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On solving liability problems

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Abstract

This paper introduces liability problems, as a generalization of bankruptcy problems, where every agent not only owns a certain amount of cash money, but also has outstanding claims and debts towards the other agents. Assuming that the agents want to cash their claims, we will analyze liability rules which prescribe how the total available amount of cash should be allocated among the agents.

In particular we focus on bankruptcy rule based bilateral transfer schemes. Existence of these schemes is established and it is seen that within the class of hierarchical liability problems, such a transfer scheme is unique. Although in general a bankruptcy rule based bilateral transfer scheme need not be unique, we show that the resulting bankruptcy rule based transfer allocation is. This leads to the definition of bankruptcy rule based liability rules. For hierarchical liability problems an alternative characterization of such liability rules is provided. Moreover it is shown that the axiomatic characterization of the Aumann-Maschler bankruptcy rule on the basis of consistency can be extended to the corresponding liability rule. We conclude with a discussion of alternative approaches to solve liability problems.

Keywords: Liability problems, bankruptcy.

JEL classification number: C71, G33.

1 Introduction

The classical bankruptcy problem, consisting of a single estate and multiple claimants, is introduced by O'Neill (1982). A bankruptcy rule prescribes, for each bankruptcy problem, how to divide the estate over the claimants. In the literature one can find a wide variety of bankruptcy rules, which arise from both an axiomatic as well as a game-theoretic analysis, see for an overview, Thomson (2003). The classical bankruptcy problem has been extended in different ways, e.g. to multi-issue allocation situations in which the estate has to be divided among a group of agents with claims stemming from different issues, see Calleja et al. (2005), to stochastic bankruptcy games (Habis and Herings, 2011) and to allocating the losses due to financial distress within a business sector (van Gulick, 2010). Lately, a main trending topic is multiple estates. In a current work of Bjorndal and Jorsten (2009) a bankruptcy problem with multiple banks (estates) is represented by a flow model. The banks can have separate claims on each other and there is a set of agents having separate claims on those banks. Palvolgyi et al. (2010) consider the case of agents with non-homogeneous preferences over multiple estates. Here, the agents have a single claim, but the utility per estate differs. The problem is analyzed from a non-cooperative perspective and focusses on how the agents should divide their claim into subclaims over

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the estates. Moulin and Sethuraman (2012) analyze bipartite rationing problems with multiple estates, agents with a single claim, but in which the agents are not necessarily compatible with all estates. These compatibilities are represented by a bipartite graph. By analyzing the flows in the graph and using a consistency axiom, bankruptcy rules are extended to this setting.

In this paper we introduce liability problems with multiple estates of a rather different nature. In financial accounting a *liability* is defined as an obligation of “an entity arising from past transactions or events, the settlement of which may result in the transfer or use of assets, provision of services or other yielding of economic benefits in the future”³. Usually a liability is associated with an uncertainty, but this need not be the case. The more creditors an agent has, the higher the liabilities. We will investigate the scenario where a group of agents is related by having mutual liabilities, but reaches the point in time where the agents want to cash their claims. Before this time moment, none of the agents worry about the possible insufficient cash in the current assets. Until, for some exogenous reason, individuals start cashing their claims. This will lead to a cascading effect and will reveal the possibly insufficient cash level of agents and the agents typically might not obtain all of what they, however rightfully, claim.

This approach can be seen as a simplified and deterministic model of the monetary inter-relationships between banks, governments and companies in case of a financial crisis and threatening bankruptcy of banks.

A *liability problem* can be represented by a matrix, in which an entry represents a claim from one agent on another agent. The diagonal entries represents the players’ cash levels. A special class of liability problems are the *hierarchical liability problems* in which the claim matrix is triangular. This implies that we can index the agents, such that every agent only claims from agents with lower index. In this sense there is a hierarchy among the agents. For an example, think of the vertical relations in a supply-chain: insufficient cash of a buyer may lead to insufficient cash of his supplier(s).

This paper will analyze liability problems from an allocation perspective: if in a liability problem the agents reach the stage that they want to cash their claims and remove all current liabilities, how should the total amount of available cash be fairly distributed among the agents? In this setting, we implicitly assume that there is an independent authority charged with the task of fairly solving the liability problem. A *liability rule* will for each liability problem prescribe how to allocate the total cash among the agents. In particular we assume each allocation to stem from a so-called *bilateral transfer scheme* that satisfies some basic requirements. More specifically, we consider *bankruptcy rule based transfer schemes*, in which the incoming plus available cash of every agent is allocated among his claimants according to a specific bankruptcy rule. We show that for every bankruptcy rule there always exists a bankruptcy rule based transfer scheme, which is not necessarily unique. Interestingly, it is seen that each bankruptcy rule based transfer scheme leads to the same *bankruptcy rule based transfer allocation*, so allocation-wise a unique outcome is provided. For the subclass of hierarchical liability problems, it is shown that there is also a unique bankruptcy rule based transfer scheme.

These results imply that each bankruptcy rule can be extended to a liability rule: a

³Loosely quoted from the framework of the *International Financial Reporting Standards Foundation*.

bankruptcy rule based liability rule. We provide an explicit characterization for the Aumann-Maschler (AM) rule (Aumann and Maschler, 1985) based liability rule, by extending the properties of consistency and the Concede & Divide-principle from the bankruptcy setting to the context of liability problems.

Profiting from the special structure of hierarchical liability problems, one can extend bankruptcy rules in an alternative recursive way into liability rules. It is shown that for each bankruptcy rule the resulting allocation coincides with the allocation prescribed by the corresponding bankruptcy based liability rule, thus providing another characterization of bankruptcy based liability rules on the class of hierarchical liability problems.

The paper concludes with a sketch of two alternative approaches to solve liability problems. The first alternative involves reducing non-hierarchical problems into more tractable hierarchical liability problems by bilaterally and cyclically leveling the claims. We will see, however, that there is no straightforward procedure how to eliminate the cycles and that different procedures may result in different reduced problems. The second alternative is inspired by the hydraulic rationing methods for claims problems (Kaminski, 2000).

The organization of this paper is as follows. In Section 2 we will formally introduce liability problems. Then, in Section 3 we will give a short introduction to bankruptcy rules, define bankruptcy rule based transfer schemes and corresponding bankruptcy rule based transfer allocations. Section 4 studies liability rules and in particular bankruptcy based liability rules in a hierarchical setting, while Section 5 analyzes bankruptcy based liability rules on the general class of liability problems, including the characterization of the AM-rule based liability rule. Section 6 concludes with two alternative ways to solve liability problems.

2 Liability problems and liability rules

A classical bankruptcy problem involves an estate E that has to be divided among a finite group of agents N , all having a nonnegative claim d_i , $i \in N$, on the estate. We summarize these claims into a vector $d = (d_i)_{i \in N}$. The set of all bankruptcy problems (E, d) on N is denoted by $\mathcal{B}^N$.

In a liability problem, a finite group of economic agents, denoted by N , have been interacting for a certain time period. Their past economic transactions have resulted in a situation in which the agents have claims on each other (think of debtors and creditors or accounts payable and receivable). As in bankruptcy problems, we assume that these claims are known, rightful and justifiable. Further, every agent has a certain nonnegative cash level or cash reserve with which he can (partially) pay his possible debtors. A *liability problem* can be represented by a nonnegative matrix $C \in \mathbb{R}_+^{N \times N}$. Here each cell $c_{ij} \in C$ represents the claim of agent j on agent i , $i \neq j$, and c_{ii} represents the cash level of agent i . If

$$\sum_{i \in N} c_{ii} < \sum_{i, j \in N, i \neq j} c_{ij},$$

there is not sufficient cash to fulfill all the claims. If for some agent $i \in N$,

$$\sum_{j \in N} c_{ji} - \sum_{j \in N \setminus \{i\}} c_{ij} < 0,$$

agent i will never be able to satisfy all his claimants. We will, however, not impose any restrictions except nonnegativity on the matrix C beforehand. The main question is how to divide $\sum_{i \in N} c_{ii}$ over the agents in N .

We denote by $\mathcal{L}^N$ the set of all liability problems on N . A *liability rule* (LR) $f : \mathcal{L}^N \rightarrow \mathbb{R}^N$ is such that $f(C) \geq 0$ and $\sum_{i \in N} f_i(C) = \sum_{i \in N} c_{ii}$ for all $C \in \mathcal{L}^N$.

We will distinguish a class of liability problems with a special triangular structure. A liability problem $C \in \mathcal{L}^N$ is called a *hierarchical liability problem* if, by reordering the agents, C can be transformed into an upper triangular matrix with zeros below the diagonal. The set $\mathcal{L}^{N,\Delta}$ contains all hierarchical liability problems on N . A liability rule that is defined on the domain of hierarchical liability problems is called a *hierarchical liability rule* (HLR).

Example 2.1. Let $N = \{1, 2, 3\}$ and $C \in \mathcal{L}^N$ be given by

$$C = \begin{matrix} & \begin{matrix} 1 & 2 & 3 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \end{matrix} & \begin{bmatrix} 3 & 1 & 4 \\ 2 & 2 & 6 \\ 1 & 0 & 1 \end{bmatrix} \end{matrix}.$$

The matrix should be interpreted in the following way. Agent 1 has a cash level of 3. He has a claim of 2 on agent 2 and a claim of 1 on agent 3, while agent 2 and 3 have a claim of 1 and 4 on agent 1. Agent 2 has a cash level of 2. He has no further claims, than the 1 on agent 1 we already mentioned, but agent 1 and 3 have a combined claim of 8 on him. This means in particular that agent 2 will never be able to pay off his debts. Agent 3 has a cash level of 1, agent 1 has a claim of 1 on his cash, while agent 3 has a claim of 4 on agent 1 and a claim of 6 on agent 2. $\blacklozenge$

Example 2.2. Let $N = \{1, 2, 3, 4\}$ and $C \in \mathcal{L}^N$ be given by

$$C = \begin{matrix} & \begin{matrix} 1 & 2 & 3 & 4 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} & \begin{bmatrix} 4 & 2 & 4 & 4 \\ 0 & 3 & 0 & 1 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & 2 \end{bmatrix} \end{matrix}.$$

The claim matrix is upper triangular, since agent 1 only faces claims and has no claims on agents 2, 3 or 4. Furthermore, agent 2 has a claim on agent 1 but faces claims only from agents 3 and 4. Agent 3 has a claim on agent 1 and faces a claim of only agent 4, while agent 4 faces no claims at all, but he has a claim on all other three agents. $\blacklozenge$

Liability problems can be seen as a generalization of bankruptcy problems. Each bankruptcy problem $(E, d) \in \mathcal{B}^N$ with $N = \{1, 2, \dots, n\}$ corresponds to a hierarchical liability problem $C(E, d) \in \mathcal{L}^{\bar{N}}$ with $\bar{N} = N \cup \{0\}$ given by

$$C(E, d) = \begin{matrix} & \begin{matrix} 0 & 1 & \dots & n \end{matrix} \\ \begin{matrix} 0 \\ 1 \\ \vdots \\ n \end{matrix} & \begin{bmatrix} E & d_1 & \dots & d_n \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix} \end{matrix}.$$

3 Bankruptcy rule based transfer schemes

Before elaborating on bankruptcy rule based transfer schemes, we provide some details on bankruptcy rules and the Aumann-Maschler rule in particular.

3.1 On the Aumann-Maschler rule

A *bankruptcy rule* $\varphi : \mathcal{B}^N \rightarrow \mathbb{R}^N$ assigns to every bankruptcy problem $(E, d) \in \mathcal{B}^N$ a vector $\varphi(E, d) \in \mathbb{R}^N$, such that

$$\sum_{i \in N} \varphi_i(E, d) = \min\{E, \sum_{j \in N} d_j\}, \quad (1)$$

$0 \leq \varphi(E, d) \leq d$ and such that *monotonicity* is satisfied: for all $(E, d) \in \mathcal{B}^N$ and all $(E', d) \in \mathcal{B}^N$ with $E' \geq E$, we have $\varphi(E, d) \leq \varphi(E', d)$. Note that the class $\mathcal{B}^N$ also contains bankruptcy problems (E, d) in which E is sufficient to fulfill the claims d and in that case $\varphi(E, d) = d$ for any bankruptcy rule φ .⁴

Note that any bankruptcy rule is *continuous* in the estate (cf. Yeh, 2008): for a sequence of nonnegative estates $E_1, E_2, \dots$ that converges to E and for any nonnegative claim vector $d \in \mathbb{R}^N$, the sequence $\varphi(E_1, d), \varphi(E_2, d), \dots$ converges to $\varphi(E, d)$.

For a detailed overview on bankruptcy rules we refer to Thomson (2003). Our focus will be mainly on the Aumann-Maschler rule AM (Aumann and Maschler, 1985), which is based on the Constrained Equal Awards rule.

The Constrained Equal Awards rule CEA is, for all $(E, d) \in \mathcal{B}^N$ and all $i \in N$, defined by

$$CEA_i(E, d) = \min\{\lambda, d_i\},$$

where $\lambda \in \mathbb{R}$ is such that $\sum_{i \in N} \min\{\lambda, d_i\} = \min\{E, \sum_{j \in N} d_j\}$.

The Aumann-Maschler rule AM is, for all $(E, d) \in \mathcal{B}^N$, defined by

$$AM(E, d) = \begin{cases} d & \text{if } \sum_{j \in N} d_j \leq E, \\ d - CEA(\sum_{j \in N} d_j - E, \frac{1}{2}d) & \text{if } E < \sum_{j \in N} d_j < 2E, \\ CEA(E, \frac{1}{2}d) & \text{if } \sum_{j \in N} d_j \geq 2E. \end{cases}$$

For bankruptcy problems involving two agents, AM satisfies the Concede & Divide-principle $C\&D$. This means that for $(E, d) \in \mathcal{B}^N$ with $N = \{1, 2\}$,

$$AM_1(E, d) = \begin{cases} (E - d_2)^+ + \frac{E - (E - d_1)^+ - (E - d_2)^+}{2} & \text{if } d_1 + d_2 \geq E, \\ d_1 & \text{if } d_1 + d_2 < E, \end{cases}$$

where $(x)^+ = \max\{x, 0\}$ for all $x \in \mathbb{R}$. Here $(E - d_2)^+$ represents the part of the estate conceded to agent 1 by agent 2, while $\frac{E - (E - d_1)^+ - (E - d_2)^+}{2}$ indicates that the total amount of the estate that is not conceded, is divided equally.

So far, bankruptcy rules are defined on a fixed but arbitrary finite agent set N . Alternatively, bankruptcy rules can also be viewed as rules on the class $\mathcal{B}$ of bankruptcy problems with arbitrary but finite N . On the class $\mathcal{B}$, AM can be characterized by means of the

⁴Chun (1988) introduced this set of problems as *rights problems*.

$C\&D$ -principle and the property of consistency .

Here, a bankruptcy rule φ on $\mathcal{B}$ is called *consistent* if for each finite agent set N , each $(E, d) \in \mathcal{B}^N$ and all $T \in 2^N \setminus \{\emptyset\}$, we have that

$$\varphi(E, d)|_T = \varphi\left(\sum_{j \in T} \varphi_j(E, d), d|_T\right).$$

Note that $(\sum_{j \in T} \varphi_j(E, d), d|_T) \in \mathcal{B}^T$.

Consistency of a rule requires that a possible reallocation of the total amount which has been allocated to a coalition T , on the basis of to the same bankruptcy rule, does not change the initial individual allocations within this coalition.

3.2 Towards transfer schemes

To devise liability rules, we will explicitly consider bilateral monetary transfer schemes on which the allocations prescribed by the rule are based.

Let $C \in \mathcal{L}^N$. Then, the matrix $P = (p_{ij}) \in \mathbb{R}^{N \times N}$ is a *transfer scheme* for C , if

- (i) for all $i \in N$, $p_{ii} = c_{ii}$,
- (ii) for all $i, j \in N$ with $i \neq j$, $0 \leq p_{ij} \leq c_{ij}$,
- (iii) for all $i \in N$, $\sum_{j \in N \setminus \{i\}} p_{ij} \leq p_{ii} + \sum_{j \in N \setminus \{i\}} p_{ji}$.

The interpretation is the following: p_{ij} , $i \neq j$, corresponds to the monetary transfer from agent i to j . For technical reasons and for computational convenience we require (i). The second condition states that the payment p_{ij} is nonnegative, but not higher than claim c_{ij} of agent j on i . The third condition requires that the sum of outgoing payments of i does not exceed his available cash plus incoming payments.

Let $\mathcal{P}(C)$ denote the set of all possible transfer schemes for the liability problem $C \in \mathcal{L}^N$.

A transfer scheme directly leads to an allocation of the available cash. Let $C \in \mathcal{L}^N$ and let $P \in \mathcal{P}(C)$. Then, we define $\alpha^P \in \mathbb{R}^N$ as the P -based *transfer allocation*, i.e. for all $i \in N$

$$\alpha_i^P = p_{ii} + \sum_{j \in N \setminus \{i\}} (p_{ji} - p_{ij}). \quad (2)$$

Note that because of (iii), $\alpha^P \geq 0$ and that

$$\begin{aligned} \sum_{i \in N} \alpha_i^P &= \sum_{i \in N} \left[p_{ii} + \sum_{j \in N \setminus \{i\}} (p_{ji} - p_{ij}) \right] \\ &= \sum_{i \in N} p_{ii} + \sum_{i \in N} \sum_{j \in N \setminus \{i\}} p_{ji} - \sum_{i \in N} \sum_{j \in N \setminus \{i\}} p_{ij} \\ &= \sum_{i \in N} p_{ii} = \sum_{i \in N} c_{ii}. \end{aligned}$$

Example 3.1. Reconsider the liability problem $C \in \mathcal{L}^N$ of Example 2.1 with $N = \{1, 2, 3\}$ and C given by

$$C = \begin{bmatrix} 3 & 1 & 4 \\ 2 & 2 & 6 \\ 1 & 0 & 1 \end{bmatrix}.$$

An example of a transfer scheme for C is

$$P = \begin{bmatrix} 3 & 1 & 4 \\ 1.5 & 2 & 1.5 \\ 1 & 0 & 1 \end{bmatrix}.$$

The first two conditions (i) and (ii) can easily be checked. To verify condition (iii), observe that

$$\begin{aligned} p_{12} + p_{13} &= 5 \leq p_{11} + p_{21} + p_{31} = 5.5 \\ p_{21} + p_{23} &= 3 \leq p_{22} + p_{12} + p_{32} = 3 \\ p_{31} + p_{32} &= 1 \leq p_{33} + p_{13} + p_{23} = 6.5. \end{aligned}$$

Note that P leads to the P -based transfer allocation $\alpha^P = (0.5, 0, 5.5)$. ◆

Next, we introduce a specific type of transfer schemes, called bankruptcy rule based transfer schemes.

Let $C \in \mathcal{L}^N$ and let φ be a bankruptcy rule. For all $i \in N$, define $d^i(C) \in \mathbb{R}^N$ by

$$d_j^i(C) = \begin{cases} c_{ij} & \text{if } j \neq i, \\ 0 & \text{if } j = i, \end{cases} \quad (3)$$

as the vector of claims on agent i . Then, $P = (p_{ij}) \in \mathbb{R}^{N \times N}$ is called a φ -based transfer scheme for C if,

- (i) for all $i \in N$, $p_{ii} = c_{ii}$,
- (ii) for all $i, j \in N$ with $i \neq j$,

$$p_{ij} = \varphi_j \left(p_{ii} + \sum_{k \in N \setminus \{i\}} p_{ki}, d^i(C) \right).$$

We denote by $\mathcal{P}^\varphi(C)$ the set of all φ -based transfer schemes.

Example 3.2. Let $N = \{1, 2, 3\}$ and consider the liability problem $C \in \mathcal{L}^N$ given by

$$C = \begin{bmatrix} 2 & 100 & 0 \\ 100 & 4 & 12 \\ 0 & 0 & 0 \end{bmatrix}.$$

An AM -based transfer scheme is given by

$$P = \begin{bmatrix} 2 & 8 & 0 \\ 6 & 4 & 6 \\ 0 & 0 & 0 \end{bmatrix}.$$

For this, observe e.g. that $d^2(C) = (100, 0, 12)$ and that $p_{23} = AM_3(p_{22} + p_{12}, d^2(C)) = AM_3(12, (100, 0, 12)) = 6$. Furthermore $\alpha^P = (0, 0, 6)$.

One can check that also the matrix $\tilde{P}$ given by

$$\tilde{P} = \begin{bmatrix} 2 & 22 & 0 \\ 20 & 4 & 6 \\ 0 & 0 & 0 \end{bmatrix}$$

belongs to $\mathcal{P}^{AM}(C)$. Note that $\alpha^{\tilde{P}} = \alpha^P$. ♦

The next lemma shows that a φ -based transfer scheme is indeed a transfer scheme.

Lemma 3.3. *Let $C \in \mathcal{L}^N$ and let φ be a bankruptcy rule. Then, $\mathcal{P}^\varphi(C) \subset \mathcal{P}(C)$.*

Proof. Take $P = (p_{ij}) \in \mathcal{P}^\varphi(C)$. It is sufficient to show that condition (ii) of a φ -based transfer scheme implies conditions (ii) and (iii) of a transfer scheme.

We start with showing (ii). Since φ is a bankruptcy rule, we have that for all $i, j \in N$ with $i \neq j$

$$0 \leq \varphi_j \left(p_{ii} + \sum_{k \in N \setminus \{i\}} p_{ki}, d^i(C) \right) \leq d_j^i(C) = c_{ij},$$

which implies that

$$0 \leq p_{ij} \leq c_{ij}.$$

Next we show condition (iii), using the basic properties of a bankruptcy rule. For all $i \in N$,

$$\begin{aligned} \sum_{j \in N \setminus \{i\}} p_{ij} &= \sum_{j \in N \setminus \{i\}} \varphi_j \left(p_{ii} + \sum_{k \in N \setminus \{i\}} p_{ki}, d^i(C) \right) \\ &= \sum_{j \in N} \varphi_j \left(p_{ii} + \sum_{k \in N \setminus \{i\}} p_{ki}, d^i(C) \right) \\ &\leq p_{ii} + \sum_{k \in N \setminus \{i\}} p_{ki}. \end{aligned} \quad \square$$

A φ -based transfer scheme P satisfies an attractive property: in the corresponding φ -based transfer allocation α^P an agent can only receive a positive amount if he paid off all his claimants.

Lemma 3.4. *Let $P \in \mathcal{P}^\varphi(C)$ for some $C \in \mathcal{L}^N$. Let $i \in N$. If $\alpha_i^P > 0$, then*

$$p_{ij} = c_{ij} \text{ for all } j \in N \setminus \{i\}.$$

Proof. Let $\alpha_i^P > 0$. Then by (2)

$$p_{ii} + \sum_{j \in N \setminus \{i\}} (p_{ji} - p_{ij}) > 0,$$

i.e.

$$\sum_{j \in N \setminus \{i\}} p_{ij} < p_{ii} + \sum_{j \in N \setminus \{i\}} p_{ji}. \quad (4)$$

Moreover, since P is a φ -based transfer scheme, for all $j \in N \setminus \{i\}$

$$p_{ij} = \varphi_j \left(p_{ii} + \sum_{k \in N \setminus \{i\}} p_{ki}, d^i(C) \right).$$

and consequently

$$\sum_{j \in N \setminus \{i\}} p_{ij} = \min \left\{ p_{ii} + \sum_{k \in N \setminus \{i\}} p_{jk}, \sum_{j \in N \setminus \{i\}} c_{ij} \right\}. \quad (5)$$

Using (4) it must be that

$$\sum_{j \in N \setminus \{i\}} p_{ij} = \sum_{j \in N \setminus \{i\}} c_{ij}$$

and using (ii) of φ -based transfer schemes, it follows that $p_{ij} = c_{ij}$, for all $j \in N \setminus \{i\}$. $\square$

The next theorem shows that one can always find at least one φ -based transfer scheme.

Theorem 3.5. *Let $C \in \mathcal{L}^N$ and let φ be a bankruptcy rule. Then, $\mathcal{P}^\varphi(C) \neq \emptyset$.*

Proof. Using the following iterative procedure we construct a φ -based transfer scheme for C .

For all $i \in N$, set $d^i = d^i(C)$ and set $E^i(0) = c_{ii}$.

Then, recursively define, for all $i \in N$ and $k \in \{1, 2, \dots\}$,

$$E^i(k+1) = c_{ii} + \sum_{j \in N \setminus \{i\}} \varphi_i(E^j(k), d^j). \quad (6)$$

Note that

$$E^i(1) = c_{ii} + \sum_{j \in N \setminus \{i\}} \varphi_i(c_{jj}, d^j) \geq c_{ii} = E^i(0).$$

Let $k \geq 1$ and assume that $E^i(k) \geq E^i(k-1)$. Then, by monotonicity of φ we find that

$$E^i(k+1) = c_{ii} + \sum_{j \in N \setminus \{i\}} \varphi_i(E^j(k), d^j) \geq c_{ii} + \sum_{j \in N \setminus \{i\}} \varphi_i(E^j(k-1), d^j) = E^i(k).$$

Hence, by induction, for all $i \in N$

$$E^i(0) \leq E^i(1) \leq E^i(2) \leq \dots \quad (7)$$

Consider $P = (p_{ij}) \in \mathbb{R}^{N \times N}$, given by

$$p_{ij} = \begin{cases} c_{ii} & \text{for all } i, j \in N \text{ with } i = j, \\ \lim_{k \rightarrow \infty} \varphi_j(E^i(k), d^i) & \text{for all } i, j \in N \text{ with } i \neq j. \end{cases} \quad (8)$$

Note that the limit in (8) exists, because $\{E^i(k)\}_{k=0}^\infty$ is an increasing sequence, while φ is monotonic and bounded from above.

Moreover, condition (ii) of a φ -based transfer scheme is satisfied since for all $i, j \in N$ with $i \neq j$, we have that

$$\begin{aligned} p_{ij} &= \lim_{k \rightarrow \infty} \varphi_j(E^i(k), d^i) \\ &= \varphi_j\left(\lim_{k \rightarrow \infty} E^i(k), d^i\right) \\ &= \varphi_j\left(c_{ii} + \lim_{k \rightarrow \infty} \sum_{\ell \in N \setminus \{i\}} \varphi_i(E^\ell(k), d^\ell), d^i\right) \\ &= \varphi_j\left(c_{ii} + \sum_{\ell \in N \setminus \{i\}} p_{\ell i}, d^i\right). \end{aligned}$$

The second equality follows from continuity of φ , the third equality follows from (6) and the last equality follows from (8). $\square$

4 Hierarchical liability problems

As Example 3.2 shows, a general φ -based transfer scheme need not to be unique. For hierarchical liability problems, however, there is a unique φ -based transfer scheme.

Theorem 4.1. *Let $C \in \mathcal{L}^{N,\Delta}$ and let φ be a bankruptcy rule. Then, $|\mathcal{P}^\varphi(C)| = 1$.*

Proof. Let $N = \{1, \dots, n\}$ and let $C \in \mathcal{L}^{N,\Delta}$. By the upper triangularity of C , we can assume, w.l.o.g., that $c_{ij} = 0$ if $i > j$. Let $P = (p_{ij})$ and $\tilde{P} = (\tilde{p}_{ij})$ both be φ -based transfer schemes for C .

Clearly, if $i > j$,

$$p_{ij} = \tilde{p}_{ij} = 0. \quad (9)$$

And since

$$c_{11} + \sum_{j \in N \setminus \{1\}} p_{j1} = c_{11} + \sum_{j \in N \setminus \{1\}} \tilde{p}_{j1} = c_{11},$$

the fact that P and $\tilde{P}$ are φ -based transfer schemes implies for all $j \in N \setminus \{1\}$

$$p_{1j} = \tilde{p}_{1j} = \varphi_j(c_{11}, d^1(C)).$$

Now consider $i \in N$ and assume that for all $g \in \{1, \dots, i-1\}$ and $h \in \{1, \dots, n\}$,

$$p_{gh} = \tilde{p}_{gh}.$$

By equation (9),

$$\begin{aligned} c_{ii} + \sum_{g \in N \setminus \{i\}} p_{gi} &= c_{ii} + \sum_{g < i} p_{gi} \\ &= c_{ii} + \sum_{g < i} \tilde{p}_{gi} \\ &= c_{ii} + \sum_{g \in N \setminus \{i\}} \tilde{p}_{gi} \end{aligned}$$

and thus for all $j \in N \setminus \{i\}$

$$\begin{aligned} p_{ij} &= \varphi_j(c_{ii} + \sum_{g \in N \setminus \{i\}} p_{gi}, d^i(C)) \\ &= \varphi_j(c_{ii} + \sum_{g \in N \setminus \{i\}} \tilde{p}_{gi}, d^i(C)) = \tilde{p}_{ij}. \end{aligned} \quad \square$$

This theorem implies that on $\mathcal{L}^{N,\Delta}$ a φ -based transfer allocation is uniquely defined for every bankruptcy rule φ . Hence, we can extend each bankruptcy rule to a hierarchical liability rule.

Let φ be a bankruptcy rule. The corresponding *hierarchical φ -based liability rule* $\rho^\varphi : \mathcal{L}^{N,\Delta} \rightarrow \mathbb{R}^N$ is defined by

$$\rho^\varphi(C) = \alpha^P,$$

for all $C \in \mathcal{L}^{N,\Delta}$ where P is the unique φ -based transfer scheme for C .

An alternative way of using a bankruptcy rule to solve hierarchical liability problems, is the following recursive procedure that we first illustrate by means of an example.

Example 4.2. Let $N = \{1, \dots, 4\}$ and consider $C \in \mathcal{L}^{N,\Delta}$, given by

$$C = \begin{bmatrix} 4 & 2 & 4 & 4 \\ 0 & 3 & 0 & 1 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & 2 \end{bmatrix}.$$

In the recursive procedure we start with agent 1, who has no claims on the other agents. His cash, $c_{11} = 4$, is divided on the basis of a bankruptcy problem with estate 4 and claims, 2, 4 and 4. Hence we treat this subproblem of the liability problem as a bankruptcy problem $(4, (2, 4, 4))$. Selecting $\varphi = AM$ as an appropriate bankruptcy rule, $AM(4, (2, 4, 4)) = (1, 1.5, 1.5)$. Thus agent 2 gets 1 from agent 1's cash and agents 3 and 4 both receive 1.5. Correspondingly we can update our (partly) solved liability problem into

$$C^1 = \begin{bmatrix} 4 - 1 - 1.5 - 1.5 & 0 & 0 & 0 \\ 0 & 3 + 1 & 0 & 1 \\ 0 & 0 & 2 + 1.5 & 3 \\ 0 & 0 & 0 & 2 + 1.5 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 1 \\ 0 & 0 & 3.5 & 3 \\ 0 & 0 & 0 & 3.5 \end{bmatrix}.$$

In C^1 agent 2 has no claim on agent 1 anymore and we allocate $c_{22}^1 = 4$ on the basis of the bankruptcy problem $(4, (0, 1))$. Since $AM(4, (0, 1)) = (0, 1)$, this means that 1 is transferred to agent 4 while agent 2 keeps an amount of 3. Updating leads to

$$C^2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 3.5 & 3 \\ 0 & 0 & 0 & 4.5 \end{bmatrix}.$$

In the next step an amount of 3 is transferred from 3 to 4, and updating gives

$$C^3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 0.5 & 0 \\ 0 & 0 & 0 & 7.5 \end{bmatrix}.$$

The diagonal, i.e $(0, 3, 0.5, 7.5)$, of this matrix can be viewed as an allocation which solves this hierarchical liability problem based on a recursive application of the AM -rule.

Implicitly, we derived the following corresponding bilateral transfer scheme for C

$$P = \begin{bmatrix} 4 & 1 & 1.5 & 1.5 \\ 0 & 3 & 0 & 1 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & 2 \end{bmatrix}.$$

Note that this is the AM -based transfer scheme and that $\alpha^P = (0, 3, 0.5, 7.5)$. ◆

The formal definition of how to extend liability rules in the recursive way as described in the previous example is provided below.

Let $C \in \mathcal{L}^{N,\Delta}$ and let φ be a bankruptcy rule. Set $N = \{1, \dots, n\}$ and assume that $c_{ij} = 0$ for all $i, j \in N$ with $i > j$. Set $C^0 = C$. Recursively, for $j \in \{1, \dots, n-1\}$, define $C^j \in \mathcal{L}^{N,\Delta}$ by

$$\begin{aligned}
c_{ii}^j &= \begin{cases} c_{ii}^{j-1} & \text{if } i < j \\ c_{ii}^{j-1} + \varphi_i(c_{jj}^{j-1}, (c_{jk})_{k \in \{j+1, \dots, n\}}) & \text{if } i > j. \end{cases} \\
c_{jj}^j &= c_{jj}^{j-1} - \sum_{k > j} \varphi(c_{jj}^{j-1}, (c_{jk})_{k \in \{j+1, \dots, n\}}) \\
c_{ik}^j &= \begin{cases} 0 & \text{if } i = j \text{ and } k \neq i, \\ c_{ik}^{j-1} & \text{if } i \neq j \text{ and } k \neq i. \end{cases}
\end{aligned} \tag{10}$$

Finally set

$$C^{rec} = C^{n-1}.$$

Correspondingly, the hierarchical *recursive* φ -based liability rule $\xi^\varphi : \mathcal{L}^{N,\Delta} \rightarrow \mathbb{R}^N$ is defined by

$$\xi^\varphi(C) = \text{diag}(C^{rec})$$

for each $C \in \mathcal{L}^{N,\Delta}$.

Interestingly, for every bankruptcy rule φ , the recursive φ -based liability rule ξ^φ and the hierarchical φ -based liability rule ρ^φ coincide.

Theorem 4.3. *For all bankruptcy rules φ ,*

$$\rho^\varphi = \xi^\varphi.$$

Proof. Let $C \in \mathcal{L}^{N,\Delta}$ and let φ be a bankruptcy rule. Set $N = \{1, \dots, n\}$ and assume that $c_{ij} = 0$ for all $i, j \in N$ with $i > j$. Let $P = (p_{ij})$ be the unique φ -based transfer scheme for C . Then, we have that for all $i \in N$

$$\begin{aligned}
\rho_i^\varphi(C) &= \alpha_i^P = c_{ii} + \sum_{j \in N \setminus \{i\}} p_{ji} - \sum_{j \in N \setminus \{i\}} p_{ij} \\
&= c_{ii} + \sum_{j=1}^{i-1} p_{ji} - \sum_{j=i+1}^n p_{ij}.
\end{aligned}$$

Moreover, for all $i \in N$, $\xi_i^\varphi(C) = c_{ii}^i$. Thus it is sufficient to show that for all $i \in N$

$$c_{ii}^i = c_{ii} + \sum_{j=1}^{i-1} p_{ji} - \sum_{j=i+1}^n p_{ij}. \tag{11}$$

For $i = 1$, (11) is satisfied since

$$\begin{aligned}
c_{11}^1 &= c_{11} - \sum_{j=2}^n \varphi_j(c_{11}, (c_{1k})_{k \in \{2, \dots, n\}}) \\
&= c_{11} - \sum_{j=2}^n \varphi_j(c_{11} + \sum_{k \in N \setminus \{1\}} p_{k1}, d^1(C)) \\
&= c_{11} - \sum_{j=2}^n p_{1j}.
\end{aligned}$$

The first equality follows from (10), the second equality holds because $p_{k1} = 0$ for all $k \in N \setminus \{1\}$ and the last equality follows from condition (ii) of φ -based transfer schemes.

Note that, for all $j \in N \setminus \{1\}$

$$\begin{aligned} c_{jj}^1 &= c_{jj} + \varphi_j(c_{11}, (c_{1k})_{k \in \{2, \dots, n\}}) \\ &= c_{jj} + p_{1j}. \end{aligned}$$

The proof continues by means of induction. Let $i \leq n - 1$ and assume that

$$c_{jj}^{i-1} = \begin{cases} c_{jj} + \sum_{k=1}^{i-1} p_{kj} & \text{if } j > i - 1 \\ c_{jj} + \sum_{k=1}^{j-1} p_{kj} - \sum_{k=j+1}^n p_{jk} & \text{if } j \leq i - 1. \end{cases}$$

We will prove that

$$c_{jj}^i = \begin{cases} c_{jj} + \sum_{k=1}^i p_{kj} & \text{if } j > i \\ c_{jj} + \sum_{k=1}^{i-1} p_{kj} - \sum_{k=j+1}^n p_{jk} & \text{if } j = i. \end{cases}$$

For $j \in \{i + 1, \dots, n\}$, we have that

$$\begin{aligned} c_{jj}^i &= c_{jj}^{i-1} + \varphi_j(c_{ii}^{i-1}, (c_{ik})_{k \in \{i+1, \dots, n\}}) \\ &= c_{jj} + \sum_{k=1}^{i-1} p_{kj} + \varphi_j(c_{ii} + \sum_{k=1}^{i-1} p_{ki}, (c_{ik})_{k \in \{i+1, \dots, n\}}) \\ &= c_{jj} + \sum_{k=1}^{i-1} p_{kj} + \varphi_j(c_{ii} + \sum_{k=1}^{i-1} p_{ki}, d^i(C)) \\ &= c_{jj} + \sum_{k=1}^{i-1} p_{kj} + p_{ij} \\ &= c_{jj} + \sum_{k=1}^i p_{kj}, \end{aligned}$$

where the first equality follows from the definition of ξ and the second equality is based on the induction assumption. Similarly one finds

$$\begin{aligned} c_{ii}^i &= c_{ii}^{i-1} - \sum_{k=i+1}^n \varphi(c_{ii}^{i-1}, (c_{ik})_{k \in \{i+1, \dots, n\}}) \\ &= c_{ii} + \sum_{k=1}^{i-1} p_{ki} - \sum_{k=i+1}^n \varphi(c_{ii} + \sum_{k=1}^{i-1} p_{ki}, d^i(C)) \\ &= c_{ii} + \sum_{k=1}^{i-1} p_{ki} - \sum_{k=i+1}^n p_{ik}. \end{aligned}$$

□

5 General liability problems

As seen in Example 3.2, the *AM* bankruptcy rule allows for multiple *AM*-based transfer schemes for a non-hierarchical liability problem. For an arbitrary bankruptcy rule, however, there is always a unique φ -based transfer allocation.

Theorem 5.1. Let $C \in \mathcal{L}^N$, let φ be a bankruptcy rule and let $P, \tilde{P} \in \mathcal{P}^\varphi(C)$. Then,

$$\alpha^P = \alpha^{\tilde{P}}.$$

Proof. On the contrary suppose that $\alpha^P \neq \alpha^{\tilde{P}}$. For notational convenience, set $\alpha^P = \alpha$ and $\alpha^{\tilde{P}} = \tilde{\alpha}$.

Let $N = \{1, \dots, n\}$. Without loss of generality we assume that, $\alpha_1 < \tilde{\alpha}_1$. Since $0 \leq \alpha_1 < \tilde{\alpha}_1$, Lemma 3.4 implies that, for all $j \in N \setminus \{1\}$

$$\tilde{p}_{1j} = c_{1j}. \quad (12)$$

Since

$$\tilde{\alpha}_1 = \tilde{p}_{11} + \sum_{j \in N \setminus \{1\}} (\tilde{p}_{j1} - \tilde{p}_{1j}) = c_{11} + \sum_{j \in N \setminus \{1\}} (\tilde{p}_{j1} - \tilde{p}_{1j}) > c_{11} + \sum_{j \in N \setminus \{1\}} (p_{j1} - p_{1j}) = \alpha_1$$

there must be an agent $j \in N \setminus \{1\}$ for which

$$\tilde{p}_{j1} - \tilde{p}_{1j} > p_{j1} - p_{1j}.$$

Therefore, by (12),

$$\tilde{p}_{j1} - p_{1j} \geq \tilde{p}_{j1} - c_{1j} = \tilde{p}_{j1} - \tilde{p}_{1j} > p_{j1} - p_{1j}$$

and hence

$$\tilde{p}_{j1} > p_{j1}.$$

Without loss of generality we assume that $j = 2$.

Note that $p_{21} < \tilde{p}_{21} \leq c_{21}$. Thus, by Lemma 3.4, $\alpha_2 = 0$ and therefore $\alpha_1 + \alpha_2 < \tilde{\alpha}_1 + \tilde{\alpha}_2$, i.e.

$$c_{11} + c_{22} + \sum_{j \in N \setminus \{1\}} (p_{j1} - p_{1j}) + \sum_{j \in N \setminus \{2\}} (p_{j2} - p_{2j}) < c_{11} + c_{22} + \sum_{j \in N \setminus \{1\}} (\tilde{p}_{j1} - \tilde{p}_{1j}) + \sum_{j \in N \setminus \{2\}} (\tilde{p}_{j2} - \tilde{p}_{2j})$$

and consequently

$$\sum_{j \in N \setminus \{1,2\}} (p_{j1} - p_{1j}) + \sum_{j \in N \setminus \{1,2\}} (p_{j2} - p_{2j}) < \sum_{j \in N \setminus \{1,2\}} (\tilde{p}_{j1} - \tilde{p}_{1j}) + \sum_{j \in N \setminus \{1,2\}} (\tilde{p}_{j2} - \tilde{p}_{2j}).$$

Thus there must be an agent $\ell \in N \setminus \{1, 2\}$ with

$$\begin{aligned} \tilde{p}_{\ell 1} - \tilde{p}_{1\ell} &> p_{\ell 1} - p_{1\ell} \\ \text{or} \quad \tilde{p}_{\ell 2} - \tilde{p}_{2\ell} &> p_{\ell 2} - p_{2\ell}. \end{aligned}$$

Without loss of generality we assume that $\ell = 3$ and that $\tilde{p}_{31} - \tilde{p}_{13} > p_{31} - p_{13}$. Then,

$$\tilde{p}_{31} - p_{13} \geq \tilde{p}_{31} - c_{13} = \tilde{p}_{31} - \tilde{p}_{13} > p_{31} - p_{13}.$$

Thus we conclude that $\tilde{p}_{31} > p_{31}$ and, using Lemma 3.4, that $\alpha_3 = 0$ and that $\alpha_1 + \alpha_2 + \alpha_3 < \tilde{\alpha}_1 + \tilde{\alpha}_2 + \tilde{\alpha}_3$.

We can continue with this reasoning with respect to agent $4, 5, \dots, n$. As a result we will find that $\alpha_1 + \alpha_2 + \dots + \alpha_n < \tilde{\alpha}_1 + \tilde{\alpha}_2 + \dots + \tilde{\alpha}_n$, which is not possible because of efficiency of a φ -based transfer allocation. $\square$

Hence, we can introduce φ -based rules for general liability problems. Let φ be a bankruptcy rule. The corresponding φ -based liability rule $\rho^\varphi : \mathcal{L}^N \rightarrow \mathbb{R}^N$ is defined by

$$\rho^\varphi(C) = \alpha^P,$$

for all $C \in \mathcal{L}^N$ where P is a φ -based transfer scheme for C .

The final part of this section will provide an axiomatic characterization of ρ^{AM} as a φ -based liability rule on the class $\mathcal{L}$ of all liability problems with an arbitrary but finite set of players by extending the properties *C&D* and consistency for bankruptcy rules to general liability rules.

In bankruptcy problems the principle of Concede & Divide is defined for problems with two claimants. However, in a liability problem with two agents, every agent faces only one (possible) claimant. For such liability problems the allocation prescribed by any φ -based liability rule is unique. This is shown in the following lemma.

Lemma 5.2. *Let $C \in \mathcal{L}^N$ with $N = \{1, 2\}$. Let φ^I and φ^{II} be bankruptcy rules. Then,*

$$\rho^{\varphi^I}(C) = \rho^{\varphi^{II}}(C).$$

Proof. Note that for all $i \in N$, $d^i(C)$ has at most one positive claim. Take $P = (p_{ij}) \in \mathcal{P}^{\varphi^I}(C)$. Then, with $N = \{i, j\}$,

$$\begin{aligned} p_{ij} &= \rho_j^{\varphi^I}(p_{ii} + p_{ji}, d^i(C)) \\ &= \rho_j^{\varphi^{II}}(p_{ii} + p_{ji}, d^i(C)). \end{aligned}$$

Hence, $P \in \mathcal{P}^{\varphi^{II}}(C)$ and thus $\rho^{\varphi^I}(C) = \rho^{\varphi^{II}}(C)$. $\square$

Instead, we will define a Concede & Divide-principle for liability problems with 3 agents, in which every agent has two (possible) claimants. A liability rule f satisfies the *Concede & Divide-principle C&D* if for each N with $|N| = 3$ and for each $C \in \mathcal{L}^N$, there exists an underlying transfer scheme $P \in \mathcal{P}(C)$ such that $f(C) = \alpha^P$ and for each player $i \in N$, his ‘estate’ $e^i = c_{ii} + \sum_{\ell \neq i} p_{\ell i}$ is allocated among the remaining two players, j, k , respecting the bankruptcy Concede & Divide-principle, i.e.

$$p_{ij} = \begin{cases} c_{ij} & \text{if } e^i \geq c_{ij} + c_{ik}, \\ (e^i - c_{ik})^+ + \frac{e^i - (e^i - c_{ik})^+ - (e^i - c_{ij})^+}{2} & \text{otherwise.} \end{cases} \quad (13)$$

Example 5.3. Reconsider the liability problem $C \in \mathcal{L}^N$ of Example 2.1 with $N = \{1, 2, 3\}$ and C given by

$$C = \begin{bmatrix} 3 & 1 & 4 \\ 2 & 2 & 6 \\ 1 & 0 & 1 \end{bmatrix}.$$

Take $P \in \mathcal{P}^{AM}(C)$ given by

$$P = \begin{bmatrix} 3 & 1 & 4 \\ 1 & 2 & 2 \\ 1 & 0 & 1 \end{bmatrix}$$

with $\rho^{AM}(C) = \alpha^P = (0, 0, 6)$. We check that the entries in P satisfy (13). Here, $e^1 = p_{11} + p_{21} + p_{31} = 5$, $e^2 = 3$ and $e^3 = 7$. Both player 1's and player 3's estate are sufficient to satisfy their claimants, hence $p_{12} = c_{12} = 1$, $p_{13} = 4$ and $p_{31} = 1$. Player 2's estate is not sufficient, therefore

$$p_{21} = (e^2 - c_{23})^+ + \frac{e^2 - (e^2 - c_{21})^+ - (e^2 - c_{23})^+}{2} = 0 + \frac{3 - 1 - 0}{2} = 1$$

and $p_{23} = 2$. ♦

Next, we define the property of consistency for a liability rule. This property is defined on the class $\mathcal{L}$ of liability problems with arbitrary but finite N . Consistency requires that a reallocation of the total amount which has been allocated to a coalition T , on the basis of that rule and an underlying transfer scheme, does not change the initial individual allocations within this coalition. A liability rule f for $\mathcal{L}$ is called *consistent* if for all N and for all $C \in \mathcal{L}^N$ there exists a $P \in \mathcal{P}(C)$ such that $f(C) = \alpha^P$ and such that for all $T \in 2^N \setminus \{\emptyset\}$ with $C^{T,P} \in \mathcal{L}^T$,

$$f(C^{T,P}) = f(C)|_T, \quad (14)$$

where $C^{T,P} \in \mathbb{R}^{T \times T}$ is defined, for all $i, j \in T$, by

$$c_{ij}^{T,P} = \begin{cases} c_{ij} & \text{if } i \neq j, \\ c_{ii} + \sum_{k \in N \setminus T} (p_{ki} - p_{ik}) & \text{if } i = j. \end{cases} \quad (15)$$

Note that there is only a consistency requirement for T if $C^{T,P} \in \mathcal{L}^T$. As is seen in the following example, it can indeed happen that $C^{T,P} \notin \mathcal{L}^T$.

Example 5.4. Let $N = \{1, 2, 3, 4\}$. Reconsider the hierarchical liability problem $C \in \mathcal{L}^N$ of Example 2.2, given by

$$C = \begin{bmatrix} 4 & 2 & 4 & 4 \\ 0 & 3 & 0 & 1 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & 2 \end{bmatrix}.$$

The unique AM -based transfer scheme P for C is given by

$$P = \begin{bmatrix} 4 & 1 & 1.5 & 1.5 \\ 0 & 3 & 0 & 1 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & 2 \end{bmatrix}.$$

and $\rho^{AM} = (0, 3, 0.5, 7.5)$.

With $T = \{1, 2, 4\}$ we have

$$C^{T,P} = \begin{bmatrix} 2.5 & 2 & 4 \\ 0 & 3 & 1 \\ 0 & 0 & 5 \end{bmatrix},$$

which is a liability problem and the unique AM -based transfer scheme P^T for $C^{T,P}$ is given by

$$P^T = \begin{bmatrix} 2.5 & 1 & 1.5 \\ 0 & 3 & 1 \\ 0 & 0 & 5 \end{bmatrix},$$

while $\rho^{AM}(C^{T,P}) = (0, 3, 7.5)$. We see that the consistency requirement for this T is satisfied. However with $T = \{1, 2, 3\}$, we obtain

$$C^{T,P} = \begin{bmatrix} 2.5 & 2 & 4 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

Since $C^{T,P}$ contains negative entries, it is not a liability problem and therefore does not impose a consistency requirement. $\blacklozenge$

The AM -based liability rule satisfies consistency and $C\&D$.

Theorem 5.5. ρ^{AM} is consistent and satisfies $C\&D$.

Proof. We start with proving $C\&D$. Let $C \in \mathcal{L}^N$ with $|N| = 3$. Let $i \in N$ and set $N \setminus \{i\} = \{j, k\}$. Consider an arbitrary $P \in \mathcal{P}^{AM}(C)$. Obviously $\rho^{AM}(C) = \alpha^P$ by Theorem 5.1. Moreover,

$$\begin{aligned} p_{ij} &= AM_j(p_{ii} + p_{ji} + p_{ki}, d^i(C)) \\ &= AM_j(e^i, (c_{ij}, c_{ik})). \end{aligned}$$

Since the bankruptcy rule AM satisfies the $C\&D$ principle for bankruptcy problems, we find that

$$p_{ij} = \begin{cases} c_{ij} & \text{if } e^i \geq c_{ij} + c_{ik}, \\ (e^i - c_{ik})^+ + \frac{e^i - (e^i - c_{ik})^+ - (e^i - c_{ij})^+}{2} & \text{otherwise.} \end{cases}$$

Next, we show consistency. For this, let $C \in \mathcal{L}^N$, consider an arbitrary $P \in \mathcal{P}^{AM}(C)$ and let $T \in 2^N \setminus \{\emptyset\}$ be such that $C^{T,P} \in \mathcal{L}^T$. It suffices to show that $\rho^{AM}(C)|_T = \rho^{AM}(C^{T,P})$.

Define $P^T = (p_{ij}^T) \in \mathbb{R}^{T \times T}$ by

$$p_{ij}^T = \begin{cases} p_{ij} & \text{if } i \neq j \\ p_{ii} + \sum_{k \in N \setminus T} (p_{ki} - p_{ik}) & \text{if } i = j. \end{cases} \quad (16)$$

We will first show that $P^T \in \mathcal{P}^{AM}(C^{T,P})$, which implies that $\alpha^{P^T} = \rho^{AM}(C^{T,P})$.

For this, note that $c_{ii}^{T,P} = p_{ii}^T$ for all $i \in T$. It remains to prove that for all $i \in T$ and $j \in T \setminus \{i\}$,

$$p_{ij}^T = AM_j \left(p_{ii}^T + \sum_{k \in T \setminus \{i\}} p_{ki}^T, d^i(C^{T,P}) \right).$$

This is true because for each $i \in T$ and $j \in T \setminus \{i\}$

$$\begin{aligned}
p_{ij}^T = p_{ij} &= AM_j \left(p_{ii} + \sum_{k \in N \setminus \{i\}} p_{ki}, d^i(C) \right) \\
&= AM_j \left(p_{ii} + \sum_{k \in N \setminus \{i\}} p_{ki} - \sum_{k \in N \setminus T} AM_k \left(p_{ii} + \sum_{k \in N \setminus \{i\}} p_{ki}, d^i(C) \right), d^i(C)|_T \right) \\
&= AM_j \left(p_{ii} + \sum_{k \in N \setminus \{i\}} p_{ki} - \sum_{k \in N \setminus T} p_{ik}, d^i(C)|_T \right) \\
&= AM_j \left(p_{ii} + \sum_{k \in N \setminus T} (p_{ki} - p_{ik}) + \sum_{k \in T \setminus \{i\}} p_{ki}, d^i(C)|_T \right) \\
&= AM_j \left(p_{ii}^T + \sum_{k \in T \setminus \{i\}} p_{ki}^T, d^i(C^{T,P}) \right),
\end{aligned}$$

where the third equality follows from consistency of AM , the fourth equality follows from the fact that $P \in \mathcal{P}^{AM}(C)$, while the last equality follows from (16).

The proof is finished if we show that $\alpha^{P^T} = \rho^{AM}(C)|_T$. For this, note that with $i \in T$

$$\begin{aligned}
\alpha_i^{P^T} &= p_{ii}^T + \sum_{j \in T \setminus \{i\}} (p_{ji}^T - p_{ij}^T) \\
&= p_{ii} + \sum_{j \in N \setminus T} (p_{ji} - p_{ij}) + \sum_{j \in T \setminus \{i\}} (p_{ji} - p_{ij}) \\
&= p_{ii} + \sum_{j \in N \setminus \{i\}} (p_{ji} - p_{ij}) \\
&= \alpha_i^P = \rho_i^{AM}(C).
\end{aligned}$$

□

We conclude this section with a characterization of the AM -based liability rule.

Theorem 5.6. *Let φ be a bankruptcy rule. Then, $\rho^\varphi = \rho^{AM}$ if and only if ρ^φ satisfies consistency and $C\&D$.*

Proof. For the ‘only if’-part, we refer to Theorem 5.5. To prove the ‘if’-part, let φ be a bankruptcy rule such that ρ^φ satisfies consistency and $C\&D$. As we have seen before, the class $\mathcal{B}$ of bankruptcy problems is a subclass of $\mathcal{L}$ by identifying each $(E, d) \in \mathcal{B}^N$ with $N = \{1, \dots, n\}$, with $C(E, d) \in \mathcal{L}^{N \cup \{0\}, \Delta}$ given by

$$C(E, d) = \begin{matrix} & 0 & 1 & \cdots & n \\ \begin{matrix} 0 \\ 1 \\ \vdots \\ n \end{matrix} & \begin{bmatrix} E & d_1 & \cdots & d_n \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & & \cdots & 0 \end{bmatrix} \end{matrix}.$$

Let P be the unique φ -based transfer scheme for $C(E, d)$. Then,

$$P = \begin{bmatrix} E & p_{01} & \cdots & p_{0n} \\ 0 & 0 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & & \cdots & 0 \end{bmatrix},$$

with

$$\alpha_i^P = \begin{cases} p_{0i} & \text{if } i \in N, \\ E - \sum_{j \in N} p_{0j} & \text{if } i = 0. \end{cases}$$

Moreover, for all $i \in N$

$$\rho_i^\varphi(C(E, d)) = \alpha_i^P = p_{0i} = \varphi(c_{00}, d^0(C)) = \varphi_i(E, d). \quad (17)$$

Thus $\varphi(E, d) = \rho^\varphi(C(E, d))|_N$.

If we can show that

- (i) $C\&D$ of ρ^φ on $\mathcal{L}$ implies $C\&D$ of φ on $\mathcal{B}$,
- (ii) consistency of ρ^φ on $\mathcal{L}$ implies consistency of φ on $\mathcal{B}$.

Then, $\varphi = AM$ (cf. Aumann and Maschler, 1985) and consequently $\rho^\varphi = \rho^{AM}$.

For this, we first show that P is the unique transfer scheme for $C(E, d)$ that leads to the transfer allocation α^P and for this reason $C\&D$ and consistency of ρ^φ can only have implications on P .

Let $\tilde{P} = (\tilde{p}_{ij}) \in \mathcal{P}(C(E, d))$ be an arbitrary transfer scheme for $C(E, d)$ with $\tilde{P} \neq P$. Then,

$$\tilde{P} = \begin{bmatrix} E & \tilde{p}_{01} & \cdots & \tilde{p}_{0n} \\ 0 & 0 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & & \cdots & 0 \end{bmatrix}$$

and there must be a player $i \in N$ with $\tilde{p}_{0i} \neq p_{0i}$. Hence, $\alpha^{\tilde{P}} \neq \alpha^P$.

With respect to (i), let $N = \{1, 2\}$ and $(E, d) \in \mathcal{B}^N$. Let $i \in N$ and $\{j\} = N \setminus \{i\}$. We need to show that

$$\varphi_i(E, d) = \begin{cases} d_i & \text{if } E \geq d_1 + d_2, \\ (E - d_j)^+ + \frac{E - (E - d_i)^+ - (E - d_j)^+}{2} & \text{otherwise.} \end{cases}$$

$C\&D$ on $\mathcal{L}$ and (17) imply that, with $c_{0i} = C_{0i}(E, d)$ and $c_{0j} = C_{0j}(E, d)$,

$$\begin{aligned} \varphi_i(E, d) &= \rho_i^\varphi(C(E, d)) = \begin{cases} c_{0i} & \text{if } e^0 \geq c_{01} + c_{02}, \\ (e^0 - c_{0j})^+ + \frac{e^0 - (e^0 - c_{0i})^+ - (e^0 - c_{0j})^+}{2} & \text{otherwise,} \end{cases} \\ &= \begin{cases} d_i & \text{if } E \geq d_1 + d_2, \\ (E - d_j)^+ + \frac{E - (E - d_i)^+ - (E - d_j)^+}{2} & \text{otherwise.} \end{cases} \end{aligned}$$

With respect to (ii), let $(E, d) \in \mathcal{B}^N$ and $T \in 2^N \setminus \{\emptyset\}$. We have to prove that

$$\varphi(E, d)|_T = \varphi\left(\sum_{j \in T} \varphi_j(E, d), d|_T\right).$$

Let $T = \{k_1, \dots, k_t\}$. Then, using (15) and (17),

$$C^{T \cup \{0\}, P}(E, d) = \begin{matrix} & 0 & k_1 & \cdots & k_t \\ \begin{matrix} 0 \\ k_1 \\ \vdots \\ k_t \end{matrix} & \left[\begin{array}{cccc} E - \sum_{j \in N \setminus T} \varphi_j(E, d) & d_{k_1} & \cdots & d_{k_t} \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{array} \right] \end{matrix}$$

Clearly $C^{T \cup \{0\}, P}(E, d) \in \mathcal{L}^{T \cup \{0\}, \Delta}$ and

$$C^{T \cup \{0\}, P}(E, d) = C(E - \sum_{j \in N \setminus T} \varphi_j(E, d), d|_T).$$

Using consistency, we find for all $i \in T$ that

$$\rho_i^\varphi(C(E, d))|_{T \cup \{0\}} = \rho_i^\varphi(C(E - \sum_{j \in N \setminus T} \varphi_j(E, d), d|_T)).$$

By equation (17), for all $i \in T$

$$\begin{cases} \rho_i^\varphi(C(E, d)) & = \varphi_i(E, d) \\ \rho_i^\varphi(C(E - \sum_{j \in N \setminus T} \varphi_j(E, d), d|_T)) & = \varphi_i(E - \sum_{j \in N \setminus T} \varphi_j(E, d), d|_T) \end{cases}$$

and therefore,

$$\varphi((E, d)|_T) = \varphi(E - \sum_{j \in N \setminus T} \varphi_j(E, d), d|_T) = \varphi\left(\sum_{j \in T} \varphi_j(E, d), d|_T\right),$$

where the last equality follows from (1). $\square$

6 Alternative approaches

In this section we discuss two alternative approaches to analyze liability problems: a reduction approach and a hydraulic approach.

6.1 Reduction approach

In the reduction approach a general liability problem is reduced to a more tractable hierarchical liability problem. The main difference between hierarchical and non-hierarchical liability problems is the (non-)existence of cycles of claims.

In this section we will show that, by eliminating these cycles, it is possible to reduce a general liability problem to a hierarchical liability problem, but that such a reduction is not possible without changing the nature of the liability problem. There are choices to be made. Different reduction choices can result in different reduced hierarchical liability problems.

The possibilities regarding reduction steps and the subsequent effects will be illustrated in the following example.

Example 6.1. Let $N = \{1, 2, 3, 4\}$ and let $C \in \mathcal{L}^N$ be given by

$$C = \begin{bmatrix} 4 & 5 & 8 & 7 \\ 1 & 8 & 3 & 12 \\ 9 & 6 & 6 & 2 \\ 1 & 1 & 5 & 7 \end{bmatrix},$$

with $\rho^{AM}(C) = (0, 2\frac{1}{3}, 3\frac{1}{3}, 19\frac{1}{3})$.

A natural first step in reducing a general liability problem is to assume that on a bilateral level the claims are already settled. Thus for all pairs $i, j \in N$ with $i \neq j$, $c_{ij}c_{ji} = 0$. The bilaterally leveled claim matrix $\bar{C} = (\bar{c}_{ij}) \in \mathcal{L}^N$ is obtained from C in the following way

$$\bar{c}_{ij} = \begin{cases} [c_{ij} - c_{ji}]^+ & \text{if } j \neq i \\ c_{ii} & \text{if } j = i. \end{cases}$$

Thus, we eliminate cycles of length 2 and obtain

$$\bar{C} = \begin{bmatrix} 4 & 4 & 0 & 6 \\ 0 & 8 & 0 & 11 \\ 1 & 3 & 6 & 0 \\ 0 & 0 & 3 & 7 \end{bmatrix}$$

which is still a non-hierarchical liability problem.

Not only can we level claims bilaterally, we can also do this for longer cycles. In the matrix $\bar{C}$ we can find multiple cycles of claims. The longest one, with length 4, goes from player 1 to player 2, then from player 2 to player 4, from player 4 to player 3 and from player 3 back to player 1, see the bold entries in $\bar{C}$ below:

$$\bar{C} = \begin{bmatrix} 4 & \mathbf{4} & 0 & 6 \\ 0 & 8 & 0 & \mathbf{11} \\ \mathbf{1} & 3 & 6 & 0 \\ 0 & 0 & \mathbf{3} & 7 \end{bmatrix}.$$

Since the lowest claim in this cycle is 1 ($\bar{c}_{31} = 1$), we can reduce the cycle by 1, which results in the following non-hierarchical liability problem C^1 :

$$C^1 = \begin{bmatrix} 4 & 3 & 0 & 6 \\ 0 & 8 & 0 & \mathbf{10} \\ 0 & \mathbf{3} & 6 & 0 \\ 0 & 0 & \mathbf{2} & 7 \end{bmatrix}.$$

In C^1 we detect another cycle: from 2 to 4, to 3 and back to 2. We can reduce the claims by an amount of 2, with the hierarchical liability problem $C^{1,\Delta}$ as a result. Here,

$$C^{1,\Delta} = \begin{bmatrix} 4 & 3 & 0 & 6 \\ 0 & 8 & 0 & 8 \\ 0 & 1 & 6 & 0 \\ 0 & 0 & 0 & 7 \end{bmatrix}$$

is a hierarchical liability problem since $C^{1,\Delta} \in \mathcal{L}^{\bar{N}}$ with $\bar{N} = \{1, 3, 2, 4\}$ is an upper triangular matrix. We have that $\rho^{AM}(C^{1,\Delta}) = (0, 2.5, 5, 17.5) \neq \rho^{AM}(C)$.

In the liability problem $\bar{C}$, we can also start with another cycle: from 2 to 4, then from 4 to 3 and from 3 back to 2 as shown by the bold entries in $\bar{C}$ below:

$$\bar{C} = \begin{bmatrix} 4 & 4 & 0 & 6 \\ 0 & 8 & 0 & \mathbf{11} \\ 1 & \mathbf{3} & 6 & 0 \\ 0 & 0 & \mathbf{3} & 7 \end{bmatrix}.$$

In this case we can reduce all claims with an amount of 3 and we would immediately end up with the hierarchical liability problem $C^{2,\Delta}$ given by

$$C^{2,\Delta} = \begin{bmatrix} 4 & 4 & 0 & 6 \\ 0 & 8 & 0 & 8 \\ 1 & 0 & 6 & 0 \\ 0 & 0 & 0 & 7 \end{bmatrix}.$$

If we rearrange the players such that $\bar{N} = \{3, 1, 2, 4\}$, the matrix $C^{2,\Delta} \in \mathcal{L}^{\bar{N}}$ is upper triangular. Note that $\rho^{AM}(C^{2,\Delta}) = (0, 2, 5, 18)$ which is different from both $\rho^{AM}(C)$ and from $\rho^{AM}(C^{1,\Delta})$. $\blacklozenge$

6.2 Hydraulic approach

Inspired by Kaminski (2000) one could introduce hydraulic solutions for hierarchical liability problems. We illustrate a hydraulic method by means of an example.

Example 6.2. Let $N = \{1, 2, 3, 4\}$. Consider the liability problem $C \in \mathcal{L}^{N,\Delta}$, given by

$$C = \begin{bmatrix} 6 & 3 & 4 & 5 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Every agent's cash can be seen as a vessel of liquid that is connected to its claimants by a system of equally wide tubes. The tube from 1 to 3 can only transfer $c_{13} = 4$ units of liquid, after this amount the tube is closed or disconnected. Hence c_{13} is the capacity of tube₁₃. The same holds for all other entries in C . This is represented in Figure 6.2, in the left system of connected vessels. The digit in a shaded area indicates the content level of the vessel and the digit in a rhombus represent the capacity of a tube. We open all the vessels simultaneously and let the liquid flow, until no flow is possible anymore.

In this way, for agent 1, through every outgoing tube, an amount of 2 will be transferred to vessels 2, 3 and 4. The remaining capacity of the tubes are 2, 1 and 3 respectively. The initial content of the second vessel, 1, is divided equally among 3 and 4, but at the same time an extra amount of 2 flows into his vessel via tube₁₂ and this amount is also divided among 3 and 4. In this way we can continue with vessel 3 and 4. The final result is shown in Figure 6.2, in the right system of connected vessels. The final allocation equals $(0, 0, 2.5, 6.5)$.

This hydraulic scheme fits with the *CEA* idea and in fact it can be shown that for this example $\rho^{CEA}(C) = (0, 0, 2.5, 6.5)$. $\blacklozenge$

Figure 1: A hydraulic solution for Example 6.2

References

- Aumann, R. and Maschler, M. (1985). Game theoretical analysis of bankruptcy problems from the talmud. *Journal of Economic Theory*, 36:195–213.
- Bjorndal, E. and Jornsten, K. (2009). Flow sharing and bankruptcy games. *International Journal of Game Theory*, 39:11–28.
- Calleja, P., Borm, P., and Hendrickx, R. (2005). Multi-issue allocation situations. *European Journal of Operational Research*, 164:730–747.
- Chun, Y. (1988). The proportional solution for rights problems. *Mathematical Social Sciences*, 15:231–246.
- Habis, H. and Herings, P. (2011). Stochastic bankruptcy games. Research Memorandum 045, Maastricht University. To appear in *International Journal of Game Theory*.
- Kaminski, M. (2000). ‘Hydraulic’ rationing. *Mathematical Social Sciences*, 40:131–155.
- Moulin, H. and Sethuraman, J. (2012). The bipartite rationing problem. Mimeo Rice University. To appear in *Operations Research*.
- O’Neill, B. (1982). A problem of rights arbitration from the talmud. *Mathematical Social Sciences*, 2:345–371.

- Palvolgyi, D., Peters, H., and Vermeulen, D. (2010). A strategic approach to estate division problems with non-homogeneous preferences. Research Memorandum 036, Maastricht University.
- Thomson, W. (2003). Axiomatic and game-theoretic analysis of bankruptcy and taxation problems: a survey. *Mathematical social sciences*, 45:249–297.
- van Gulick, G. (2010). *Game Theory and Applications in Finance*. PhD thesis, Tilburg University.
- Yeh, C. (2008). Secured lower bound, composition up, and minimal rights first for bankruptcy problems. *Journal of Mathematical Economics*, 44:925–932.