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**THE COMMUNICATION TREE VALUE FOR
TU-GAMES WITH GRAPH COMMUNICATION**

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The communication tree value for TU-games with graph communication

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Abstract

A new solution is presented for transferable utility games with graph communication where the cooperation possibilities are represented by a graph. Players are only able to cooperate and obtain some worth in a coalition if they form a connected set in the given graph. To determine the payoff for each player, a single-valued solution, the communication tree value, is proposed for this class of games. The idea is that to form the grand coalition of all players a player can join a set of players only if this player is connected in the graph to at least one of the players in the set. To a set of players, starting with an arbitrary single player, from each maximally connected subset of remaining players one player joins who is connected to one or more of the players in that set. In this way a (rooted) tree on the set of players is obtained, called a communication tree. For a given game each communication tree of the graph induces a marginal contribution vector, in which any player receives a payoff equal to what he contributes in worth when he joins his subordinates in the tree. The average payoff over all communication trees of the graph determines the value. In case the underlying graph is cycle-complete the value coincides with the average tree solution. When there is complete communication between all players, which is the special case of cycle-completeness, players join one by one, yielding the Shapley value. A weak form of convexity is introduced, under which the value is guaranteed to be an element of the core. For games with complete graph communication the condition coincides with convexity and in case the underlying graph is cycle-free it is weaker than super-additivity.

Keywords: Cooperative game, communication structure, Myerson value, core stability, convexity, rooted tree.

AMS subject classification: 90B18, 91A12, 91A43.

JEL code: C71.

1 Introduction

Cooperative games with transferable utility or simply TU-games have been subject of study of those economic environments in which agents can take joint actions to create surplus. Formally, a TU-game consists of a finite set of players and a characteristic function assigning to each subset of players a real number, called the worth of the coalition. In a TU-game any subset of players is assumed to be able to cooperate and produce its worth. The problem is how much payoff every player should obtain. The best known single-valued solution concept for TU-games is the Shapley value ([13]), which takes as solution the average of all marginal contribution vectors of a game.

Myerson [11] is the first to introduce a communication structure to TU-games by combining a TU-game with a graph on the set of players, being a set of communication links between pairs of players. Myerson assumes that a coalition of players is able to act jointly and obtain its worth only if it is a connected set of players in the given graph, meaning that any pair of different players in the set is able to communicate directly or by means of a sequence of adjacent links in the coalition. This is a special case of a standard approach to wider environments where the cooperation possibilities are restricted. TU-games with more general or different sets of restricted coalitions have been studied intensively in the literature, e.g., see [12] and [1]. In this paper we focus on TU-games with restricted communication structure where the communication is represented by an (undirected) graph, as in [11], and call these games TU-games with graph communication.

Myerson [11] introduces a value for the class of TU-game with graph communication which is known as the Myerson value. To this end first the so called graph-restricted standard TU-game is derived from a given TU-game with graph communication and then the Shapley value is applied on this game. The Myerson value is characterized by component efficiency and fairness axioms. Borm, Owen, and Tijs [3] introduce the so called position value for the class of TU-games with graph communication, see also Meessen [10]. They give other characterizations for the Myerson value and a characterization for the position value where the underlying graph is cycle-free. Slikker [14] gives a characterization of the position value for the whole class of TU-games with graph communication by means of component efficiency and balanced total threats axioms. In Herings, van der Laan, Talman and Yang [8] a different solution has been proposed, called the average tree solution. This solution is a generalization of [7] for the class of TU-games where the underlying graph is cycle-free to the class of all TU-games with graph communication. In [7], also a characterization of the average tree solution for the class of games with cycle-free graph communication is given by means of component efficiency and component fairness. The average tree solution is the average of those marginal contribution vectors that correspond to a specific collection of spanning trees for the underlying graph, see also [2]. In Koshevoy

and Talman [9] the average of all different marginal contribution vectors is taken as solution concept. This solution, called the gravity center solution, (generically) coincides with the gravity center of the convex hull of all marginal contribution vectors of the game.

In this paper a different generalization of the average tree solution from the class of TU-games with cycle-free graph communication to the class of TU-games with graph communication is introduced. We define a new solution and call it the *communication tree value*. This value takes the average of all marginal contribution vectors induced by what we call communication trees. The idea of the construction of a communication tree is similar to the well-known interpretation of the Shapley value that players successively join each other in any order and receive as payoff the average of their marginal contributions. When the graph is not complete we allow a player to join only if this player is connected to someone who joined before. More precisely, one of the players enters first and becomes the root of a communication tree. The remaining players form a subgraph which may consist of several different components. Then, from every component one of the players who is directly connected in the graph to the root, becomes linked in the tree to the root. In this way a connected coalition of players is formed. In the next step, from every component of the remaining players one of the players who are directly connected in the graph to a player in the coalition, is linked in the tree to the (unique) player who joined the coalition in the previous step and is connected to that component. Again a connected coalition is formed and the same step is applied to this coalition, and so on. Since the number of players is finite and in each step at least one player is linked to the existing coalition, the process will terminate after a finite number of steps leading to a rooted tree. Such a tree is called a communication tree because for each player it holds that this player and all his subordinates in the tree form a connected set in the underlying graph and also he and all the other players, which includes the root, form a connected set. Moreover, these other players themselves form a connected set in the graph. However, a communication tree is not necessarily a spanning tree for the graph, like all the trees are that define the average tree solution.

Given a TU-game with graph communication, the marginal contribution of a player for a communication tree is given by the difference in total worths when this player joins his subordinates in the tree. In particular, the player who is the root of the tree receives as payoff the difference of the worth of the grand coalition, being the coalition of all players, and the sum of the worths of the components of the other players. The communication tree value then allocates to the players the average of the marginal contribution vectors over all communication trees. We show that the communication tree value coincides with the average tree solution when the underlying graph is cycle-complete and thus coincides with both the average tree solution and the Shapley value when the underlying graph is

complete. In this way on the class of TU-games with arbitrary graph communication the communication tree value can be seen as another generalization of the average tree solution for TU-games with cycle-free graph communication as introduced in [7] and also of the Shapley value for TU-games with complete graph communication. We further introduce a weak kind of convexity for the class of TU-games with graph communication, called graph-convexity, which guarantees that the communication tree value is in the core. For TU-games with complete graph communication graph-convexity coincides with convexity, while for TU-games with cycle-free graph communication the notion of graph-convexity is weaker than superadditivity.

This paper is organized as follows. Section 2 gives the preliminaries on TU-games with graph communication. In Section 3 the communication tree value is introduced. Section 4 is devoted to the analysis of comparison of the communication tree value with the average tree solution. In Section 5 the notion of graph-convexity is introduced for TU-games with graph communication and it is shown that the communication tree value lies in the core if the game is graph-convex. Section 6 concludes.

2 TU-games with graph communication

A TU-game with graph communication is a triple (N, v, L) , with $N = \{1, \dots, n\}$ a finite set of players, $v: 2^N \rightarrow \mathbb{R}$ a characteristic function, and (N, L) a graph with node set the set of players N and a set of links L , being a subset of $\{\{i, j\} \mid i \neq j, i, j \in N\}$, the set of all subsets of N of size 2. The set L stands for the collection of communication links between the players. If $\{i, j\}$ belongs to L , then players i and j are said to be linked and this link is interpreted as communication ability between the two players. The pair (N, v) is called a *standard TU-game* or simply a *TU-game*. For $S \subset N$, the game (N, u_S) , where $u_S(X) = 1$, when $S \subset X$, and $u_S(X) = 0$ otherwise, is called a *unanimity game on S*.

A sequence of different nodes $(i_1, \dots, i_{k'})$ is a *path* in a graph (N, L) from i_1 to $i_{k'}$ if $\{i_k, i_{k+1}\} \in L$ for $k = 1, \dots, k' - 1$. A coalition of players $S \in 2^N$ is *connected* in (N, L) if for any $i, j \in S$, $i \neq j$, there is a path in (N, L) from i to j satisfying that all nodes of the path are in S . Notice that the empty set and all singleton coalitions are connected coalitions by definition. Any two members of a connected coalition are assumed to be able to communicate with each other either directly via a link between them or indirectly through a path of nodes in the coalition. A coalition S of players is a *component* in the graph (N, L) if S is a connected coalition in (N, L) and S cannot form a larger connected coalition in (N, L) with any other player $j \in N \setminus S$. A graph (N, L) is *complete* if $L = \{\{i, j\} \mid i \neq j, i, j \in N\}$, i.e. every player is able to communicate with every other player. A path in a graph (N, L) of at least three different nodes $(i_1, \dots, i_{k'})$ is a *cycle* in (N, L) if

$\{i_{k'}, i_1\} \in L$. A graph (N, L) is *cycle-free* if it does not contain any cycles. For a given graph (N, L) , each $K \in 2^N$ induces a subgraph $(K, L(K))$, with $L(K) = \{\{i, j\} \in L \mid i, j \in K\}$ the set of links on K . Notice that by definition $L(N) = L$. A graph (N, L) is *cycle-complete* if for any cycle $(i_1, \dots, i_{k'})$ in (N, L) the subgraph $(\{i_1, \dots, i_{k'}\}, L(\{i_1, \dots, i_{k'}\}))$ is a complete graph. The concepts defined above for the graph (N, L) are defined similarly for any subgraph $(K, L(K))$, $K \in 2^N$. For $K \in 2^N$, $C^L(K)$ denotes the collection of connected coalitions in the subgraph $(K, L(K))$ and $\widehat{C}^L(K)$ denotes the collection of components in $(K, L(K))$.

A directed graph on N is a pair (N, D) such that $D \subseteq \{(i, j) \in N \times N \mid i \neq j\}$ is a collection of *directed links*. A player i is a *predecessor* of player j and j a *successor* of i in D if $(i, j) \in D$. A sequence of different nodes $(i_1, \dots, i_{k'})$ is a *directed path* from i_1 to $i_{k'}$ in the directed graph (N, D) if $(i_k, i_{k+1}) \in D$ for $k = 1, \dots, k' - 1$. A *tree* (N, T) is a directed graph with T a collection of exactly $n - 1$ directed edges, such that from exactly one node, called the *root*, there is a unique directed path to every other node. In a tree the root has no predecessor and any other node has exactly one predecessor. A node j is a *subordinate* of node i and i is a *superior* of j in T if T contains a directed path from i to j . The set $B_i(T)$ consists of all subordinates of player $i \in N$ in T and $\overline{B}_i(T)$ consists of the subordinates of player i in T together with player i , i.e., $\overline{B}_i(T) = B_i(T) \cup \{i\}$. Given an undirected graph (N, L) , a tree (N, T) is a *spanning tree* for (N, L) if $(i, j) \in T$ implies $\{i, j\} \in L$. For a given directed graph (N, D) , each $K \in 2^N$ induces a directed subgraph $(K, D(K))$, with $D(K) = \{(i, j) \in D \mid i, j \in K\}$ the set of directed links on K . Notice that by definition $D(N) = D$. The concepts defined for the directed graph (N, D) are defined similarly for any subgraph $(K, D(K))$, $K \in 2^N$.

In a TU-game with graph communication (N, v, L) it is assumed that a coalition S of players can only cooperate and realize its worth $v(S)$ if S is a connected coalition. In the rest of the paper, we assume without loss of generality that the grand coalition N itself is a connected coalition, so N can realize its worth $v(N)$. Otherwise, the analysis can be done analogously for each component in the graph (N, L) . When each pair of players can communicate directly, i.e., $L = \{\{i, j\} \mid i, j \in N, i \neq j\}$, (N, v, L) is said to be a TU-game with *complete graph communication*. We associate such game with the standard TU-game (N, v) , and with some abuse of notation denote this special case of a TU-game with graph communication by (N, v) as well.

A *payoff vector* $x \in \mathbb{R}^n$ is an n -dimensional vector allocating a payoff $x_i \in \mathbb{R}$ to every player $i \in N$. We write $x(S) = \sum_{i \in S} x_i$ for $S \in 2^N$. For a TU-game with graph communication (N, v, L) , a payoff vector $x \in \mathbb{R}^n$ is *efficient* if $x(N) = v(N)$, i.e., it fully distributes the worth $v(N)$ of the grand coalition N to all players. A *solution* for TU-games with graph communication is a mapping F that assigns to every TU-game with

graph communication (N, v, L) a set of payoff vectors $F(N, v, L) \subset \mathbb{R}^n$. A solution F is *efficient* if for any (N, v, L) every element of $F(N, v, L)$ is efficient.

The *core* is perhaps the best-known multi-valued solution for standard TU-games, see Gillies [5]. It assigns to every TU-game (N, v) the set

$$C(N, v) = \{x \in \mathbb{R}^n \mid x(N) = v(N), \text{ and } x(S) \geq v(S) \text{ for all } S \in 2^N\} \quad (2.1)$$

of undominated efficient payoff vectors. With complete graph communication, a payoff vector x is dominated if there exists a coalition S such that $x(S) < v(S)$. However, in general for TU-games with graph communication (N, v, L) , the members of a coalition S can only cooperate if S is connected and so a payoff vector x can only be dominated by connected coalitions. Consequently, for TU-games with graph communication (N, v, L) the core becomes equal to the set $C(N, v, L)$ given by

$$C(N, v, L) = \{x \in \mathbb{R}^n \mid x(N) = v(N), \text{ and } x(S) \geq v(S) \text{ for all } S \in C^L(N)\}, \quad (2.2)$$

i.e., the core is the set of efficient payoff vectors that are not dominated by any connected coalition. Notice that $C(N, v, L) = C(N, v^L)$, where v^L stands for the characteristic function of the so-called *Myerson restricted* standard TU-game (N, v^L) induced by (N, v, L) and defined by

$$v^L(S) = \sum_{K \in \hat{C}^L(S)} v(K), \quad S \in 2^N,$$

i.e., the worth of a coalition S equals the sum of the worths of its components in the subgraph $(S, L(S))$.

Perhaps the best-known single-valued solution for standard TU-games (N, v) is the *Shapley value*, see [13], which assigns to every standard TU-game (N, v) the average $\phi(N, v)$ of all $n!$ marginal vectors $m^\pi(N, v) \in \mathbb{R}^n$ of the game (N, v) , where $\pi = (\pi(1), \dots, \pi(n))$ is a permutation $\pi: N \rightarrow N$ assigning a unique number $\pi(i) \in N$ to every $i \in N$ and $m^\pi(N, v) = (m_1^\pi(N, v), \dots, m_n^\pi(N, v))$ with, for every $j \in N$, $m_j^\pi(N, v) = v(\pi^j \cup \{j\}) - v(\pi^j)$ and $\pi^j = \{i \in N \mid \pi(i) < \pi(j)\}$. The *Myerson value*, μ , see [11], is a single-valued solution assigning to every TU-game with graph communication (N, v, L) the Shapley value $\phi(N, v^L)$ of the Myerson restricted game, i.e.,

$$\mu(N, v, L) = \phi(N, v^L).$$

Another single-valued solution for TU-games with graph communication is the *average tree solution* introduced in Herings *et al.* [8]. For a graph (N, L) , let $\mathcal{A}^L$ be the set of n -tuples $A = (A_1, A_2, \dots, A_n)$ of subsets of N satisfying the following conditions:

- (1) For all $i \in N$, $i \in A_i$, and for some $j \in N$, $A_j = N$;

(2) For every $K \in \widehat{C}^L(A_i \setminus \{i\})$, $i \in N$, it holds that $K = A_j$ for some $j \in N$ with $\{i, j\} \in L$. The average tree solution, AT , is defined as the average of the marginal contribution vectors induced by all $A \in \mathcal{A}^L$, i.e.,

$$AT(N, v, L) = \frac{1}{|\mathcal{A}^L|} \sum_{A \in \mathcal{A}^L} m^A(N, v, L),$$

where for $A \in \mathcal{A}^L$ the marginal contribution vector $m^A(N, v, L)$ is given by

$$m_i^A(N, v, L) = v(A_i) - \sum_{K \in \widehat{C}^L(A_i \setminus \{i\})} v(K), \quad i \in N.$$

3 The communication tree value

In this section we present a new single-valued solution for TU-games with graph communication. The solution is based on the idea of a combinatorial formation of the grand coalition. A well-known interpretation of the Shapley value is that starting with any of the players the grand coalition is formed in steps; at each step one of the remaining players is allowed to enter a “room” and joins the player who previously entered. Here we adopt this line of interpretation by allowing players to enter partially unordered, depending on the structure of the underlying graph, and a player may enter only if he has a link to someone who entered before. The grand coalition is formed as follows. Starting with any of the players, from each component of the remaining players one player who is linked to a player who entered before enters and he joins the last player who entered before and is linked to the component. After all players have entered, possibly partially unordered, a tree on the set of players is obtained. We call such a tree a communication tree. In case the graph is complete the players will enter one by one and each player joins the player that just entered before, yielding a line tree, which corresponds to a complete ordering or permutation as in the interpretation of the Shapley value.

Definition 3.1 For a given graph (N, L) , a tree (N, T) is a *communication tree* for (N, L) if for every $i \in N$ and $K \in \widehat{C}^L(B_i(T))$ there is a unique $(i, j) \in T$ such that $K = \overline{B}_j(T)$ and $\{j, h\} \in L$ for some $h \in N$ satisfying $j \in B_h(T)$.

For a communication tree for the graph (N, L) it holds that if player (node) j is a successor of player i then in the graph player j and his subordinates form a component of the set of subordinates of player i and, moreover, player j is linked to at least one of his superiors, which may or may not be player i . The root of a communication tree is any of the players and has as successors from each component of the set of other players precisely

one player that is linked to him. On its turn, each of the latter players has as successors from each component of the set of remaining players exactly one player that is linked to him or another superior, and so on. In this way, starting with an arbitrary player, each player is joined by one player from every component of the remaining players that is linked to him or to some player that joined before.

The notion of communication tree for a graph has the following characterization.

Lemma 3.2 *For a given graph (N, L) , a tree (N, T) with root $r \in N$ is a communication tree if and only if the following conditions hold:*

- (1) *For every $i \in N$ both $\overline{B}_i(T)$ and $N \setminus B_i(T)$ are in $C^L(N)$.*
- (2) *For all $(i, j) \in T$ and $(i, h) \in T$, $j \neq h$, it holds that $\overline{B}_j(T) \cup \overline{B}_h(T) \notin C^L(N)$, i.e., for any $k \in \overline{B}_j(T)$ and $l \in \overline{B}_h(T)$ it holds that $\{k, l\} \notin L$.*

Proof. Let (N, T) be a communication tree for (N, L) with root r . Since $N \in C^L(N)$ and $\overline{B}_r(T) = N$, both $\overline{B}_r(T)$ and $N \setminus B_r(T) = \{r\}$ are in $C^L(N)$. Suppose $i \neq r$. Then $\overline{B}_i(T) \in \widehat{C}^L(B_j(T))$ for some $j \in N$, and so $\overline{B}_i(T) \in C^L(N)$. To show that $N \setminus B_i(T)$, $i \neq r$, is in $C^L(N)$, take any $j \notin B_i(T)$. Then all superiors of j in (N, T) are not in $B_i(T)$ and, by definition of T , there exists a path in $(N \setminus B_i(T), L(N \setminus B_i(T)))$ of j and superiors of j in (N, T) from j to the root r . Therefore, $N \setminus B_i(T)$ is a connected coalition. Next, let $(i, j) \in T$ and $(i, h) \in T$ with $h \neq j$. Then $\overline{B}_j(T)$ and $\overline{B}_h(T)$ are two different components of $B_i(T)$ and therefore their union is not a connected coalition, i.e., $\overline{B}_j(T) \cup \overline{B}_h(T) \notin C^L(N)$.

To prove the converse, let (N, T) be a tree such that (1) and (2) are satisfied. For $i \in N$, let $j_1, \dots, j_k$ be such that $(i, j_h) \in T$ for $h = 1, \dots, k$. Since (N, T) is a tree, the sets $\overline{B}_{j_1}(T), \dots, \overline{B}_{j_k}(T)$ form a partition of $B_i(T)$. From condition (1) it follows that every $\overline{B}_{j_h}(T)$, $h = 1, \dots, k$, is a connected coalition and together with (2) it follows that $\overline{B}_{j_1}(T), \dots, \overline{B}_{j_k}(T)$ are the components of $B_i(T)$, i.e., for every $K \in \widehat{C}^L(B_i(T))$, $i \in N$, there is a unique $(i, j) \in T$ such that $K = \overline{B}_j(T)$. Take any $\overline{B}_j(T) \in \widehat{C}^L(B_i(T))$, then from condition (1) it follows that $N \setminus B_j(T)$ is a connected coalition and so since $\overline{B}_j(T) \neq N$ there exists $h \in N \setminus \overline{B}_j(T)$ such that $\{j, h\} \in L$. From condition (2) it follows that $j \in B_h(T)$. $\square$

The property in condition (1) states that in any communication tree of a graph every player together with his subordinates forms a connected coalition and also with all the other players this player forms a connected coalition, and so, the player is linked to at least one of these other players. The property in condition (2) says that if in a communication tree K_1 and K_2 are two different components in the set of subordinates of some player, then there is no link in the graph between any player of K_1 and any player of K_2 . This means that if in the tree two players are not subordinate of each other, they are not able to communicate directly with each other. In a communication tree the player at the root

is therefore able to communicate with any other player by a unique (directed) path of links in the tree, while that player himself is able to transmit information he obtains (from the root) to his subordinates because he and his subordinates form a connected coalition in the underlying graph.¹ However, a player is not able to communicate directly with any other subordinate of his superiors, but only indirectly via a unique path of superiors of them, one being a common superior.

Corollary 3.3 *For a communication tree (N, T) for a graph (N, L) , for any $i \in N$ it holds that $N \setminus \overline{B}_i(T) \in C^L(N)$.*

Proof. Take any $j \in N \setminus \overline{B}_i(T)$, $i \in N$. Then, by definition of T , there exists a path in $(N \setminus \overline{B}_i(T), L(N \setminus \overline{B}_i(T)))$ of j and superiors of j in (N, T) from j to the root r . Therefore $N \setminus \overline{B}_i(T) \in C^L(N)$. $\square$

The corollary implies that for any communication tree for a graph it holds that the complement of the set consisting of a player and his subordinates is always connected. Therefore for a communication tree it holds for any player that both the set of this player together with his subordinates and the set of remaining players are connected and that the player himself is connected to the latter set.

The following example illustrates the concept of communication trees.

Example 3.4 Let (N, L) be given by $N = \{1, 2, 3, 4\}$ and $L = \{\{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 4\}, \{3, 4\}\}$. Consider the case where player 1 is the root, so $B_1(T) = \{2, 3, 4\}$ is the set of subordinates of player 1. $B_1(T)$ is a connected coalition in (N, L) and has therefore one component $\{2, 3, 4\}$. All players in $\{2, 3, 4\}$ are linked to player 1 and we take anyone to join player 1. First, let player 2 join, then $B_2(T) = \{3, 4\}$ is the set of subordinates of player 2 with unique component $\{3, 4\}$. Both players 3 and 4 in $\{3, 4\}$ are linked to players 1 or 2. Notice that player 4 is linked to player 1, but not to player 2, while player 3 is linked to both player 1 and player 2. If player 3 joins first, he joins player 2 and $B_3(T) = \{4\}$ is the set of subordinates of player 3, consisting of player 4 only, who then joins as last player, yielding communication tree $T_1^1 = \{(1, 2), (2, 3), (3, 4)\}$. If player 4 joins first instead of player 3, then player 4 joins player 2 and $B_4(T) = \{3\}$ is the set of subordinates of player 4, consisting of player 3 only, who then joins as last player, yielding communication tree $T_2^1 = \{(1, 2), (2, 4), (4, 3)\}$. Notice that in the graph player 4 is not linked to player 2 but to player 1. Next, let player 3 join player 1, then the set of subordinates of player 3 is $B_3(T) = \{2, 4\}$, however now there are two different components, $\widehat{C}^L(\{2, 4\}) = \{\{2\}, \{4\}\}$. Since both components contain single elements both player 2 and player 4 must join the previously entered player 3, yielding communication tree $T_3^1 = \{(1, 3), (3, 2), (3, 4)\}$. Finally, let player 4 join player 1, then $B_4(T) = \{2, 3\}$ is the set of remaining players 4 with unique

¹This property clarifies the name of such trees.

component $\{2, 3\}$. Both players 2 and 3 are linked to the players 1 or 4. Notice that player 2 is linked to player 1, but not to player 4, while player 3 is linked to both player 1 and player 4. If player 2 joins first, he joins player 4 and $B_2(T) = \{3\}$ is the set of subordinates of player 2, consisting of player 3 only, who then joins as last player, yielding communication tree $T_4^1 = \{(1, 4), (4, 2), (2, 3)\}$. If player 3 joins first instead of player 2, then player 3 joins player 4 and $B_3(T) = \{2\}$ is the set of subordinates of player 3, consisting of player 2 only, who then joins as last player, yielding communication tree $T_5^1 = \{(1, 4), (4, 3), (3, 2)\}$. So, there are five communication trees having player 1 as root. Since player 1 and player 3 are symmetric, there are also five communication trees having player 3 as a root, namely

$$\begin{aligned} T_1^3 &= \{(3, 2), (2, 1), (1, 4)\}, \\ T_2^3 &= \{(3, 2), (2, 4), (4, 1)\}, \\ T_3^3 &= \{(3, 1), (1, 2), (1, 4)\}, \\ T_4^3 &= \{(3, 4), (4, 1), (1, 2)\}, \\ T_5^3 &= \{(3, 4), (4, 2), (2, 1)\}. \end{aligned}$$

Similarly, we find four communication trees having player 2 as root and by symmetry also four communication trees having player 4 as root, namely

$$\begin{aligned} T_1^2 &= \{(2, 1), (1, 4), (4, 3)\}, \\ T_2^2 &= \{(2, 1), (1, 3), (3, 4)\}, \\ T_3^2 &= \{(2, 3), (3, 4), (4, 1)\}, \\ T_4^2 &= \{(2, 3), (3, 1), (1, 4)\}, \\ T_1^4 &= \{(4, 1), (1, 2), (2, 3)\}, \\ T_2^4 &= \{(4, 1), (1, 3), (3, 2)\}, \\ T_3^4 &= \{(4, 3), (3, 2), (2, 1)\}, \\ T_4^4 &= \{(4, 3), (3, 1), (1, 2)\}. \end{aligned}$$

In total there are 18 communication trees for this graph. □

As shown in the example, the set of communication trees is typically a proper subset of the collection of all trees on N . Moreover, a communication tree is not necessarily a spanning tree, as we will discuss in the next section. The number of communication trees in a graph depends on the structure of the graph.

To define the new value on the class of games with graph communication we only consider communication trees. For a TU-game with graph communication the marginal contribution vector corresponding to a communication tree is defined as follows. Let $\mathcal{T}^L$ denote the collection of all communication trees for a graph (N, L) .

Definition 3.5 For a TU-game with graph communication (N, v, L) , the *marginal contribution vector* $m^T(N, v, L) \in \mathbb{R}^n$ corresponding to communication tree $T \in \mathcal{T}^L$ is the vector of payoffs given by

$$m_i^T(N, v, L) = v(\bar{B}_i(T)) - \sum_{\{j \in N \mid (i, j) \in T\}} v(\bar{B}_j(T)), \quad i \in N.$$

Notice that one can equivalently define the payoff vector $m^T(N, v, L)$ by

$$m_i^T(N, v, L) = v(\bar{B}_i(T)) - \sum_{K \in \hat{\mathcal{C}}^L(B_i(T))} v(K), \quad i \in N.$$

At payoff vector $m^T(N, v, L)$ player $i \in N$ receives a payoff equal to the worth of the connected coalition $\bar{B}_i(T)$ minus the total worths of the components of $(B_i(T), L(B_i(T)))$. With respect to the corresponding communication tree (N, T) , the marginal contribution vector assigns to a player the worth of the connected coalition consisting of himself and his subordinates minus the total worth of the components of his subordinates. Notice that a marginal contribution vector $m^T(N, v, L)$ is a marginal vector $m^\pi(N, v^L)$ of the restricted game (N, v^L) for any permutation π on N satisfying $B_i(T) \subset \pi^i$ for all $i \in N$. We now define the communication tree value as the average of all marginal contribution vectors that correspond to a communication tree for the underlying graph.

Definition 3.6 On the class of TU-games with graph communication the *communication tree value* assigns to a TU-game with graph communication (N, v, L) the payoff vector $CT(N, v, L)$ given by

$$CT(N, v, L) = \frac{1}{|\mathcal{T}^L|} \sum_{T \in \mathcal{T}^L} m^T(N, v, L).$$

Example 3.7 We consider a TU-game with graph communication (N, v, L) defined as in Example 3.4 and with v equal to the unanimity game u_N on N . For any communication tree T with root r , $r \in N$, we have $m_r^T(N, u_N, L) = 1$ and $m_i^T(N, u_N, L) = 0$ for $i \neq r$. Thus, the communication tree value for this game equals $CT(N, v, L) = (\frac{5}{18}, \frac{4}{18}, \frac{5}{18}, \frac{4}{18})$.

4 Comparison with the average tree solution

In this section we compare the communication tree value with the average tree solution and analyze the value for TU-games with graph communication in case the underlying graph is cycle-complete. We first give an alternative definition of the average tree solution in terms of trees and prove its equivalence with the definition of the average tree solution given above.

Definition 4.1 For a given graph (N, L) , a tree (N, T) is a *spanning communication tree* for (N, L) if for every $K \in \widehat{C}^L(B_i(T))$, $i \in N$, there is a unique $(i, j) \in T$ such that $K = \overline{B}_j(T)$ and $\{i, j\} \in L$.

The set of spanning communication trees of a graph (N, L) is denoted by $\mathcal{ST}^L$ and the set of spanning trees of (N, L) by $\mathcal{S}^L$.

Lemma 4.2 For a graph (N, L) it holds that $\mathcal{ST}^L = \mathcal{T}^L \cap \mathcal{S}^L$.

Proof Take any $T \in \mathcal{ST}^L$. By definition $(i, j) \in T$ implies $\{i, j\} \in L$, which means that $T \in \mathcal{S}^L$. Moreover, since $i \notin B_i(T)$, we may take $h = i$ in the definition of communication tree, and so $T \in \mathcal{T}^L$.

To show the converse, let $T \in \mathcal{T}^L \cap \mathcal{S}^L$. By definition of communication tree, for any $K \in \widehat{C}^L(B_i(T))$, $i \in N$, there is unique $(i, j) \in T$ such that $K = \overline{B}_j(T)$ and $\{j, h\} \in L$ with $j \in B_h(T)$. Since T is a spanning tree of (N, L) , $(i, j) \in T$ implies $\{i, j\} \in L$. Since $i \notin B_i(T)$, we may take $h = i$, and so $K = \overline{B}_j(T)$ and $\{i, j\} \in L$, which implies $T \in \mathcal{ST}^L$. $\square$

The lemma states that if a communication tree is also a spanning tree then it is a spanning communication tree and that each spanning communication tree is both a communication tree and a spanning tree. For a spanning communication tree (N, T) for a graph (N, L) it holds that $(i, j) \in T$ implies that $\{i, j\} \in L$, while for a communication tree (N, T) for (N, L) it holds that $(i, j) \in T$ implies $\{j, h\} \in L$ for some superior h of j , which is not necessarily i .

The next theorem shows that for any TU-game with graph communication the average tree solution is the average of the marginal contribution vectors over all spanning communication trees.

Theorem 4.3 For a TU-game with graph communication (N, v, L) it holds that the average tree solution is equal to

$$AT(N, v, L) = \frac{1}{|\mathcal{ST}^L|} \sum_{T \in \mathcal{ST}^L} m^T(N, v, L).$$

Proof We first show that there is a bijection $f : \mathcal{ST}^L \rightarrow \mathcal{A}^L$ from which it will follow that $m^T(N, v, L) = m^{f(T)}(N, v, L)$ for all $T \in \mathcal{ST}^L$. Let

$$f(T) = (\overline{B}_1(T), \overline{B}_2(T), \dots, \overline{B}_n(T)), \text{ for } T \in \mathcal{ST}^L.$$

Clearly, for each $i \in N$, $i \in \overline{B}_i(T)$, and for the root r of T , we have $\overline{B}_r(T) = N$. So, condition (1) in the definition of the average tree solution is satisfied. Take any $i \in N$ and $K \in \widehat{C}^L(B_i(T))$. Then there exists $j \in N$ such that $\overline{B}_j(T) = K$ and since $T \in \mathcal{ST}^L$, $\{i, j\} \in L$, so that condition (2) in the definition of the average tree solution is also satisfied. Hence, the mapping f is well-defined.

Next, we show that f is one-to-one. Take two different elements from $\mathcal{ST}^L$, say T and T' . Suppose that $\overline{B}_i(T) = \overline{B}_i(T')$ for all $i \in N$. Since $T \neq T'$, there is $(i, j) \in T$, but $(i, j) \notin T'$ for some $i, j \in N$. So it holds that $\overline{B}_j(T) \subset \overline{B}_i(T)$. Since $(i, j) \in T$, there is no $h \in N$ such that $\overline{B}_j(T) \subset \overline{B}_h(T) \subset \overline{B}_i(T)$. By assumption, $\overline{B}_j(T') \subset \overline{B}_i(T')$. However, since $(i, j) \notin T'$, there exists $h \in N$ such that $(i, h) \in T'$ and $j \in \overline{B}_h(T')$ and so $\overline{B}_j(T') \subset \overline{B}_h(T') \subset \overline{B}_i(T')$, yielding a contradiction, and therefore $f(T) \neq f(T')$. To show that for any $A \in \mathcal{A}^L$ there exists a $T \in \mathcal{ST}^L$ with $f(T) = A$, let $T = \{(i, j) | A_j \in \widehat{C}^L(A_i \setminus \{i\}), i \in N\}$. Take any $j \in \overline{B}_i(T)$, $i \in N$. If $j = i$, then by definition we have $j \in A_i$. If $j \neq i$, then there exists a directed path from i to j . Denote this path by $(i, l_1, \dots, l_k, j)$. Then, by definition of T , $j \in A_{l_k} \setminus \{l_k\}$. Also we have $A_{l_k} \setminus \{l_k\} \subset A_{l_{k-1}} \setminus \{l_{k-1}\} \subset \dots \subset A_{l_1} \setminus \{l_1\} \subset A_i \setminus \{i\} \subset A_i$. Thus $j \in A_i$, so $\overline{B}_i(T) \subset A_i$. Now take any $j \in A_i$. If i is such that $A_i = N$, there is no $j' \in N$ such that $A_i \in \widehat{C}^L(A_{j'} \setminus \{j'\})$, which implies that $(j', i) \notin T$. So, i is the root of T and $\overline{B}_i(T) = N$, and thus $A_i \subset \overline{B}_i(T)$. Now let i be such that $A_i \neq N$. Take any $j \in A_i$. If $i = j$, then by definition $A_i \subset \overline{B}_i(T)$. Let $i \neq j$, then $A_j \subset A_i$, which by construction implies that there is a directed path from i to j in T , and so $A_i \subset \overline{B}_i(T)$. Thus, for all $i \in N$ it holds that $A_i = \overline{B}_i(T)$. From this it follows that f is a bijection and that $m^{f(T)}(N, v, L) = m^T(N, v, L)$ for all $T \in \mathcal{ST}^L$. $\square$

The next example shows the difference between the communication tree value and the average tree solution.

Example 4.4 Consider again the graph analyzed in Example 3.4. Of the 18 communication trees there are 14 spanning communication trees, namely

$$T_1^1 = \{(1, 2), (2, 3), (3, 4)\},$$

$$T_3^1 = \{(1, 3), (3, 2), (3, 4)\},$$

$$T_5^1 = \{(1, 4), (4, 3), (3, 2)\},$$

$$T_1^2 = \{(2, 1), (1, 4), (4, 3)\},$$

$$T_2^2 = \{(2, 1), (1, 3), (3, 4)\},$$

$$T_3^2 = \{(2, 3), (3, 4), (4, 1)\},$$

$$T_4^2 = \{(2, 3), (3, 1), (1, 4)\},$$

$$T_1^3 = \{(3, 2), (2, 1), (1, 4)\},$$

$$T_3^3 = \{(3, 1), (1, 2), (1, 4)\},$$

$$T_4^3 = \{(3, 4), (4, 1), (1, 2)\},$$

$$T_1^4 = \{(4, 1), (1, 2), (2, 3)\},$$

$$T_2^4 = \{(4, 1), (1, 3), (3, 2)\},$$

$$T_3^4 = \{(4, 3), (3, 2), (2, 1)\},$$

$$T_4^4 = \{(4, 3), (3, 1), (1, 2)\}.$$

For the TU-game with graph communication (N, u_N, L) defined as in Example 3.7 the communication tree value is the average of 18 marginal vectors and equals $CT(N, v, L) = (\frac{5}{18}, \frac{4}{18}, \frac{5}{18}, \frac{4}{18})$, while the average tree solution is the average of 14 marginal vectors and equals $AT(N, v, L) = (\frac{3}{14}, \frac{4}{14}, \frac{3}{14}, \frac{4}{14})$. The GC solution in [9], being the average of all 22 different marginal vectors, is equal to $GC(N, v, L) = (\frac{5}{22}, \frac{6}{22}, \frac{5}{22}, \frac{6}{22})$. Observe that in spite of the fact that player 1 is linked to any other player in the graph he gets in both the average tree solution and the GC solution less than player 2 who is linked to only player 1 and player 3. However, the communication tree value gives to player 1 more than to player 2. For the Myerson value all players get the same payoff, $\frac{1}{4}$. $\square$

On the subclass of TU-games with cycle-complete graph communication the communication tree value and the average tree solution coincide because in this class all communication trees are spanning trees, as is shown in the next lemma.

Lemma 4.5 *Let (N, L) be a cycle-complete graph, then $\mathcal{T}^L = \mathcal{ST}^L$*

Proof From Lemma 4.2 it follows that we have to show that if (N, L) is a cycle-complete graph then $\mathcal{T}^L \subseteq \mathcal{S}^L$. Take any $T \in \mathcal{T}^L$. Suppose $T \notin \mathcal{S}^L$, then there exists a link $(i, j) \in T$ such that $\{i, j\} \notin L$. Since $T \in \mathcal{T}^L$, there exists $h \in N \setminus B_i(T)$ such that $\{j, h\} \in L$. Clearly, $h \neq i$. Since $\overline{B}_i(T)$ is a connected coalition in (N, L) there exists a path from i to j in $(\overline{B}_i(T), L(\overline{B}_i(T)))$. Denote this path by $(p_1, p_2, \dots, p_m)$ with $p_1 = i$, $p_m = j$ and $\{p_l, p_{l+1}\} \in L$, $l = 1, 2, \dots, m-1$. Since $N \setminus B_i(T)$ is a connected coalition there exists also a path in $(N \setminus B_i(T), L(N \setminus B_i(T)))$ from h to i . Denote this path by

$(q_1, q_2, \dots, q_{m'})$ with $q_1 = h$, $q_{m'} = i$ and $\{q_l, q_{l+1}\} \in L$, $l = 1, 2, \dots, m' - 1$. Since the intersection of $N \setminus B_i(T)$ and $\overline{B}_i(T)$ only contains node i and since $p_1 = i$ and $\{q_{m'-1}, i\} \in L$, the path $(p_1, p_2, \dots, p_m, h, q_2, \dots, q_{m'-1})$ is a cycle in (N, L) . By cycle-completeness of (N, L) it follows that $\{i, j\} \in (N, L)$, a contradiction. $\square$

The lemma states that on the class of cycle-complete graphs the set of spanning trees coincides with the set of spanning communication trees.

Corollary 4.6 *For any cycle-free or complete communication graph (N, L) it holds that $\mathcal{T}^L = \mathcal{ST}^L$.*

Obviously both cycle-free and complete graphs are cycle-complete, so that we get the above corollary as a special case of Lemma 4.5.

Theorem 4.7 *For any TU-game with cycle-complete graph communication (N, v, L) it holds that $CT(N, v, L) = AT(N, v, L)$. Moreover, for a TU-game with complete graph communication (N, v, L) it holds that $CT(N, v, L) = \phi(N, v)$.*

Proof From Lemma 4.5, Theorem 4.3 and Theorems 1 and 2 in [8] the result follows immediately. $\square$

The theorem implies that on the class of TU-games with cycle-complete graph communication the communication tree value is equal to the average tree solution and that on the class of TU-games with complete graph communication both solutions coincide with the Shapley value. In this way the communication tree value can be seen as an alternative generalization of the average tree solution on the class of TU-games with cycle-free graph communication and of the Shapley value on the class of standard TU-games. Observe that for a complete graph there are $n!$ communication trees and that each communication tree is a directed line graph, i.e., each player has exactly one successor and one predecessor, except the root who has no predecessor and the unique leaf who has no successor. Any directed line graph corresponds to a unique permutation in which the last chosen player enters first, the second last chosen player enters second, and so on. For any cycle-free graph there are only n communication trees which are precisely the spanning trees of the graph. For an arbitrary graph the number of communication trees lies between these two numbers and is at least equal to the number of spanning communication trees. The communication tree value is in general the average of a larger number of marginal contribution vectors than the average tree solution is. However, the number of spanning trees of an arbitrary graph is not directly related to the number of communication trees but is larger than the number of spanning communication trees in case the graph is not cycle-free.

5 Graph-convexity and core properties

In this section we provide conditions for TU-games with graph communication under which the communication tree value lies in the core of a game. For a standard TU-game (N, v) it is well-known that the Shapley value is in the core $C(N, v)$ if the game is convex, i.e., $v(U) + v(W) \leq v(U \cup W) + v(U \cap W)$ for all $U, W \subset N$. A standard TU-game is superadditive if these inequalities are satisfied for every U and W such that $U \cap W = \emptyset$. Superadditivity is insufficient to ensure that a standard TU-game has a non-empty core. We say that a game with graph communication (N, v, L) is convex (superadditive) if its Myerson restricted game (N, v^L) is convex (superadditive). A superadditive TU-game with cycle-free graph communication has a non-empty core. In particular, it follows from Demange [4] that every marginal contribution vector $m^A(N, v, L)$, $A \in \mathcal{A}^L$, is in $C(N, v^L)$ if (N, v, L) is a superadditive TU-game with cycle-free graph communication. In fact, when the game (N, v^L) is superadditive, then (N, v^L) is permutationally convex for every permutation induced by any spanning tree $T \in \mathcal{S}^L$ on (N, L) and, according to Granot and Huberman [6], $m^T(N, v, L)$ is in $C(N, v^L)$. So, for superadditive TU-games with cycle-free graph communication the communication tree value is in the core, because the core is a convex set. Also for TU-games with cycle-free graph communication, Talman and Yamamoto [15] provide a condition even weaker than superadditivity under which the average tree solution and thus the communication tree value is in the core.

We next introduce the notion of graph-convexity, which will be shown to assure that the communication tree value is in the core for an arbitrary TU-game with graph communication.

Definition 5.1 A TU-game with graph communication (N, v, L) is *graph-convex* if

$$v(U) + v(W) \leq v(U \cup W) + \sum_{K \in \widehat{C}^L(U \cap W)} v(K),$$

for any $U, W \in C^L(N)$ that satisfy:

- (1) Both $U \cup W$ and $N \setminus U$ are in $C^L(N)$;
- (2) For any $K \in \widehat{C}^L(U \cap W)$ the set $N \setminus K$ is in $C^L(N)$;
- (3) There exists $i \in U \setminus W$ such that $\{i\} \cup (N \setminus U)$ is in $C^L(N)$.

Graph-convexity requires that the convexity condition only holds for any two connected sets of which the union is connected, the complements of all the components of the intersection are connected, the complement of one of the two sets is connected, and this complement is

connected to some element of this set which is not also an element of the other set. Graph-convexity reduces to convexity for the class of games with graph communication where the underlying graph is complete, because for those games all subsets of N are connected. We illustrate the concept of graph-convexity with an example.

Example 5.2 Let the TU-game with graph communication be defined through (N, v, L) , where N and L are the same as in Example 3.4, $v(S) = 2$ if $S = \{1, 2, 3\}$ or N , $v(S) = 1$ if $S = \{1, 3, 4\}$, and $v(S) = 0$ otherwise. The game is superadditive and graph-convex, but not convex. To see the latter, let $U = \{1, 2, 3\}$ and $W = \{1, 3, 4\}$, then $v(U) + v(W) = 3 > 2 = v(U \cap W) + v(U \cup W)$. Since $v = v^L$ and the game (N, v) is superadditive, the game (N, v, L) is superadditive. For graph-convexity, we first observe that for the case when U or W is not equal to one of the sets in $\{1, 2, 3\}$, $\{1, 3, 4\}$ or N , the inequality for graph-convexity is satisfied trivially. Moreover, if $U \subset W$ or $W \subset U$, i.e., if one of the sets is a subset of the other, then again the inequality is satisfied. So, we are only left with the case when $U = \{1, 2, 3\}$ and $W = \{1, 3, 4\}$. Then $U \cup W \in C^L(N)$, $U \cap W = \{1, 3\} \in C^L(N)$, but $N \setminus (U \cap W) = \{2, 4\} \notin C^L(N)$. This shows that the game is graph-convex. The Myerson value equals $\mu(N, v, L) = (\frac{18}{24}, \frac{9}{24}, \frac{18}{24}, \frac{3}{24})$ and the communication tree value equals $CT(N, v, L) = (\frac{16}{18}, \frac{4}{18}, \frac{16}{18}, 0)$. For $x = (x_1, x_2, x_3, x_4)$ to be a core element of this game we necessarily have $x_4 = 0$, since $v(\{1, 2, 3\}) = v(N)$, which implies that $\mu(N, v, L) \notin C(N, v, L)$. The communication tree value, however, is an element of core.

□

The next theorem shows that the communication tree value is in the core if the game is graph-convex.

Theorem 5.3 *If a TU-game with graph communication (N, v, L) is graph-convex, then $CT(N, v, L) \in C(N, v, L)$.*

Proof. We show that under graph-convexity every marginal contribution vector $m^T = m^T(N, v, L)$, $T \in \mathcal{T}^L$, is an element of $C(N, v, L)$, which proves the result since the core is a convex set.

Consider any $T \in \mathcal{T}^L$. We have by definition of m^T that $\sum_{i \in N} m_i^T = v(N)$. To prove that $\sum_{i \in S} m_i^T \geq v(S)$ for $S \in C^L(N)$, take any $S \in C^L(N)$ and let $S_1, \dots, S_{k'}$ be the components of S in the directed subgraph $(S, T(S))$. Denote $B_i(T)$ by B_i and $\overline{B}_i(T)$ by $\overline{B}_i$ for all $i \in N$. Since S_k is connected and (N, T) is a tree, the directed subgraph $(S_k, T(S_k))$ is a tree and there exists a unique $r_k \in S_k$ such that $S_k \subset \overline{B}_{r_k}$, $k = 1, \dots, k'$. Notice that r_k is the root of $(S_k, T(S_k))$. We define $I = \{r_1, \dots, r_{k'}\}$. By Property (2) of Lemma 3.2, either $\overline{B}_i \subset \overline{B}_j$ or $\overline{B}_j \subset \overline{B}_i$ or $\overline{B}_i \cap \overline{B}_j = \emptyset$ for every pair $i, j \in I$. Since S is connected,

(N, T) is a tree, and because of Property (2) of Lemma 3.2 there is one node, say, $r_{k'}$, such that for every $i \in I \setminus \{r_{k'}\}$ the node i is a subordinate of $r_{k'}$ and therefore $\overline{B}_i \subset B_{r_{k'}}$. Since $S_i \subset \overline{B}_i$, for all $i \in I$, we have that $S \subset \overline{B}_{r_{k'}}$.

Next, for $k = 1, \dots, k'$, let $F_{r_k} = \{j \in N \setminus S \mid (i, j) \in T \text{ for some } i \in S_k\}$ be the set of successors of nodes in S_k in the tree (N, T) outside S . We define $J = \cup_{i \in I} F_i$ as the set of all such successors. For $j \in J$, let

$$H_j = \{i \in I \mid \overline{B}_i \subset \overline{B}_j \text{ and } \nexists i' \in I \setminus \{i\} \text{ such that } \overline{B}_i \subset \overline{B}_{i'} \subset \overline{B}_j\}.$$

We define $\widehat{N} = I \cup J$ and the directed graph $(\widehat{N}, \widehat{T})$ by

$$\widehat{T} = \{(i, j) \mid j \in F_i, i \in I\} \cup \{(j, i) \mid i \in H_j, j \in J\}.$$

Clearly, $(\widehat{N}, \widehat{T})$ is a tree with root $r_{k'}$.

For $k = 1, \dots, k'$, it holds that $\overline{B}_{r_k} = S_k \cup (\cup_{j \in F_{r_k}} \overline{B}_j)$, so

$$\sum_{i \in S} m_i^T = \sum_{k=1}^{k'} [v(\overline{B}_{r_k}) - \sum_{j \in F_{r_k}} v(\overline{B}_j)]. \quad (5.3)$$

Without loss of generality, let $1, \dots, k'$ be such that $1 \leq k_1 < k_2 \leq k'$ implies $\overline{B}_{r_{k_1}} \subset \overline{B}_{r_{k_2}}$ or $\overline{B}_{r_{k_1}} \cap \overline{B}_{r_{k_2}} = \emptyset$. For $k = 0, \dots, k'$, let $B^k = \overline{B}_{r_1} \cup \dots \cup \overline{B}_{r_k}$, so it follows that $B^0 = \emptyset$.

Consider some $k \in \{1, \dots, k'\}$ and write $F_{r_k} = \{j_1^k, \dots, j_{\ell_k}^k\}$. If $F_{r_k} = \emptyset$, then we define ℓ_k to be zero. When $\ell_k \geq 1$, then, for $\ell = 1, \dots, \ell_k$, consider the two sets $U = \overline{B}_{j_\ell^k}$ and $W = S \cup B^{k-1} \cup (\overline{B}_{j_1^k} \cup \dots \cup \overline{B}_{j_{\ell-1}^k})$. We show that the sets U and W satisfy the conditions of Definition 5.1. By property (1) of Lemma 3.2, the set U is connected and by Corollary 3.3 the set $N \setminus U$ is connected. To see that the sets W and $U \cup W$ are connected, it is sufficient to show by using induction that $U \cup W$ is connected if W is connected, since for $k = 1$ and $\ell = 1$ the set $W = S$ is connected. Because $j_\ell^k \in F_{r_k}$, there exists $i \in S_k$ such that $(i, j_\ell^k) \in T$. Then $i \in W$ and, by property (1) of Lemma 3.2, $\overline{B}_i \in C^L(N)$, and so $i \in W$ is connected to $U = \overline{B}_{j_\ell^k}$ since $\overline{B}_{j_\ell^k} \in \widehat{C}^L(B_i)$. Since by induction W is connected, this implies that $U \cup W$ is also connected. As for condition (2) of Definition 5.1 for any $K \in \widehat{C}^L(U \cap W)$ it holds that $K = \overline{B}_h$ for some $h \in H_{j_\ell^k}$, which implies from Corollary 3.3 that $N \setminus K$ is connected. Concerning condition (3) of Definition 5.1, $j_\ell^k \in U \setminus W$ and $\{j_\ell^k\} \cup (N \setminus U) = N \setminus B_{j_\ell^k}$ is connected according to property (1) of Lemma 3.2. This shows that U and W satisfy all conditions of Definition 5.1. From graph-convexity it now follows that

$$v(S \cup B^{k-1} \cup (\overline{B}_{j_1^k} \cup \dots \cup \overline{B}_{j_{\ell-1}^k})) + v(\overline{B}_{j_\ell^k}) \leq v(S \cup B^{k-1} \cup (\overline{B}_{j_1^k} \cup \dots \cup \overline{B}_{j_\ell^k})) + \sum_{i \in H_{j_\ell^k}} v(\overline{B}_i).$$

By repeated application of this argument for $\ell = 1, \dots, \ell_k$ and since $F_{r_k} = \{j_1^k, \dots, j_{\ell_k}^k\}$ and $S \cup B^{k-1} \cup (\cup_{j \in F_{r_k}} \bar{B}_j) = S \cup B^k$, it follows that for $k = 1, \dots, k'$

$$v(S \cup B^{k-1}) + \sum_{j \in F_{r_k}} v(\bar{B}_j) \leq v(S \cup B^k) + \sum_{j \in F_{r_k}} \sum_{i \in H_j} v(\bar{B}_i).$$

Notice that this formula is also valid if $F_{r_k} = \emptyset$, since then $S \cup B^{k-1} = S \cup B^k$. By repeated application of the last inequality for $k = 1, \dots, k'$, we find that

$$v(S) + \sum_{k=1}^{k'} \sum_{j \in F_{r_k}} v(\bar{B}_j) \leq v(S \cup B^{k'}) + \sum_{k=1}^{k'} \sum_{j \in F_{r_k}} \sum_{i \in H_j} v(\bar{B}_i).$$

Since $S \cup B^{k'} = \bar{B}_{r_{k'}}$ and $(\hat{N}, \hat{T})$ is a tree, it follows that every $\bar{B}_{r_k}$, $k = 1, \dots, k' - 1$, appears exactly once in the summation in the right-hand side, and we obtain

$$v(S) + \sum_{k=1}^{k'} \sum_{j \in F_{r_k}} v(\bar{B}_j) \leq \sum_{k=1}^{k'} v(\bar{B}_{r_k}). \quad (5.4)$$

From equations (5.3) and (5.4) it follows that

$$v(S) \leq \sum_{i \in S} m_i^T,$$

which completes the proof. $\square$

From this theorem the next corollary follows immediately.

Corollary 5.4 *If a TU-game with graph communication (N, v, L) is graph-convex, then the core $C(N, v, L)$ is non-empty.*

The next lemma shows that under weaker conditions than superadditivity, a TU-game with cycle-free graph communication is graph-convex.

Lemma 5.5 *A TU-game with cycle-free graph communication (N, v, L) is graph-convex if and only if for every $W \in C^L(N)$ it holds that*

$$v(U) + v(W) \leq v(U \cup W) \text{ for all } U \in \hat{C}^L(N \setminus W).$$

Proof. Let $U, W \in C^L(N)$ satisfy the conditions in Definition 5.1 with $N \setminus U$ being connected. We first show that $U \cap W = \emptyset$. Suppose $U \cap W \neq \emptyset$. Take any $k \in U \cap W$. From condition (3) of Definition 5.1 there exists $i \in U \setminus W$ and $j \in N \setminus U$ such that $\{i, j\} \in L$. We distinguish between the cases $j \in W$ and $j \notin W$. If $j \in W$, then there exists a path from j to k in $(W, L(W))$ and a path from k to i in $(U, L(U))$, since both U and W

are connected. This implies that there is a path in (N, L) from j to i containing k . This contradicts the fact that the graph is cycle-free. Now, let $j \notin W$. Take any $h \in W \setminus U$. Then there is a path from i to k in $(U, L(U))$, a path from k to h in $(W, L(W))$, and a path from h to j in $(N \setminus U, L(N \setminus U))$, since $N \setminus U$ is connected. This implies that there is a path in (N, L) from i to j containing h and k . This contradicts again that the graph is cycle-free. Therefore, $U \cap W = \emptyset$. Then, since $N \setminus U$ is connected and (N, L) is cycle-free, U is a component of $N \setminus W$, i.e., $U \in \widehat{C}^L(N \setminus W)$.

Conversely, suppose $W \in C^L(N)$ and $U \in \widehat{C}^L(N \setminus W)$. Then, since (N, L) is cycle-free, we have that $U \in C^L(N)$, $U \cup W \in C^L(N)$, and $N \setminus U \in C^L(N)$, so that condition (1) of Definition 5.1 is satisfied. Since $U \in \widehat{C}^L(N \setminus W)$, it holds that $U \cap W = \emptyset$, which implies that condition (2) of Definition 5.1 is satisfied. Condition (3) is satisfied because $U \setminus W = U$ and (N, L) is connected. $\square$

The lemma shows that the superadditivity condition $v(U) + v(W) \leq v(U \cup W)$ only has to hold for connected W and connected U being a component of $(N \setminus W, L(N \setminus W))$. For a game with cycle-free graph communication superadditivity requires that $v(U) + v(W) \leq v(U \cup W)$ for all disjoint U and W satisfying that U , W , and $U \cup W$ are connected. So, the next corollary follows immediately.

Corollary 5.6 *A TU-game with cycle-free graph communication (N, v, L) is graph-convex if (N, v, L) is superadditive.*

The following example illustrates that graph-convexity is strictly weaker than superadditivity and also that the Myerson value may not be in the core if the game is graph-convex.

Example 5.7 Let (N, v, L) be such that $N = \{1, 2, 3, 4\}$, $L = \{\{1, 2\}, \{2, 3\}, \{3, 4\}\}$, $v(S) = -1$ if $S = \{2, 3\}$, $v(S) = 0$ otherwise. The game is not superadditive. To see this take $U = \{2\}$ and $W = \{3\}$, then $v(\{2, 3\}) = -1 < 0 = v(\{2\}) + v(\{3\})$. Because both $N \setminus \{2\}$ and $N \setminus \{3\}$ are not connected, this condition is redundant for graph-convexity. The only condition where $\{2, 3\}$ shows up is when $U = \{1\}$ and $W = \{2, 3\}$ or $U = \{4\}$ and $W = \{2, 3\}$. Notice that the complement of $\{2, 3\}$ is not connected and therefore cannot be equal to U . In either case we have that the desired inequality is satisfied, since $-1 = v(U) + v(W) < v(U \cup W) = 0$. The communication tree value equals $CT(N, v, L) = (0, 0, 0, 0)$ and is the unique core element, while the Myerson value equals $\mu(N, v, L) = (\frac{1}{12}, \frac{-1}{12}, \frac{-1}{12}, \frac{1}{12})$, which is outside the core. $\square$

6 Concluding remarks

In this paper the communication tree value is proposed for the class of TU-games with graph communication, i.e., a TU-game paired with an undirected graph, representing the communication or the cooperation possibilities of the players. This solution generalizes both the solution introduced by Herings *et al.* [7] for the class of TU-games with cycle-free graph communication and the Shapley value for the class of TU-games with complete graph communication. We introduce the condition of graph-convexity under which the communication tree value is an element of the core. For the class of TU-games with cycle-free graph communication, graph-convexity is weaker than superadditivity. In general, graph-convexity is weaker than convexity, and only coincides with it for games with complete graph communication.

References

- [1] Algaba, E., J.M Bilbao, and J.J. López (2000), A unified approach to restricted games, *Theory and Decision* 50, 333–345.
- [2] Baron, R., S. Béal, E. Rémy, and P. Solal (2010), Average tree solution and the distribution of Harsanyi dividends, *International Journal of Game Theory* 40, 331–349.
- [3] Borm, P., G. Owen, and S. Tijs (1992), On the position value for communication situations, *SIAM Journal of Discrete Mathematics* 5, 305–320.
- [4] Demange, G. (2004), On group stability in hierarchies and networks, *Journal of Political Economy* 112, 754–778.
- [5] Gillies, D.B. (1953), Some Theorems on n -Person Games, PhD thesis, Princeton University, Princeton.
- [6] Granot, G., and G. Huberman (1982), The relationship between convex games and minimal cost spanning tree games: A case for permutationally convex games, *SIAM Journal of Algebraic and Discrete Methods*, 3, 288–292.
- [7] Herings, P.J.J., G. van der Laan, and A.J.J. Talman (2008), The average tree solution for cycle-free graph games, *Games and Economic Behavior* 62, 77–92.
- [8] Herings, P.J.J., G. van der Laan, A.J.J. Talman, and Z. Yang (2010), The average tree solution for cooperative games with communication structure, *Games and Economic Behavior* 68, 626–632.

- [9] Koshevoy, G.A. and A.J.J. Talman (2011), Solution concepts for games with general coalitional structure, CentER Discussion Paper 2011-122, Tilburg University, Tilburg.
- [10] Meessen, R. (1988), Communication Games, Master Thesis, University of Nijmegen, Nijmegen (in Dutch).
- [11] Myerson, R.B. (1977), Graphs and cooperation in games, *Mathematics of Operations Research* 2, 225–229.
- [12] Myerson, R.B. (1980), Conference structures and fair allocation rules, *International Journal of Game Theory* 9, 169–182.
- [13] Shapley, L. (1953), A value for n -person games, in: H.W. Kuhn and A.W. Tucker (eds.), *Contributions to the Theory of Games II*, Princeton University Press, Princeton, pp. 307–317.
- [14] Slikker, M. (2005), A characterization of the position value, *International Journal of Game Theory* 33, 505–514.
- [15] Talman, A.J.J. and Y. Yamamoto (2007), Average tree solution and subcore for acyclic graph games. *Journal of the Operations Research Society of Japan* 51, 203–212.
- [16] Velzen, S. van (2005), Cooperation in Network and Scheduling, PhD Thesis, Tilburg University, Tilburg.