is not a convex combination of $\mathcal{F}(N, b)$ and $\mathcal{D}(N, b)$. $\triangleleft$

Also note that the weight on ED is large enough for letting K have the same property as ED : any cost share in some component not containing 0 always strictly dominates a cost share in a larger component (in size), even if these components correspond to different elementary cost vectors. This property is essential for proving that $\mathcal{DF}$ satisfies CM and PM and is formalized in the following lemma.

Lemma 4 For any $(N, b), (N', b') \in \mathcal{E}$ and any $i \in N, j \in N'$ such that $0 \notin C_i(N, b)$ and $n_i(N, b) < n_j(N', b')$ we have $K_i(N, b) > K_j(N', b')$.

Proof: If $0 \in C_j(N', b')$ we have $K_i(N, b) > 0 = K_j(N', b')$, so assume $0 \notin C_j(N', b')$. Then we have

$$\begin{aligned}K_i(N, b) &= ED_i(N, b) + \frac{1}{2(n_i(N, b) + 1)^2} (deg_i(N, b) - ED_i(N, b)) \\ &= \frac{1}{n_i(N, b)} + \frac{1}{2(n_i(N, b) + 1)^2} (deg_i(N, b) - ED_i(N, b)) \\ &\geq \frac{1}{n_i(N, b)} - \frac{1}{2(n_i(N, b) + 1)^2} \\ &> \frac{1}{n_i(N, b)} - \frac{1}{2n_i(N, b)(n_i(N, b) + 1)} \\ &= \frac{1}{2} \left(\frac{1}{n_i(N, b)} + \frac{1}{n_i(N, b) + 1} \right) \\ &\geq \frac{1}{2} \left(\frac{1}{n_j(N', b') - 1} + \frac{1}{n_j(N', b')} \right) \\ &= \frac{1}{n_j(N', b')} + \frac{1}{2n_j(N', b')(n_j(N', b') - 1)} \\ &> \frac{1}{n_j(N', b')} + \frac{1}{2(n_j(N', b') + 1)^2} \\ &\geq \frac{1}{n_j(N', b')} + \frac{1}{2(n_j(N', b') + 1)^2} (deg_j(N', b') - ED_j(N', b'))\end{aligned}$$

$$\begin{aligned}
&= ED_j(N', b') + \frac{1}{2(n_j(N', b') + 1)^2} (deg_j(N', b') - ED_j(N', b')) \\
&= K_j(N', b').
\end{aligned}$$

□

Now we provide a solution to the open problem posed in Bogomolnaia and Moulin (2010).

Theorem 5 $\mathcal{DF}$ satisfies *Co*, *CM*, *PM*, *Ra* and *ASR*.

Proof: *Co* follows from (3), any partial solution extends to a continuous solution.

In order to show that $\mathcal{DF}$ satisfies *CM*, *PM*, and *Ra* it is sufficient, using Proposition 6 of Bogomolnaia and Moulin (2010), that partial solution K obeys the required properties on $\mathcal{E}$. First we show that K is cost monotonic, i.e. that for any finite $N \subseteq \mathbb{N}$, $b, b' \in \mathcal{E}^N$, $i \in N$, $k \in N_0 \setminus \{i\}$ with $b_{ik} = 0$ and $b'_{ik} = 1$ and $b_e = b'_e$ for all $e \in E_{N_0} \setminus \{ik\}$ we have $K_i(N, b) \leq K_i(N, b')$. If $0 \in C_i(N, b)$ we trivially have $K_i(N, b) = 0 \leq K_i(N, b')$. So from now we assume that $0 \notin C_i(N, b)$. Then $k \neq 0$. We distinguish between two cases. First assume that $k \in C_i(N, b')$. Then $C_i(N, b) = C_i(N, b')$, so $n_i(N, b) = n_i(N, b')$, and hence $ED_i(N, b) = ED_i(N, b')$. Moreover, we have $d_i(N, b') = d_i(N, b) + 1$, $d_k(N, b') = d_k(N, b) + 1$ and $d_j(N, b') = d_j(N, b)$ for every $j \in N \setminus \{i, k\}$. Now

$$\begin{aligned}
d_i(N, b) &= 1 + |\{j \in C_i(N, b) \setminus \{i\} : b_{ij} = 1\}| \\
&\leq \sum_{j \in C_i(N, b) \setminus \{i\}} (1 + |\{l \in C_j(N, b) \setminus \{j\} : b_{jl} = 1\}|) \\
&= \sum_{j \in C_i(N, b) \setminus \{i\}} d_j(N, b)
\end{aligned}$$

and hence

$$\frac{d_i(N, b)}{\sum_{j \in C_i(N, b)} d_j(N, b)} \leq \frac{d_i(N, b) + 1}{\sum_{j \in C_i(N, b)} d_j(N, b) + 2} = \frac{d_i(N, b')}{\sum_{j \in C_i(N, b')} d_j(N, b')},$$

which is equivalent with $deg_i(N, b) \leq deg_i(N, b')$. So, $K_i(N, b) \leq K_i(N, b')$. Second, assume that $k \notin C_i(N, b')$. Then $C_i(N, b) = C_i(N, b') \cup C_k(N, b')$ and hence $n_i(N, b) > n_i(N, b')$. According to Lemma 4 we have $K_i(N, b) < K_i(N, b')$.

Now we show that K is population monotonic, i.e. that for any finite $N \subseteq \mathbb{N}$, $i \in N$, $k \in \mathbb{N} \setminus N$, $b \in \mathcal{E}^N$, $b' \in \mathcal{E}^{N \cup \{k\}}$ such that $b'_N = b$, we have $K_i(N \cup \{k\}, b') \leq K_i(N, b)$. Note that $C_i(N, b) \subseteq C_i(N \cup \{k\}, b')$ and hence $n_i(N, b) \leq n_i(N \cup \{k\}, b')$. If $0 \in C_i(N \cup \{k\}, b')$ we have $K_i(N \cup \{k\}, b') = 0 \leq K_i(N, b)$ by definition. If $0 \notin C_i(N \cup \{k\}, b')$ and $n_i(N, b) < n_i(N \cup \{k\}, b')$ then $K_i(N \cup \{k\}, b') < K_i(N, b)$ according to Lemma 4. So assume $0 \notin C_i(N \cup \{k\}, b')$ and $n_i(N, b) = n_i(N \cup \{k\}, b')$. Then $C_i(N, b) = C_i(N \cup \{k\}, b')$, so $ED_i(N, b) = ED_i(N \cup \{k\}, b')$ and $deg_i(N, b) = deg_i(N \cup \{k\}, b')$. Hence $K_i(N, b) = K_i(N \cup \{k\}, b')$.

In order to show that K obeys ranking, let $N \subseteq \mathbb{N}$ be a finite set, $b \in \mathcal{E}^N$, and $i, j \in N$ such that $b_{ik} \leq b_{jk}$ for every $k \in N_0 \setminus \{i, j\}$. We have to show that $K_i(N, b) \leq K_j(N, b)$. First assume that $C_j(N, b) = \{j\}$. Then $ED_i(N, b) \leq 1 = ED_j(N, b)$ and $deg_i(N, b) \leq 1 = deg_j(N, b)$, so $K_i(N, b) \leq K_j(N, b)$. Now assume that $C_j(N, b) \neq \{j\}$. Then there is a $k \in N_0 \setminus \{j\}$ with $b_{jk} = 0$. If $k = i$ then $b_{ij} = 0$, otherwise we have $b_{ik} = b_{jk} = 0$. In both situations we have $j \in C_i(N, b)$ so $C_i(N, b) = C_j(N, b)$. If $0 \in C_i(N, b) (= C_j(N, b))$ then $K_i(N, b) = K_j(N, b) = 0$ by definition. If $0 \notin C_i(N, b) (= C_j(N, b))$ then $ED_i(N, b) = ED_j(N, b)$. Moreover, for all $k \in C_i(N, b) \setminus \{i, j\}$ we have that $b_{ik} = 1$ implies $b_{jk} = 1$, so $d_i(N, b) \leq d_j(N, b)$. Hence $deg_i(N, b) \leq deg_j(N, b)$ and consequently $K_i(N, b) \leq K_j(N, b)$.

Finally, let us show that $\mathcal{DF}$ satisfies *ASR*. Let $N \subseteq \mathbb{N}$ be a finite set with $|N| \geq 3$, $c \in \mathcal{RC}^N$, and $i, j \in N$ such that $c_{ik} < c_{jk}$ for all $k \in N \setminus \{i, j\}$ and $c_{0i} \leq c_{0j}$. We have to show that $\mathcal{DF}_i(N, c) < \mathcal{DF}_j(N, c)$. Note that from ranking it follows that $\mathcal{DF}_i(N, c) \leq \mathcal{DF}_j(N, c)$. In fact, we have shown above that $K_i(N, b) \leq K_j(N, b)$ for all $b \in \mathcal{E}^N$ with $\gamma(c, b) > 0$. Hence it is sufficient to show that there is at least one $b \in \mathcal{E}^N$ with $\gamma(c, b) > 0$ such that $K_i(N, b) < K_j(N, b)$. Let $c^* = \max_{k \in N \setminus \{i, j\}} c_{jk}$ and let $b \in \mathcal{E}^N$ be such that $Ca(b) = \{e \in E_{N_0} : c_e \geq c^*\}$. Then $\gamma(c, b) > 0$, $b_{ik} = 0$ for all $k \in N \setminus \{i, j\}$ and there is a $k \in N \setminus \{i, j\}$ with $b_{jk} = 1$. Moreover $b_{0l} = 1$ for all $l \in N$, so $0 \notin C_i(N, b)$ and $0 \notin C_j(N, b)$. Note that if $j \notin C_i(N, b)$, i.e. if $b_{jl} = 1$ for all $l \in N \setminus \{j\}$, then $C_j(N, b) = \{j\}$. Hence in that case $ED_i(N, b) = 1/(|N| - 1) < 1 = ED_j(N, b)$ and $deg_i(N, b) \leq 1 = deg_j(N, b)$, so consequently $K_i(N, b) < K_j(N, b)$. Now suppose $j \in C_i(N, b)$. Then $C_i(N, b) = C_j(N, b)$, hence $ED_i(N, b) = ED_j(N, b)$. Moreover $d_i(N, b) < d_j(N, b)$ (if $b_{ij} = 0$ then $d_i(N, b) = 1$ and $d_j(N, b) \geq 2$ whereas if $b_{ij} = 1$ we have $d_i(N, b) = 2$ and $d_j(N, b) \geq 3$), so $deg_i(N, b) < deg_j(N, b)$ and consequently $K_i(N, b) < K_j(N, b)$. $\square$

4 The cost adjusted folk solution

As argued in Bogomolnaia and Moulin (2010) solutions that are not piecewise linear can be constructed by generalizing (3) to

$$\psi^{\tilde{L}}(N, c) = \sum_{b \in \mathcal{E}^N} \gamma(c, b) \cdot \tilde{L}(N, b, c), \quad (4)$$

where, for every $(N, b) \in \mathcal{E}$ and $c \in \mathcal{C}^N$ the vector $\tilde{L}(N, b, c) \in \mathbb{R}_+^N$ is such that $\sum_{i \in N} \tilde{L}_i(N, b, c) = C(N, b)$. If, for every $(N, b) \in \mathcal{E}$, the map $\tilde{L}(N, b, \cdot)$ is continuous on $\mathcal{C}^N$, then $\psi^{\tilde{L}}$ satisfies *Co*. If $\tilde{L}(N, b, c)$ can be computed in polynomial time for every $(N, b) \in \mathcal{E}$ and $c \in \mathcal{C}^N$ then it follows again from (1) that $\psi^{\tilde{L}}(N, c)$ can be computed in polynomial time as well for every $c \in \mathcal{C}^N$.

Inspired by the definition immediately preceding Proposition 9 in Bogomolnaia and Moulin (2010), we define for every finite $(N, b) \in \mathcal{E}$, $c \in \mathcal{C}^N$ and $i \in N$ with $0 \notin C_i(N, b)$ the number

$$\delta_i(N, b, c) = (1 + c_{0i}) \cdot \prod_{j \in C_i(N, b), j \neq i} (1 + c_{ij}) \quad (5)$$

with the convention that empty products equal 1, i.e. if $C_i(N, b) = \{i\}$ then $\delta_i(N, b, c) = 1 + c_{0i}$.⁵ Subsequently, for every finite $(N, b) \in \mathcal{E}$ and $c \in \mathcal{C}^N$ the vector $D(N, b, c) \in \mathbb{R}_+^N$ is defined by

$$D_i(N, b, c) = \begin{cases} \frac{\delta_i(N, b, c)}{\sum_{j \in C_i(N, b)} \delta_j(N, b, c)} & \text{if } 0 \notin C_i(N, b) \\ 0 & \text{if } 0 \in C_i(N, b), \end{cases}$$

for every $i \in N$.

Definition 6 For every $(N, b) \in \mathcal{E}$, $c \in \mathcal{C}^N$, and $i \in N$ we define

$$\tilde{K}_i(N, b, c) = \left(1 - \frac{1}{2(n_i(N, b) + 1)^2}\right) \cdot ED_i(N, b) + \frac{1}{2(n_i(N, b) + 1)^2} \cdot D_i(N, b, c).$$

The resulting solution $\mathcal{CF} = \psi^{\tilde{K}}$ is called the *cost adjusted folk solution*.

⁵Main difference of this approach with that of Bogomolnaia and Moulin (2010) is that here only connection costs of agent i with other agents in $C_i(N, b)$ (instead of $N \setminus \{i\}$) are taken into account.

Note that for every $(N, b) \in \mathcal{E}$, the map $\tilde{K}(N, b, \cdot)$ is continuous on $\mathcal{C}^N$, and that $\tilde{K}(N, b, c)$ can be computed in polynomial time for every $(N, b) \in \mathcal{E}$ and $c \in \mathcal{C}^N$. So $\mathcal{CF}$ satisfies *Co* and $\mathcal{CF}(N, c)$ can be computed in polynomial time for every $c \in \mathcal{C}^N$. Lemma 4 can be generalized to this setting. The proof is the same and hence omitted.

Lemma 7 *For any $(N, b), (N', b') \in \mathcal{E}$, any $c \in \mathcal{C}^N$, $c' \in \mathcal{C}^{N'}$ and any $i \in N$, $j \in N'$ such that $0 \notin C_i(N, b)$ and $n_i(N, b) < n_j(N', b')$ we have $\tilde{K}_i(N, b, c) > \tilde{K}_j(N', b', c')$.*

It can also be shown easily that $\mathcal{CF}$ satisfies *PM*, *Ra* and *ASR* by adjusting the corresponding proofs for $\mathcal{DF}$ in Theorem 5 in a straightforward way. In order to show that $\mathcal{CF}$ satisfies *CM* we have to do some more effort. We will also show that $\mathcal{CF}$ satisfies the axiom of Strict Cost Monotonicity as defined in Bogomolnaia and Moulin (2010): solution φ satisfies *Strict Cost Monotonicity (SCM)* if for all finite $N \subseteq \mathbb{N}$, all $c, c' \in \mathcal{RC}^N$, all $i \in N$, and all $k \in N_0 \setminus \{i\}$ with $c_{ik} < c'_{ik}$ and $c_e = c'_e$ for all $e \in E_{N_0} \setminus \{ik\}$, we have $\varphi_i(N, c) < \varphi_i(N, c')$.

Lemma 8 *$\mathcal{CF}$ satisfies *CM* and *SCM*.*

Proof: First we prove that $\mathcal{CF}$ satisfies *CM*. Let $N \subseteq \mathbb{N}$ be a finite set, $c, c' \in \mathcal{C}^N$, $i \in N$ and $k \in N_0 \setminus \{i\}$ such that $c_{ik} < c'_{ik}$ and $c_e = c'_e$ for all $e \in E_{N_0} \setminus \{ik\}$. Let $\alpha_1 < \alpha_2 < \dots < \alpha_l$ be the distinct positive values of c and c' together, i.e. $\{\alpha_1, \alpha_2, \dots, \alpha_l\} = \{c_e : e \in E_{N_0}, c_e > 0\} \cup \{c'_{ik}\}$, and for all $p \in \{1, \dots, l\}$, let $b_p, b'_p \in \mathcal{E}^N$ be such that $Ca(b_p) = \{e \in E_{N_0} : c_e \geq \alpha_p\}$ and $Ca(b'_p) = \{e \in E_{N_0} : c'_e \geq \alpha_p\}$. Note that $Ca(b'_p) = Ca(b_p)$ or $Ca(b'_p) = Ca(b_p) \cup \{ik\}$ for all $p \in \{1, \dots, l\}$. Now

$$\mathcal{CF}_i(N, c) - \mathcal{CF}_i(N, c') = \sum_{p=1}^l \beta_p \cdot (\tilde{K}_i(N, b_p, c) - \tilde{K}_i(N, b'_p, c')), \quad (6)$$

where $\beta_1 = \alpha_1$ and, for all $p \in \{2, \dots, l\}$ (if any), $\beta_p = \alpha_p - \alpha_{p-1}$. It is sufficient to prove that $\tilde{K}_i(N, b_p, c) \leq \tilde{K}_i(N, b'_p, c')$ for all $p \in \{1, \dots, l\}$.

So, let $p \in \{1, \dots, l\}$ and let $\lambda = \frac{1+c'_{ik}}{1+c_{ik}}$. Obviously $\lambda > 1$. Note that $C_i(N, b'_p) \subseteq C_i(N, b_p)$. If $0 \in C_i(N, b_p)$ then $\tilde{K}_i(N, b_p, c) = 0 \leq \tilde{K}_i(N, b'_p, c')$ so from now we assume $0 \notin C_i(N, b_p)$ (and hence $0 \notin C_i(N, b'_p)$). First, consider the case where $C_i(N, b'_p) = C_i(N, b_p)$. If $k = 0$ then $\delta_i(N, b'_p, c') = \lambda \delta_i(N, b_p, c)$ and $\delta_j(N, b'_p, c') = \delta_j(N, b_p, c)$ for every $j \in C_i(N, b_p) \setminus \{i\}$ from which in a straightforward way it follows that $D_i(N, b_p, c) \leq D_i(N, b'_p, c')$. Since $ED_i(N, b'_p) = ED_i(N, b_p)$ we conclude that $\tilde{K}_i(N, b_p, c) \leq \tilde{K}_i(N, b'_p, c')$. If $k \neq 0$ and $k \notin C_i(N, b_p)$ then we have $\tilde{K}_i(N, b_p, c) = \tilde{K}_i(N, b'_p, c')$ by definition. If $k \neq 0$ and $k \in C_i(N, b_p)$ then $\delta_i(N, b'_p, c') = \lambda \delta_i(N, b_p, c)$, $\delta_k(N, b'_p, c') = \lambda \delta_k(N, b_p, c)$ and $\delta_j(N, b'_p, c') = \delta_j(N, b_p, c)$ for every $j \in C_i(N, b_p) \setminus \{i, k\}$. Again via straightforward algebraic operations we get $D_i(N, b_p, c) \leq D_i(N, b'_p, c')$ and hence $\tilde{K}_i(N, b_p, c) \leq \tilde{K}_i(N, b'_p, c')$. Finally, we consider the case where $C_i(N, b'_p) \neq C_i(N, b_p)$. Then we have $n_i(N, b_p) > n_i(N, b'_p)$. Now from Lemma 7 it follows that $\tilde{K}_i(N, b_p, c) \leq \tilde{K}_i(N, b'_p, c')$.

In order to show that $\mathcal{CF}$ satisfies *SCM* it is sufficient to add to the proof above the proof of the fact that there is at least one $p \in \{1, \dots, l\}$ with $\tilde{K}_i(N, b_p, c) < \tilde{K}_i(N, b'_p, c')$, of course under the extra assumption that $c, c' \in \mathcal{RC}^N$. In the trivial case that $N = \{i\}$ we have $\tilde{K}_i(N, b_i, c) = 0 < 1 = \tilde{K}_i(N, b'_i, c')$, hence we assume that $|N| \geq 2$. Choose $p^* \in \{1, \dots, l\}$ such that $\alpha_{p^*} = \min_{j \in N} c_{0j}$, which is positive since $c \in \mathcal{RC}^N$. Now $b_{p^*} = b'_{p^*}$ and $Ca(b_{p^*}) = Ca(b'_{p^*}) = \{0j : j \in N\}$, so $C_i(N, b_{p^*}) = C_i(N, b'_{p^*}) = N$ and hence $ED_i(N, b_{p^*}) = ED_i(N, b'_{p^*})$. If $k = 0$ we have $\delta_i(N, b'_{p^*}, c') = \lambda \delta_i(N, b_{p^*}, c)$ and $\delta_j(N, b'_{p^*}, c') = \delta_j(N, b_{p^*}, c)$ for every $j \in N \setminus \{i\}$ from which it follows that $D_i(N, b_{p^*}, c) < D_i(N, b'_{p^*}, c')$ and hence $\tilde{K}_i(N, b_{p^*}, c) < \tilde{K}_i(N, b'_{p^*}, c')$. If

$k \neq 0$ and $|N| \geq 3$ we have $\delta_i(N, b'_{p^*}, c') = \lambda \delta_i(N, b_{p^*}, c)$, $\delta_k(N, b'_{p^*}, c') = \lambda \delta_k(N, b_{p^*}, c)$ and $\delta_j(N, b'_{p^*}, c') = \delta_j(N, b_{p^*}, c)$ for every $j \in N \setminus \{i, k\}$. Again it follows that $D_i(N, b_{p^*}, c) < D_i(N, b'_{p^*}, c')$ and hence $\tilde{K}_i(N, b_{p^*}, c) < \tilde{K}_i(N, b'_{p^*}, c')$. The only case which remains to be considered is $k \neq 0$ and $|N| = 2$ (i.e. $N = \{i, k\}$). Choose $p \in \{1, \dots, l\}$ such that $\alpha_p = c'_{ik}$. Now $Ca(b_p) = \{0i, 0k\}$ and $Ca(b'_p) = \{0i, 0k, ik\}$, so $\tilde{K}_i(N, b_p, c) < 1 = \tilde{K}_i(N, b'_p, c')$. $\square$

Finally, we want to show that $\mathcal{CF}$ satisfies the axiom of Source-based Strict Ranking as well: solution φ satisfies *Source-based Strict Ranking (SSR)* if for all finite $N \subseteq \mathbb{N}$, all $c \in \mathcal{RC}^N$, and all $i, j \in N$ such that $c_{ik} \leq c_{jk}$ for all $k \in N \setminus \{i, j\}$ and $c_{0i} < c_{0j}$, we have $\varphi_i(N, c) < \varphi_j(N, c)$.

Lemma 9 $\mathcal{CF}$ satisfies SSR.

Proof: Let $N \subseteq \mathbb{N}$ be finite, $c \in \mathcal{RC}^N$, and $i, j \in N$ such that $c_{ik} \leq c_{jk}$ for all $k \in N \setminus \{i, j\}$ and $c_{0i} < c_{0j}$. In order to show that $\varphi_i(N, c) < \varphi_j(N, c)$ it is sufficient to show for all $b \in \mathcal{E}^N$ with $\gamma(c, b) > 0$ that $\tilde{K}_i(N, b, c) \leq \tilde{K}_j(N, b, c)$ and that there is at least one such b for which the strict inequality holds. So let $b \in \mathcal{E}^N$ be such that $\gamma(c, b) > 0$. Then $b_{ik} \leq b_{jk}$ for every $k \in N_0 \setminus \{i, j\}$. First assume that $C_j(N, b) = \{j\}$. Then $ED_i(N, b) \leq 1 = ED_j(N, b)$ and $\delta_i(N, b, c) \leq 1 = \delta_j(N, b, c)$, so $\tilde{K}_i(N, b, c) \leq \tilde{K}_j(N, b, c)$. Now assume that $C_j(N, b) \neq \{j\}$. Then there is a $k \in N_0 \setminus \{j\}$ with $b_{jk} = 0$. If $k = i$ then $b_{ij} = 0$, otherwise we have $b_{ik} = b_{jk} = 0$. In both situations we have $j \in C_i(N, b)$ so $C_i(N, b) = C_j(N, b)$. If $0 \in C_i(N, b) (= C_j(N, b))$ then $\tilde{K}_i(N, b, c) = \tilde{K}_j(N, b, c)$ by definition. If $0 \notin C_i(N, b) (= C_j(N, b))$ then $ED_i(N, b) = ED_j(N, b)$. Moreover, from (5) we infer that $\delta_i(N, b, c) < \delta_j(N, b, c)$, and consequently $K_i(N, b) < K_j(N, b)$. Now we conclude the proof by noting that for $b \in \mathcal{E}^N$ such that $Ca(b) = \{0l : l \in N\}$ we have $\gamma(c, b) > 0$ and $C_j(N, b) = N$, so $C_j(N, b) \neq \{j\}$ and $0 \notin C_j(N, b)$. $\square$

5 Conclusions

In the still growing amount of literature on mcst problems the paper by Bogomolnaia and Moulin (2010) is somehow revolutionary as it questions the so far undisputed axiom of reductionism. From an axiomatic point of view the folk solution can be considered as one of the best, perhaps the best, alternatives amongst the collection of reductionist solutions. Sacrificing the reductionism axiom however opens a new world of solutions. Bogomolnaia and Moulin (2010) introduce families of nonreductionist solutions with strict ranking power. None of these however satisfies all established axioms as well. The aim of this paper is to close this gap. It first presents the degree adjusted folk solution that answers an open question posed in Bogomolnaia and Moulin (2010). Moreover it introduces the cost adjusted folk solution that, sacrificing the rather ‘technical’ axiom of piecewise linearity, satisfies even more nice properties. The table below provides an overview of which established axioms are obeyed by the folk solution, the BHM solution⁶, and the new degree and cost adjusted folk solutions. For sure, the two new solutions are not fully characterized by the properties mentioned in this table. In fact, many more solutions can be found obeying these properties as well. Main idea underlying the construction of the degree and cost adjusted folk solutions is to consider componentwise convex combinations of the standard folk solution and the degree solution (in case of the degree adjusted folk solution) or a solution alike the degree solution (in case of the cost adjusted folk solution). Here we still have at least two degrees of freedom: the choice of

⁶Of course in the context of capacity synthesis problems.

the solution playing the role of the degree solution and the choice of the weights needed for the respective convex combinations.

The results in this paper obviously suggest two new research questions:

- 1) Is it possible to fully characterize the degree and/or cost adjusted folk solutions?
- 2) Is it possible to formulate nice extra properties for solutions that are not in conflict with the ones obeyed by the degree and cost adjusted folk solutions and which solutions do obey all of them?

	$\mathcal{F}$	$\mathcal{BHM}$	$\mathcal{DF}$	$\mathcal{CF}$
Reductionism	+	−	−	−
Piecewise Linearity	+	+	+	−
Core selection	+	+	+	+
Continuity	+	+	+	+
Polynomial Complexity	+	+	+	+
Population Monotonicity	+	−	+	+
Cost Monotonicity	+	+	+	+
Ranking	+	+	+	+
Agent-based Strict Ranking	−	+	+	+
Source-based Strict Ranking	−	n.a.	−	+
Strict Cost Monotonicity	−	+	−	+
Closed form expression	+	−	+	+

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