BARGAINING FOR OVER-THE-COUNTER RISK REDISTRIBUTIONS: THE CASE OF LONGEVITY RISK

By

Tim Boonen, Anja De Waegenaere, Henk Norde

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Tim Boonen†  Anja De Waegenaere‡  Henk Norde§

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Abstract

Existing literature regarding the natural hedge potential that arises from combining liabilities with different sensitivities focuses on the optimal liability mix, but does not address the question whether and how changes in the liability mix can be obtained. In the absence of a well-functioning market, parties could change their liability mix through Over-the-Counter risk redistributions. This, however, requires that each involved party benefits (weakly) from the redistribution. In this paper we first show that under relatively mild conditions, there is more than one risk redistribution that satisfies this criterion. We then explicitly model the bargaining process by which firms will agree to a particular redistribution. We allow for heterogeneous beliefs regarding the underlying probability distribution, which may arise from using different models to predict future mortality rates. We use this model to quantify the potential benefits.

JEL-Classification: C71, C78, G22, J11

1 Introduction

There is an increasing need for hedging non-marketed risk. Markets serve as mechanism to reallocate risk among firms. However, if for a class of risks markets are non-existent or if there are obstacles to trade, firms could benefit from risk-sharing by trading Over-the-Counter. In this paper, we investigate the extent to which involved parties can benefit from such risk redistributions.

Although the model that we introduce allows for any type of risk, our focus in this paper is on redistribution of longevity risk. Longevity risk is the systematic risk in life-contingent liabilities that arises from the fact that death rates in a population change in an unpredictable way. This risk is a major concern for pension funds, since their liabilities are directly linked with longevity. Exposure to longevity risk can be rather substantial for pension funds, as shown by, e.g., Coughlan et al. (2007) and Hári et al. (2008). Reinsurance contracts do exist, but the capacity of reinsurance is limited (see, e.g., OECD, 2005). Moreover, there

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‡ Corresponding author: Tel: +31 13 4662373, Fax: +31 13 4663280, Email address: t.j.boonen@uvt.nl, Department of Econometrics and OR, Tilburg University, CentER for Economic Research and Netspar, PO box 90153, 5000 LE Tilburg.
§ Department of Accountancy and Department of Econometrics and OR, Tilburg University, CentER for Economic Research and Netspar.
† Department of Econometrics and OR, Tilburg University, CentER for Economic Research and Netspar.
is still considerable uncertainty regarding the price of longevity risk. Despite a very active
and growing literature on pricing of longevity risk (see, e.g., Bauer et al., 2010), lack of
consensus regarding accurate pricing hampers trade.\(^1\) Equilibrium prices do not exist, and,
as a consequence, longevity-linked contracts are mainly traded Over-the-Counter (OTC).
As pointed out by Dowd et al. (2006), the market for Over-the-Counter survivor swaps is
expected to grow fast. In this paper, we analyze opportunities for pension funds and insurers
to benefit from mutual redistribution of risk via Over-the-Counter trade. Specifically, parties
bargain for a “fair” risk redistribution and a “fair” price. This option is potentially attractive
in case redistribution of risk allows parties to benefit from natural hedge potential that arises
from combining risks that are not perfectly correlated.

Redistribution of risk in order to benefit from natural hedge potential has been investigated
also in Tsai et al. (2010) and Wang et al. (2010). Our approach differs from these existing
approaches in three ways. First, the existing literature focuses on reducing the risk in the
present value of the liability payments over complete run-off. We consider instead the case
where pension funds and insurers redistribute risk in order to reduce the volatility of their
Net Asset Value (NAV) at a prespecified future date. The Net Asset Value is defined as the
difference between the value of the assets and the value of the liabilities.\(^2\) Second, we extend
eXisting literature by allowing for the case where pension funds and insurance companies may
have heterogeneous beliefs regarding the probability distribution of future mortality rates.
There exists a relatively large variety of models that can be used to predict future mortality
rates. The seminal work of Lee and Carter (1992) gave rise to an active literature on mortality
modeling (see, e.g., Brouhns et al., 2002; Cairns et al., 2006, 2007, 2008; Cossette et al., 2007;
Dowd et al., 2010; Plat, 2009). Existing literature also shows that the effect of model risk
could be substantial, i.e., predictions for future mortality rates may differ significantly when
different models are used, or when model parameters are estimated on different datasets (see,
e.g., Dowd et al., 2008). In our model, we therefore allow the parties who wish to redistribute
risk to “agree to disagree” on the appropriate forecast model. Third, our approach differs
from existing models in that we model the risk redistribution as the outcome of a bargaining
process in which the involved parties bargain for a reallocation of risk that benefits all. They
will agree to a particular redistribution only if they benefit weakly in expected utility terms.
Moreover, when more than two parties are involved, they will only reach a mutual agreement
if no subset of parties can be better o by splitting off and instead redistributing the risk
amongst each other.

The main results of the paper are as follows. We first characterize Pareto optimal redistri-
butions of risk in a relatively general setting, and show that heterogeneous beliefs regarding
the underlying probability distribution of the risks can make redistribution more attractive.
It is more likely that parties will be able to achieve Pareto improvement via redistribution
when they disagree on the underlying probability distribution. We then apply the model to
investigate the potential benefits from redistributing longevity risk between pension funds
and insurers. We model redistribution of risk via longevity swaps with a predetermined
maturity date, and quantify the benefits from a particular swap contract by the maximum
premium that the pension fund or insurer would have been willing to pay to obtain the same
risk reduction via a reinsurance contract (i.e., the indierence price). It is often argued that
major obstacles to trade are lack of agreement regarding the probability distribution of future
mortality rates, lack of capacity in the insurance market to effectively hedge longevity

\(^1\) Blake et al. (2006) address the main obstacles for trading longevity linked products in the market.
\(^2\) The focus on Net Asset Value is in line with current regulation. Under Solvency II, for example, the
regulator requires that the current level of assets is sufficient to reduce the probability of a negative Net Asset
Value on a one year horizon to a sufficiently low level.

2
risk in annuity portfolios, and reluctance to engage in contracts with a long horizon. Our results suggest that the benefits from redistribution are significant, even when the insurer is small relative to the pension fund, and even when the horizon of the swap contract is relatively short. Moreover, the results also suggest that the benefits are significantly larger when parties disagree regarding the underlying probability distribution. With homogeneous beliefs and a death benefit insurer with a portfolio with a best-estimate value equal to 20% of the best-estimate value of the annuity portfolio of the pension fund, the benefits of the optimal swap contract are significant. Over a horizon of ten years, the premium the pension fund (insurer) would be willing to pay to obtain the same degree of risk reduction amounts to 7.1% (14.6%) of the best estimate value of the liabilities. When the two parties have heterogeneous beliefs, these benefits increase to 8.6% (17.9%).

The remainder of this paper is organized as follows. We introduce the model for Over-the-Counter redistribution of risk in Section 2. In Section 3, we characterize risk redistributions that satisfy some desirable properties, and model the outcome of the bargaining process in which the involved parties agree on a particular risk redistribution. In Section 4, we use the model to numerically illustrate the extent to which pension funds and life insurance companies can benefit from redistributing longevity risk. Section 5 concludes.

2 Over-the-Counter risk redistribution

When there is no liquid market for the risk, firms can approach each other and trade risk Over-the-Counter (OTC). In this case, both the risk redistribution and the corresponding prices are set via a bargaining process. If firms behave myopically, i.e., if each firm bargains for a redistribution that maximizes its own utility, the firms will typically not reach an agreement even though each firm knows that if they behave cooperatively, they can all benefit. In this paper, we therefore consider a setting in which firms behave cooperatively, and strive to reach a redistribution of risk that benefits all. In Section 2.1, we first introduce some notation and assumptions. In Section 2.2, we define some properties that the risk redistribution will satisfy if firms behave rationally.

2.1 Notation and assumptions

In this section, we introduce some notation and assumptions that will be used throughout the paper.

- Risk is redistributed between a finite number of firms. The set of firms is denoted by \( N \).
- Firm \( i \)'s risk profile, \( i \in N \), is represented by a random variable on a set of states of nature at a prespecified future date, \( T \). The set of possible states of the world on date \( T \) is denoted by \( \Omega ; \Omega \) is finite. Firms can “agree to disagree” on the probability measure on the state space \( \Omega \). The subjective probability measure of firm \( i \in N \) is denoted by \( \mathbb{P}_i : \Omega \rightarrow \mathbb{R}^{++} \), where \( \sum_{\omega \in \Omega} \mathbb{P}_i(\{\omega\}) = 1 \).
- To evaluate risk profiles, firm \( i \in N \) uses a von Neumann-Morgenstern utility function \( u_i : D_i \rightarrow \mathbb{R} \).

\(^3\)In game theory, this phenomenon is known as the Prisoner’s dilemma. See, e.g., Boonen (2010).
where $D_i$ is the domain of the utility function. $D_i$ is open and convex. Moreover, $u_i$ is continuous on $D_i$, twice continuously differentiable, $u''_i(x) > 0$ and $u''_i(x) < 0$ on $D_i$ for all $i \in N$, and $\lim_{x \to \inf D_i} u'_i(x) = \infty$, and $\lim_{x \to \sup D_i} u'_i(x) = 0$, for all $i \in N$.

- The risk profile of firm $i$ prior to redistribution is a random variable $X_i$ with realizations in $D_i$, i.e., $X_i : \Omega \to D_i$. Firms can change their risk profile through redistribution of risk. We do not impose any restrictions on the redistribution, i.e., we allow firms to bargain for any risk redistribution that leads to posterior risk profiles $X_{i \text{post}} : \Omega \to D_i$, for $i \in N$, that satisfy

\[
\sum_{i \in N} X_{i \text{post}} = \sum_{i \in N} X_i, \quad (1)
\]

We will refer to posterior risk profiles that satisfy (1) as feasible posterior risk profiles. For any given feasible vector of posterior risk profiles $(X_{i \text{post}})_{i \in N}$, the corresponding expected utility gain of firm $i \in N$ is denoted $\Delta U_i(X_{i \text{post}})$, i.e.,

\[
\Delta U_i(X_{i \text{post}}) = \mathbb{E}_{P_i} [u_i(X_{i \text{post}}) - u_i(X_i)], \quad \text{for all } i \in N. \quad (2)
\]

- There is complete information about the risk profiles, utility functions and probability measures of all firms.

Throughout the remainder of the paper, we let the set of firms $N$, the prior risk profiles $(X_i)_{i \in N}$, the utility functions $(u_i)_{i \in N}$, and the subjective probability measures $(P_i)_{i \in N}$ be given. We investigate the extent to which the firms will be able to agree on a redistribution of risk that leads to posterior risk profiles $(X_{i \text{post}})_{i \in N}$. For notational convenience, we will use the following shorthand notations:

- For any $S \subseteq N$, we denote $D(S)$ for the set of all vectors of possible risk profiles of the firms in $S$, i.e., $D(S) = \prod_{i \in S} D_i$.

- For any $S \subseteq N$, we denote $0_S$ for the zero-vector in $\mathbb{R}^S$. Moreover, the vector operator $x \geq y$ implies $x \geq y$ componentwise and $x \neq y$.

### 2.2 Properties of risk redistributions

As argued before, we consider the case where firms can bargain for any risk redistribution that leads to feasible posterior risk profiles. When cooperation between the firms is not mandated, however, the firms will only agree to a particular redistribution if that redistribution satisfies some properties. If firms act rationally, they will only agree to a particular redistribution if the redistribution does not make them worse off in expected utility terms. Moreover, all firms have incentive to not engage in a particular redistribution if there exists another feasible redistribution that makes each firm weakly better off, and at least one firm strictly better off. To formally define these properties, we first introduce some notation.

For any set of firms $S \subseteq N$, we therefore denote $\mathcal{F}(S)$ for the set of vectors of posterior risk profiles that the firms in $S$ can reach if they redistribute their risks amongst each other, i.e.,

\[
\mathcal{F}(S) = \left\{ (X_{i \text{post}})_{i \in S} \in D(S) : \sum_{i \in S} X_{i \text{post}} = \sum_{i \in S} X_i \right\}. \quad (3)
\]
Moreover, for any subset $S \subseteq N$, we denote $\mathcal{NI}(S)$ for the set of posterior risk profiles $(X_{i}^{\text{post}})_{i \in S} \in \mathcal{D}(S)$ for the firms in $S$ such that there does not exist another feasible redistribution that yields weakly higher expected utility gains for each firm, and a strictly higher expected utility gain for at least one firm. Formally,

$$\mathcal{NI}(S) = \left\{ (X_{i}^{\text{post}})_{i \in S} : \exists i \in S \text{ s.t. } (\Delta U_i(X_{i}^{\text{post}}))_{i \in S} \geq (\Delta U_i(X_{i}^{\text{post}}))_{i \in S} \right\}. \ (4)$$

If firms act rationally, they will only agree to redistributions that lead to posterior risk profiles that satisfy the following two properties:

(1) \textbf{Individual Rationality:} a vector of feasible posterior risk profiles $(X_{i}^{\text{post}})_{i \in N} \in \mathcal{F}(N)$ satisfies Individual Rationality if and only if each firm weakly benefits in expected utility terms, i.e., if $\Delta U_i(X_{i}^{\text{post}}) \geq 0$ for all $i \in N$.

(2) \textbf{No Pareto Improvement:} a vector of feasible posterior risk profiles $(X_{i}^{\text{post}})_{i \in N} \in \mathcal{F}(N)$ satisfies No Pareto Improvement if and only if there does not exist another vector of feasible posterior risk profiles that yields weakly higher expected utility gains for all firms, and a strictly higher expected utility gain for at least one firm. Formally, $(X_{i}^{\text{post}})_{i \in N}$ satisfies No Pareto Improvement if

$$(X_{i}^{\text{post}})_{i \in N} \in \mathcal{NI}(N). \ (5)$$

Clearly, no firm would be willing to accept a redistribution that yields a posterior risk profile that does not satisfy Individual Rationality. Moreover, all firms have an incentive to choose a redistribution that satisfies No Pareto Improvement. Together, these conditions guarantee that no firm is better off if it does not engage in the redistribution (and, hence, keeps its prior risk profile), and that the firms collectively cannot reach an alternative distribution that makes at least one firm better off without harming the other firms. This, however, does not rule out the possibility that a subset of firms could be better off if they decide to redistribute risk amongst each other, excluding the other firms from the negotiation. Allowing more firms to cooperate in the redistribution has the potential advantage that the set of posterior risk profiles that a firm can reach increases. However, it may have the drawback that each firm negotiates with a larger number of other firms who each want to benefit from the redistribution. It is therefore not a priori clear that firms cannot do better by excluding some firms from the negotiation. Therefore, we consider the following condition:

(3) \textbf{Stability:} a vector of feasible posterior risk profiles $(X_{i}^{\text{post}})_{i \in N} \in \mathcal{F}(N)$ satisfies Stability if and only if for any subset $S \subseteq N$ of firms, there does not exist a redistribution of risk that satisfies $\sum_{i \in S} X_{i}^{\text{post}} = \sum_{i \in S} X_i$, and that is weakly preferred by all firms in $S$ and strictly preferred by at least one firm in $S$. Formally, Stability is satisfied iff

$$(X_{i}^{\text{post}})_{i \in S} \in \mathcal{NI}(S), \text{ for all } S \subseteq N. \ (6)$$

If a redistribution leads to posterior risk profiles that do not satisfy Stability and firms act rationally, then they will not agree to that redistribution because there exists a subset of firms $S \subset N$ that can be better off when they exclude the other firms from the negotiation and redistribute their risk amongst each other. In game-theoretic terms, this stability condition implies that the reallocation is an element of the core of the corresponding risk redistribution game. The following result follows straightforwardly.
**Proposition 1** It holds that:

(i) *Stability* implies *No Pareto Improvement* and *Individual Rationality*.

(ii) If risk is redistributed between two firms, i.e., if \( |N| = 2 \), then *Stability* is satisfied if and only if *Individual Rationality* and *No Pareto Improvement* are satisfied.

In the next section, we characterize the set of posterior risk profiles that satisfy *No Pareto Improvement*, as well as the set of posterior risk profiles that satisfy the stronger *Stability* condition.

## 3 Pareto optimality and Stability

As discussed in the previous section, if firms act rationally, they will only agree to redistributions of risk if no subset of firms could be better off by excluding the other firms from the negotiation and instead redistributing the risk amongst each other. Proposition 1(ii) shows that a necessary condition for this *Stability* condition to be satisfied is that the redistribution satisfies *No Pareto Improvement*. In Section 3.1 we therefore first characterize the set of Pareto optimal redistributions. In Section 3.2 we show that, among the set of infinitely many redistributions that are Pareto optimal, there exist feasible risk redistributions that satisfy the stronger *Stability* condition. In general, however, the set of redistributions that satisfy *Stability* will not be single-valued, and the issue arises which redistribution is selected. In Section 3.3 we model the choice of a particular redistribution as the outcome of a bargaining process that weighs the benefits of all involved parties.

### 3.1 Pareto optimal risk redistributions

A vector of posterior risk profiles \((X^\text{post}_i)_{i \in N}\) is *Pareto optimal* if it is feasible, and satisfies *No Pareto Improvement*, i.e., the set of Pareto optimal posterior risk profiles is given by

\[
\mathcal{PO}(N) = \mathcal{F}(N) \cap \mathcal{NI}(N) = \left\{(X^\text{post}_i)_{i \in N} \in \mathcal{F}(N) : \exists (\tilde{X}^\text{post}_i)_{i \in N} \in \mathcal{F}(N) \text{ s.t. } (\Delta U_i(\tilde{X}^\text{post}_i))_{i \in N} \geq (\Delta U_i(X^\text{post}_i))_{i \in N} \right\}.
\]  

(7)

All firms have an incentive to choose a redistribution in the set of Pareto optimal redistributions. If a redistribution does not satisfy Pareto optimality, there exists another vector of feasible posterior risk profiles that yields weakly higher expected utility gains for all firms, and a strictly higher expected utility gain for at least one firm.

Borch (1962) characterizes Pareto optimal risk reallocations and shows that a vector of feasible posterior risk profiles is Pareto optimal if and only if there exists a vector \(k \in \mathbb{R}^N_{++}\), such that \((X^\text{post}_i)_{i \in N}\) maximizes the weighted sum of the expected utility gains:

\[
(X^\text{post}_i)_{i \in N} \in \argmax \sum_{i \in N} k_i \Delta U_i(\tilde{X}^\text{post}_i) \quad \text{s.t.} \quad (X^\text{post}_i)_{i \in N} \in \mathcal{F}(N)
\]  

(8)

\(^4\)Borch (1962) considers utility levels. We consider expected utility gains, i.e., the difference between the expected utility of the posterior risk profile and the prior risk profile. This, however, does not affect the result.
Thus, the set of Pareto optimal posterior risk profiles can be determined by solving optimization problem (8) for all strictly positive vectors $k \in \mathbb{R}^N_+$. Applying this result to our setting yields the following result.

**Theorem 2** It holds that:

(i) $(X_{i\text{post}})_{i \in N} \in \mathcal{PO}(N)$ if and only if there exists a $k \in \mathbb{R}^N_+$ such that

$$k_i u_i'(X_{i\text{post}}(\omega)) p_i(\omega) = k_j u_j'(X_{j\text{post}}(\omega)) p_j(\omega), \quad \text{for all } \omega \in \Omega, i, j \in N, \quad (9)$$

and

$$\sum_{i \in N} X_{i\text{post}}(\omega) = \sum_{i \in N} X_i(\omega), \quad \text{for all } \omega \in \Omega, \quad (10)$$

where $X_{i\text{post}}(\omega)$ denotes the realization of $X_{i\text{post}}$ in state $\omega \in \Omega$.

(ii) For every $k \in \mathbb{R}^N_+$, there exists a unique solution to the system of equations (9) and (10).

The above theorem shows that the set of Pareto optimal posterior risk profiles can be found by solving the system of equations (9) and (10) for every $k \in \mathbb{R}^N_+$. Note that without loss of generality, we can impose as normalization that $k_i = 1$ for some $i \in N$.

The proposition shows the effect of heterogeneous beliefs regarding the probability distribution over the states of the world. For a given $k \in \mathbb{R}^N_+$, the corresponding Pareto optimal posterior risk profile for Firm $i \in N$ in state $\omega \in \Omega$ is increasing in $p_i(\{\omega\})$ and decreasing in $p_j(\{\omega\})$ for $j \neq i$. The reason is that if a firm overestimates a probability of a future state relative to other firms, this firm overvalues the outcome in this state and, so, it is Pareto optimal to assign a higher pay-off to that firm in this state. If all probability measures coincide, the effects cancel out as in Borch (1962).

Heterogeneity regarding the subjective probability distributions also has non-trivial effects on the structure of Pareto optimal redistributions. Gerber and Pafumi (1998) show that when firms have homogeneous beliefs, and use either exponential utility functions, or a power utility function with the same risk aversion parameter for all firms, the Pareto optimal posterior risk profiles are of the form

$$X_{i\text{post}}(\omega) = d_i + \delta_i \sum_{j \in N} X_j(\omega), \quad \text{for all } \omega \in \Omega, \text{ and } i \in N, \quad (11)$$

where $\delta_i \in [0, 1], i \in N$ are non-negative fractions satisfying $\sum_{i \in N} \delta_i = 1$, and $d_i \in \mathbb{R}, i \in N$ are side-payments satisfying $\sum_{i \in N} d_i = 0$. Thus, in these cases, the Pareto optimal risk redistributions consist of a proportional redistribution of risk, and deterministic side-payments $d_i \in \mathbb{R}$. In the next example we show the effect of heterogeneous beliefs on these Pareto optimal posterior risk profiles.

**Example 3** We consider Pareto optimal risk redistributions for the case where firms use either exponential utility functions, power utility functions or quadratic utility functions, allowing for heterogeneous beliefs regarding the probability distribution over the states of the world. We consider Pareto optimal redistributions for all firms $i \in N$. 


First, consider the case where the firms use exponential utility functions, i.e., the utility function of firm \(i \in N\) is given by

\[
u_i(x) = -\frac{1}{\lambda_i} \exp(-\lambda_i x), \quad \text{for all } x \in \mathbb{R},
\]

where \(\lambda_i > 0\) denotes the degree of risk aversion of firm \(i\). Now let \(\lambda = \left(\sum_{i \in N} \frac{1}{\lambda_i}\right)^{-1}\). Solving (9) and (10) yields that the Pareto optimal posterior risk profile of firm \(i \in N\) corresponding to \(k \in \mathbb{R}^{N+}_{++}\) is given by

\[
X_{i}^{\text{post}}(\omega) = \frac{\lambda}{\lambda_i} \sum_{j \in N} X_j(\omega) + \frac{\log(P_i(\omega))}{\lambda_i} - \frac{\lambda}{\lambda_i} \sum_{j \in N} \frac{\log(P_j(\omega))}{\lambda_j} + \frac{\log(k_i)}{\lambda_i} - \frac{\lambda}{\lambda_i} \sum_{j \in N} \frac{\log(k_j)}{\lambda_j}, \quad \text{for all } \omega \in \Omega.
\]

When firms have homogeneous beliefs regarding the probability distribution, i.e., when \(P_1 = P_2 = ... = P_n\), the second and the third term vanish, and the redistribution of risk is proportional, i.e., firm \(i\) is allocated a fraction \(\delta_i = \frac{\lambda_i}{\lambda}\) of the aggregate risk. The last two terms reflect deterministic side payments \(d_i\) satisfying \(\sum_{i \in N} d_i = 0\). When firms have heterogeneous beliefs regarding the probability distribution, the redistribution is no longer proportional. In addition to the fraction \(\delta_i\) of the aggregate risk, firm \(i\) is now also assigned the risk \(Y_i\) given by

\[
Y_i(\omega) = \frac{\log(P_i(\omega))}{\lambda_i} - \frac{\lambda_i}{\lambda} \sum_{j \in N} \frac{\log(P_j(\omega))}{\lambda_j}, \quad \text{for all } \omega \in \Omega.
\]

Next, we consider the case where each firm \(i \in N\) has a risk profile that takes on non-negative values only, i.e., \(D_i = \mathbb{R}^+\), and uses the constant relative risk aversion (CRRA or power) utility function given by

\[
u_i(x) = \frac{x^{1-\gamma}}{1-\gamma}, \quad \text{for all } x \in \mathbb{R}^+.
\]

where \(\gamma \in \mathbb{R}^+_+ \backslash \{1\}\) denotes the parameter of risk aversion. Solving (9) and (10) yields that the Pareto optimal posterior risk profile of firm \(i \in N\) corresponding to \(k \in \mathbb{R}^{N+}_{++}\) is given by

\[
X_{i}^{\text{post}}(\omega) = \frac{\sum_{j \in N} (k_j P_j(\omega))^{1/\gamma}}{\sum_{j \in N} (k_j P_j(\omega))^{1/\gamma}} \left( \sum_{j \in N} X_j(\omega) \right), \quad \text{for all } \omega \in \Omega.
\]

In this case, the Pareto optimal risk redistribution does not involve side payments. The redistribution of risk, however, is proportional only if \(P_1 = P_2 = ... = P_n\).

Finally, consider the case where the risk profile of firm \(i \in N\) takes values in \(D_i = (-\infty, \frac{1}{\alpha_i}]\) for \(\alpha_i > 0\), and firm \(i\) uses a quadratic utility function given by

\[
u_i(x) = x - \frac{\alpha_i}{2} x^2, \quad \text{for all } x \in \left(-\infty, \frac{1}{\alpha_i}\right).
\]

Let \(\alpha = \left(\sum_{i \in N} \frac{1}{\alpha_i}\right)^{-1}\), \(\lambda_i(\omega) = k_i P_i(\omega) \alpha_i\) and \(\lambda(\omega) = \left(\sum_{i \in N} \frac{1}{\alpha_i(\omega)}\right)^{-1}\) for all \(i \in N\) and \(\omega \in \Omega\). Solving (9) and (10) yields that the Pareto optimal posterior risk profile of firm \(i \in N\) corresponding to \(k \in \mathbb{R}^{N+}_{++}\) is given by

\[
X_{i}^{\text{post}}(\omega) = \frac{\lambda(\omega)}{\lambda_i(\omega)} \left( \sum_{j \in N} X_j(\omega) - \frac{1}{\alpha}\right) + \frac{1}{\alpha_i}, \quad \text{for all } \omega \in \Omega.
\]
As was the case when firms use exponential utility functions, the optimal redistribution involves deterministic side-payments, and the redistribution of risk is proportional only if \( P_1 = P_2 = \ldots = P_n \).

The above example illustrates the effects of heterogeneity regarding the subjective probability distributions on the Pareto optimal redistributions. Heterogeneity implies that the redistribution is unlikely to be proportional. This has some interesting implications. For example, whereas with homogeneous beliefs (i.e., when \( P_1 = \ldots = P_n \)), all Pareto optimal posterior risk profiles are risk-free for each firm if and only if \( \sum_{i \in N} X_i \) is risk-free, this is no longer the case when firms have heterogeneous beliefs. Even when pooling all the risk profiles allows to eliminate all risk (i.e., \( \sum_{i \in N} X_i \) is risk-free), different beliefs regarding the likelihood of the different states of the world imply that the Pareto optimal posterior risk profiles are not riskless for all firms.

Moreover, heterogeneous beliefs may also increase the likelihood that firms (believe that they) can benefit from redistributing their risks. Whereas clearly \( \sum_{i \in N} E_{P_i}[X_i] = \sum_{i \in N} E_{P_1}[X_i] \) for any vector of feasible posterior risk profiles when firms have homogeneous beliefs, it is possible that there exist feasible posterior risk profiles satisfying \( \sum_{i \in N} E_{P_i}[X_i^{\text{post}}] > \sum_{i \in N} E_{P_i}[X_i] \) in case of heterogeneous beliefs. Thus, heterogeneous beliefs may imply that all firms believe that they can simultaneously gain in expectation. This suggests that heterogeneous beliefs might make redistribution of risk even more attractive.

In the next proposition, we show that it is unlikely that there is no room to benefit for the firms.

**Proposition 4** There does not exist an \((X_i^{\text{post}})_{i \in N} \in \mathcal{P}(N)\) with \( \Delta U_i(X_i^{\text{post}}) \geq 0 \) for all \( i \in N \) and \( \Delta U_i(X_i^{\text{post}}) > 0 \) for at least one \( i \in N \) if and only if for all \( j \in N \setminus \{1\} \),
\[
\frac{u_i(X_1(\omega))P_i(\omega)}{u_j(X_j(\omega))P_j(\omega)} \text{ does not depend on } \omega \in \Omega.
\]

The above proposition yields a necessary and sufficient condition for the existence of a Pareto optimal risk redistribution that weakly benefits all firms and strictly benefits at least one firm. The proposition shows that heterogeneous beliefs regarding the underlying probability distribution make it more likely that such redistributions exist. For example, whereas with homogeneous beliefs, improvement cannot be obtained when the firms have the same prior risk profile and the same risk preferences (because this implies that \( \frac{u_i(X_1(\omega))P_i(\omega)}{u_j(X_j(\omega))P_j(\omega)} = 1 \) for all \( \omega \)), improvements can be achieved when the firms have heterogeneous beliefs. Hence, only in very special cases, there is no room for improvement.

### 3.2 Stable risk redistributions

For all firms to be willing to engage in a particular redistribution, the corresponding posterior risk profiles need to satisfy Stability. We therefore introduce the following definition.

**Definition 5** \( S(N) \) denotes the set of all feasible posterior risk profiles that satisfy Stability, i.e.,
\[
S(N) = \left\{ (X_i^{\text{post}})_{i \in N} \in \mathcal{F}(N) : (X_i^{\text{post}})_{i \in S} \in \mathcal{N}(S) \text{ for all } S \subseteq N \right\}.
\]

We know from Proposition 1(ii) that a necessary condition for Stability to be satisfied is that the redistribution satisfies No Pareto Improvement and Individual Rationality. Moreover,
when risk is redistributed between two firms, these conditions are also sufficient. Therefore, it holds that
\[
S(N) \subset \{ (X_{i}^{\text{post}})_{i \in N} \in \mathcal{PO}(N) : (\Delta U_i(X_{i}^{\text{post}}))_{i \in S} \geq 0 \},
\]
where the inclusion is an equality when risk is redistributed between two firms. When risk is redistributed between more than two firms, No Pareto Improvement and Individual Rationality in general is not sufficient to guarantee Stability, which implies that the inclusion in (19) is strict.

Theorem 2 shows that there exist infinitely many Pareto optimal posterior risk profiles, i.e., the set \( \mathcal{PO}(N) \) contains infinitely many posterior risk profiles. The following theorem shows that the subset of posterior risk profiles that in addition satisfy Stability is non-empty.

\textbf{Theorem 6} The set \( S(N) \) is non-empty, i.e., there exist feasible posterior risk profiles that satisfy Stability.

3.3 The bargaining problem

The previous section shows that there exist risk redistributions that benefit all firms in the sense that each firm weakly gains from the redistribution in expected utility terms, and no subset of firms \( S \subseteq N \) can be better off when they by exclude the other firms from the negotiation and redistribute their risk amongst each other. In general, however, the set of redistributions that satisfy these criteria will not be single-valued, and the issue arises which redistribution is selected. In each redistribution, all firms weakly benefit, but the extent to which a particular firm benefits will depend on the particular redistribution that is chosen. In general, the redistribution that maximizes the expected utility gain for a particular firm will not maximize the expected utility gain of another involved firm. This implies that the firms will have to bargain over the redistribution that they choose. The selection of a particular redistribution reflects a bargaining process that can be modeled via a bargaining rule (Nash, 1950).

We first consider the case in which risk is redistributed between two parties, i.e., \( |N| = 2 \). Recall from Proposition 1 that with two firms, Stability is equivalent to Pareto Optimality and Individual Rationality. Hence, the set of potential risk redistributions that the firms will bargain over is the set of redistributions that satisfy Pareto Optimality and Individual Rationality. In this case, there exists a unique bargaining rule that satisfies some desirable properties. This rule is the Nash bargaining solution (Nash, 1950; Kalai, 1977). For the risk redistribution problem that we consider, the Nash bargaining solution is given by
\[
\mathcal{NB} = \arg\max_{(X_{i}^{\text{post}})_{i \in N} \in \mathcal{F}(N) \cap \mathcal{IR}(N)} \prod_{i \in N} \Delta U_i(X_{i}^{\text{post}}),
\]
(20)
The objective function in (20) weighs the benefits of the involved parties. By construction, it yields redistributions that satisfy Individual Rationality, i.e., \( \mathcal{NB} \subset \mathcal{IR}(N) \). Moreover, while the constraint set allows any redistribution that satisfies Feasibility and Individual Rationality, it is readily be verified that the Nash bargaining solution in (20) yields redistributions that satisfy Pareto Efficiency, i.e., \( \mathcal{NB} \subset \mathcal{PO}(N) \). Moreover, under certain regularity conditions, the Nash bargaining solution is single-valued (Nash, 1950). The following proposition shows that in our case these regularity conditions are satisfied.

\textbf{Proposition 7} If \( |N| = 2 \), then there exists a unique \( (X_{i}^{\text{post}, \mathcal{NB}})_{i \in N} \in S(N) \) such that
\[
\mathcal{NB} = \{ (X_{i}^{\text{post}, \mathcal{NB}})_{i \in N} \}.
\]
(21)
Now we consider the case where risk is redistributed between more than two firms, i.e., $|N| > 2$. In this case, \textit{Stability} in general is a stronger condition than \textit{Pareto Optimality} and \textit{Individual Rationality}. Therefore, in order to reflect the fact that firms will only agree to a redistribution if it satisfies \textit{Stability}, we consider instead the constrained Nash bargaining solution, which is given by:

$$
\text{CNB} = \arg\max_{(X_{i}^{\text{post}})_{i \in N} \in S(N)} \prod_{i \in N} \Delta U_i(X_{i}^{\text{post}}).
$$

(22)

This rule is called the \textit{coalitional Nash bargaining solution} (Compte and Jehiel, 2010) in case of Transferable Utility games. The following proposition shows that there exists a solution to the optimization problem in (22). However, in contrast to the case where $|N| = 2$, the constrained Nash bargaining solution need not be single-valued when $|N| > 2$.

**Proposition 8** There exists a $(X_{i}^{\text{post,CNB}})_{i \in N} \in S(N)$ such that

$$(X_{i}^{\text{post,CNB}})_{i \in N} \in \text{CNB}.$$  

(23)

4 Redistributing longevity risk

In this section we use the model developed in the previous section to investigate the extent to which pension funds and life insurance companies can benefit from redistributing their risks. Pension funds and life insurance companies face longevity risk, which is the risk due to uncertain changes over time in survival rates of the insured population. There is substantial uncertainty regarding the future development of survival rates (see, e.g., Pitacco et al., 2009), and this uncertainty imposes significant risk on pension funds and life insurance companies. Coughlan et al. (2007), for example, show that on average every additional year of life expectancy adds approximately 3% to 4% to the value of UK pension liabilities. In the Netherlands, unexpectedly high deviations between best estimate mortality trends estimated in 2010 and those estimated in 2007 led to increases in the net premium for old-age pension annuities of up to 12%. This illustrates the potentially huge impact of unanticipated increases in survival rates. A potential way to mitigate these adverse effects of longevity risk on the liabilities of pension funds is to exploit the natural hedge potential that arises from combining life annuities and death benefit insurance.

In Section 4.1, we model the risk profiles of pension funds offering whole life annuities, and life insurance companies offering death benefit insurance. In Section 4.2, we numerically illustrate the extent to which a pension fund and a death benefit insurer can benefit from redistributing their risks. It is often argued that major obstacles to trade are potential disagreement regarding the true distribution of future mortality rates, reluctance to engage in contracts with long horizons, and insufficient capacity in the insurance market to yield significant risk reduction for pension funds. In order to investigate these issues, we focus our numerical analysis on the effects of the time horizon $T$, the relative size of the pension fund and the death benefit insurer, and potential heterogeneity in beliefs regarding the true probability distribution of future survival rates.

4.1 The risk profiles

We consider the case where the risk profile of pension funds and insurers is the net value of their assets and liabilities, referred to as the \textit{net asset value}, at a prespecified future date $T$.  

11
Thus, the prior risk profile of firm $i$ is given by
\[ X_i(T) = A_i(T) - L_i(T), \quad (24) \]
where $A_i(T)$ denotes the (market) value of the assets at time $T$, $L_i(T)$ denotes the date-$T$ value of the liabilities.

In order to focus on longevity risk, we assume a deterministic risk-free rate (see, e.g., Olivieri, 2001; Brouhns et al., 2002; Cossette et al., 2007), which we denote $r$. The asset value on date $T$ then follows from
\[ A_i(t) = (1 + r) A_i(t-1) - L_{i,t}, \quad \text{for all } t = 1, \ldots, T, \quad (25) \]
where $L_{i,t}$ denotes the (stochastic) liability payment of firm $i$ at date $t$. Combining (24) and (25) yields
\[ X_i(T) = [A_i(0) - CL_i(T)] \cdot (1 + r)^T, \quad (26) \]
where
\[ CL_i(T) = \sum_{\tau=1}^T \frac{\tilde{L}_{i,\tau}}{(1+r)^\tau} + \frac{L_i(T)}{(1+r)^T}. \quad (27) \]

Thus, firm $i$’s risk profile equals the initial asset value $A_i(0)$ increased by the return on assets, and reduced by the random variable $CL_i(T)$, where $CL_i(T)$ represents the sum of the present value of liability payments up to year $T$, and the present value of the date-$T$ best estimate value of all payments beyond date $T$.

It now remains to specify how the date-$T$ liability value $L_i(T)$ is determined. Ideally, $L_i(T)$ would represent the market value on date $T$ of the future liabilities, i.e., the value at which the liabilities can be sold to a third party. Because there is (not yet) a liquid market for longevity-linked products, however, there is not yet a market price, and pension funds and insurance companies have to value their liabilities using mark-to-model valuation instead.

We consider the case where the pension fund and the insurer value their liabilities at the best estimate value. The best estimate value of the liabilities is defined as the discounted expected value of all future claims. Recall that we allow firms to have heterogeneous beliefs regarding the probability distribution of future survival rates. Therefore, the best estimate liability value is given by
\[ L_i(T) = BEL_i(T) = E_{P_i} \left[ \sum_{\tau=1}^T \frac{\tilde{L}_{i,\tau}}{(1+r)^\tau} \mid F_T \right], \quad \text{for all } i \in N \text{ and } T \geq 0, \quad (28) \]
where $E_{P_i} \left[ \cdot \mid F_T \right]$ denotes the expectation with respect to firm $i$’s (subjective) probability distribution $P_i$, conditional on information available on date $T$, $F_T$ (i.e., information regarding mortality rates).

Alternatively, the value of the liabilities can include a market value margin. The market value margin can, for example, reflect the cost of holding a capital reserve. For a discussion of how the market value margin is formulated under Solvency II, see for example Stevens et al. (2011).

In some cases, regulators may prescribe the use of a specific probability distribution to determine the best estimate value of the liabilities for regulatory purposes. Then, depending on whether the firm’s objective is to reduce the volatility of the Net Asset Value with liability value as prescribed by the regulator, or the Net Asset Value with liability value determined according to its own probability distribution, the firm would use either the exogenously given probability distribution $\bar{P}$, or its own subjective probability distribution $(P_i)$ to value the liabilities.
On date zero, the prior risk profile \( X_i(T) \) is uncertain due to uncertainty in the liability payments \( L_{i,\tau} \), for \( \tau = 1, \ldots, T \), as well as due to uncertainty in the value of the remaining liabilities on date \( T \), \( L_i(T) \). In Subsection 4.2.1, we discuss the characteristics of the liabilities.

Via redistribution of risk, pension funds and insurers want to arrive at feasible posterior risk profiles \( (X^\text{post}_i(T))_{i \in N} \) such that they all benefit weakly in expected utility terms. For any given posterior risk profile, however, there exist infinitely many ways in which the firms can achieve the corresponding risk redistribution. One way that we consider in particular is redistribution of risk via a swap contract in which a single payment occurs on maturity date \( T \). Specifically, for any given posterior risk profiles \( (X^\text{post}_i(T))_{i \in N} \), we let \( (\chi_i(T))_{i \in N} \) be given by

\[
\chi_i(T) = X_i(T) - X^\text{post}_i(T), \quad \text{for } i \in N,
\]

i.e., \( \chi_i(T) \) reflects the net payment from firm \( i \) to the other firms on date \( T \). Then, the posterior risk profiles can be written as

\[
X^\text{post}_i(T) = [A_i(0) - CL^\text{post}_i(T)] \cdot (1 + r)^T, \quad \text{for } i \in N.
\]

where

\[
CL^\text{post}_i(T) = \sum_{\tau=1}^{T} \frac{L_{i,\tau}}{(1 + r)^\tau} + \frac{\chi_i(T) + L_i(T)}{(1 + r)^T}, \quad \text{for } i \in N.
\]

Thus, \( \chi_i(T) \) can be seen as the payoff of a swap contract with fixed maturity \( t = T \) and a single payment on date \( t = T \). Clearly, it follows from (1) and (29) that

\[
\sum_{i \in N} \chi_i(T) = 0.
\]

A special case of our model that is considered often in the literature is the case where \( T = T^{\text{max}} \), with \( T^{\text{max}} \) large enough so that the probability that all participants are deceased is 1. Then, \( L_i(T^{\text{max}}) = 0 \), and so \( CL_i(T^{\text{max}}) \) is equal to

\[
CL_i(T^{\text{max}}) = \sum_{\tau=1}^{T^{\text{max}}} \frac{L_{i,\tau}}{(1 + r)^\tau}, \quad \text{for } i \in N,
\]

i.e., it equals the date-zero present value of all future liabilities over complete run-off. This is the case which is typically considered in the literature (see, e.g., Tsai et al., 2010; Wang et al, 2010). A potential drawback of this approach is that firms agree on a redistribution of risk over complete run-off. Considering instead a shorter horizon, as we do, allows for the possibility to renegotiate. For example, firms can re-evaluate their liabilities according to new mortality data, new regulations, attrition, and new participants. So, at the future date, a new contract can be negotiated.

### 4.2 Benefits from risk redistributions

In this section we numerically illustrate the potential benefits from redistributions of longevity risk between a pension fund and a death benefit insurer. In Subsection 4.2.1, we first specify

\footnote{It is also possible to transfer longevity-linked products at time \( T \) or via swaps with maturity \( T \) and periodic pay-offs.}
the risk profiles, risk preferences, and subjective probability distributions of the pension fund and the death benefit insurer. Then, in Subsection 4.2.2 we numerically illustrate the benefits from a redistribution of risk that reflects the outcome of the bargaining process according to the Nash bargaining solution. We focus on the effects of the time horizon $T$, the relative size of the pension fund and the death benefit insurer, and potential heterogeneity in beliefs regarding the true probability distribution of future survival rates.

4.2.1 Prior risk profiles, subjective probabilities, and risk preferences

The extent to which insurers and pension funds will be able to benefit from redistributing their risks depends on the characteristics of their liabilities (i.e., the liability payments $L_i$, their risk preferences, and their (subjective) beliefs regarding the probability distribution on the underlying state space.

We start by discussing the characteristics of the liabilities. For the pension fund, $L_i$ is a random variable that equals the aggregate annuity payment in year $\tau$ to all participants who are alive and older than 65 in that year. For the insurer, $L_i$ is a random variable that equals the aggregate death benefit payment in year $\tau$ to all participants who died in that year and were younger than 65. The level of these payments is affected by two types of mortality risk. First, the payments are subject to longevity risk, which is the risk that arises due to the fact that the survival rates in a given population change over time in an unpredictable way. In addition, the liabilities are subject to individual mortality risk which arises due to the fact that, conditional on given survival rates in the population, whether a particular individual survives an additional year is uncertain. Individual mortality risk, however, becomes negligible when portfolio size is large (see, for example, Olivieri, 2001). Given the large portfolio sizes that we consider, we will assume that individual mortality risk is negligible, and focus on the impact of longevity risk. Formally, for sufficiently large portfolios, the aggregate portfolio payment $L_i$ of firm $i$ in year $\tau$ can be approximated by

$$L_i = \sum_{j \in M_i} \delta_{i,j} \cdot \tau p_{x_{i,j},0} \cdot 1_{x_{i,j}+\tau \geq 65},$$

respectively, where

- $M_i$ denotes the set of insureds of firm $i \in N$,
- $\delta_{i,j} \in \mathbb{R}_+$, and $x_{i,j} \in \mathbb{N}$ denote the insured right and the age, respectively, of insured $j \in M_i$ of firm $i \in N$;
- $z p_{x,0}$ denotes the future probability that an individual belonging to the cohort aged $x$ in year $t = 0$ will survive at least $z$ more years, i.e., $z p_{x,0} = p_{x,0} \cdot p_{x+1} \cdot \ldots \cdot p_{x+z-1}$, where $p_{x+s}$ for $s \geq 0$ denotes the future probability that an individual aged $x + s$ in year $s$ will survive at least one more year.

Details regarding the number of insureds and their insured rights are provided in Appendix B.

Longevity risk arises from the fact that the future one-year probabilities $p_{x+s}$ for $s \geq 0$ are unknown on date zero. Longevity risk affects the date-$T$ net asset value of the firms through its effect on the probability distribution of $(CL_i(T))_{i \in N}$. It follows from (27) and (28) that

$$CL_i(T) = \sum_{\tau=1}^{T} \frac{L_i}{(1 + r)^\tau} + \mathbb{E}_{\mathcal{F}_T} \left[ \sum_{\tau>T} \frac{L_i}{(1 + r)^\tau} \bigg| \mathcal{F}_T \right],$$

respectively, where

- $\mathcal{F}_T$ denotes the sigma-algebra generated by the information available up to date $T$.

14
where $F_T$ denotes the set of one-year survival probabilities up to year $T$, i.e., $F_T = \{p_{x,s} : s \leq T, x \in \mathbb{N}\}$.

We distinguish the case where the pension fund and the insurer have homogeneous beliefs regarding the underlying probability distribution and the case where they have heterogeneous beliefs. Specifically, we assume that both the pension fund and the death benefit insurer use the standard Lee and Carter (1992) model to estimate the probability distribution of future mortality rates (see Appendix C), but they may disagree on the appropriate historical time period that is used to estimate the model parameters. For the case of homogeneous beliefs, they each estimate the model parameters based on data for Dutch males as reported in the Human Mortality Database for the period 1977 to 2009. For the case of heterogeneous beliefs, however, the death benefit insurer uses data from the shorter time period from 1987 until 2009. We will refer to these models as LC(1977-2009) and LC(1987-2009), respectively. We display the corresponding parameter estimates in Appendix C.

Figure 1 displays the 95% confidence interval of $CL_i(T)$ as percentage deviation from its expected value, i.e., $(CL_i(T) - E[CL_i(T)]) / E[CL_i(T)] \cdot 100\%$, as a function of the horizon $T$. The left panel corresponds to the case where the probability distribution of future mortality rates is estimated based on LC(1977-2009). The right panel corresponds to the case where the probability distribution of future mortality rates is estimated based on LC(1987-2009).

As can be seen from Figure 1, the size of the 95% confidence interval is monotonically increasing in $T$. The size of the confidence interval for $T = 10$ is not much larger than the size of the confidence interval for $T = 1$ for both the pension fund and the death benefit insurer. A further increase in the length of the horizon has a relatively small effect on the uncertainty. Comparison of the left panel and the right panel shows that the use of a shorter
historical period to estimate the parameters of the Lee and Carter model leads to more risky risk profiles for the pension fund and the death benefit insurer, for all time horizons. For example, for $T = 1$, the 95% confidence interval for the death benefit insurer is about 2 times bigger than when the longer historical data period is used to estimate the model parameters. The same pattern is observed if one considers other risk measures such as, for example, the relative standard deviation of $CL_i(T)$. Detailed summary statistics of $CL_i(T)$ are displayed in Table 3 in Appendix D.

An important characteristic of the joint distribution of the net asset values of the pension fund and the death benefit insurer is that, for all horizons $T$, they exhibit relatively strong negative correlation. Figure 2 illustrates the strong negative correlation between $CL_1(T)$ and $CL_2(T)$ for the case where $T = 1$, and the probability distribution of mortality is estimated based on LC(1977-2009).

Table 4 in Appendix D shows that for horizons $T \in \{1, 5, 10, T_{\text{max}}\}$, the correlation between $CL_1(T)$ and $CL_2(T)$ ranges from $-0.91$ to $-0.97$ for LC(1977-2009), and from $-0.49$ to $-0.97$ for LC(1987-2009). This strong negative correlation suggests that the pension fund and the death benefit insurer could potentially benefit significantly from natural hedge potential that arises from redistributing the risks.

The extent to which the pension fund and the death benefit insurer can benefit from redistributing their risks also depends on their risk preferences. The pension fund and the death benefit insurer each use an exponential utility function (see (12)) with risk aversion parameter

$$
\lambda_i = \begin{cases} 
10^{-3} & \text{for the pension fund}, \\
5 \times 10^{-4} & \text{for the two death benefit insurer}.
\end{cases}
$$

(35)

These levels of risk aversion imply that the maximum premium that the pension fund would be willing to pay for a full buy-out of the liabilities equals 104.5% of the best estimate value.
of the liabilities. For the death benefit insurer, the corresponding maximum premium equals 144.3% of the best estimate value.\(^8\)

### 4.2.2 Benefits from redistribution

The negative correlation between the risk profiles of the pension fund and the death benefit insurer suggests strong potential for hedge benefit for all parties. In this subsection, we numerically illustrate the extent to which the pension fund and one death benefit insurer can benefit from this hedge potential by redistributing their risks amongst each other. Given their prior risk profiles from (26), we determine the set of Pareto optimal posterior risk profiles from (9) and (10). We then use (21) to select the particular Pareto optimal posterior risk profiles that reflect the outcome of a bargaining process that equally weights the benefits of both parties, i.e., the Nash bargaining solution. We note that because the pension fund and the insurer use an exponential utility function, the set of Pareto optimal posterior risk profiles is independent of the initial asset values \(A_i(0)\). Hence, the Nash bargaining solution as given in (21) is also independent of \(A_i(0)\).

We first investigate the effect of heterogeneous beliefs on the benefits from redistribution for the case where \(T = 1\). In Figure 3, we display the probability distributions of \(CL_i(1)\) (grey histograms) and the probability distributions of \(CL_i^{\text{post}}(1)\) (black histograms) for the pension fund (left figure) and for the death benefit insurer (right figure), expressed as percentage deviation from their date-zero expected values. The upper (lower) panels correspond to the case of homogeneous (heterogeneous) beliefs.

The upper panel shows that in case of homogeneous beliefs, the risk redistribution implies that the Net Asset Value of both the pension fund and the insurer becomes significantly less dispersed. Comparing the upper and the lower panel shows that, after redistribution, the probability that the death benefit insurer faces payments lower than the date-zero best estimate value is significantly higher in the case of heterogeneous beliefs. The reason is that the death benefit insurer assigns relatively lower probabilities to states of the world with less extreme outcomes for the aggregate risk, and relatively higher probabilities to states of the world with more extreme outcomes for the aggregate risk. The pension fund assigns almost no probability to these more extreme outcomes. Both parties therefore benefit if payments is scenarios with more extreme outcomes for the aggregate risk are allocated to the pension fund.

To quantify the benefits from the redistribution, we consider the following two criteria:

1. **The percentage decrease in the date-zero best estimate value of the liabilities.** We denote \(BEL_i\) for the date-zero expected value of the present value of all future payments of firm \(i\), given the prior risk profile, and \(BEL_i^{\text{post}}(T)\) for the date-zero expected present value of all future payments of firm \(i\), given the posterior risk profile. Then, we determine the percentage reduction in the best estimate value as

\[
\% \text{Red}BEL_i(T) = \frac{BEL_i - BEL_i^{\text{post}}(T)}{BEL_i}.
\]

---

\(^8\)If the Net Asset Value is divided equally over all participants (insureds), and if all participants (insureds) have the same absolute risk aversion, and if the expected utility of the pension fund (insurer) equals the expected utility of its participants (insureds), this implies an individual risk aversion parameter of approximately 50. Note that the monetary unit in our model equals the annual pension right. In case the annual pension right equals 25,000 dollar, this parameter corresponds with a risk aversion parameter of approximately 0.002 when the monetary unit is one dollar.
Figure 3: The grey (black) histograms represent the probability distributions of the prior (posterior) risk profiles of the pension fund (left figure) and the death benefit insurer (right figure), as percentage deviation from their corresponding best estimate values. The posterior distribution is given by (21). The upper (lower) panel corresponds to the case of homogeneous (heterogeneous) beliefs.

(ii) The relative zero-utility premium. Recall that the redistribution implies that firm $i$ effectively receives a net payment equal to $X_{i}^{\text{post}}(T) - X_i(T) = -\chi_i(T)$ on date $T$. The value to firm $i$ of the risk redistribution can therefore be quantified by determining the maximum premium that firm $i$ would have been willing to pay on date 0 for a contract that yields this net payment on date $T$. This maximum premium, which we denote $p_i(T) \in \mathbb{R}$, is the premium at which firm $i$ would be indifferent between buying the contract and not buying the contract, and is therefore referred to as the zero-utility premium. It is the unique solution of the following equation:

$$
\mathbb{E}_{\mathbb{P}_i} \left[ u_i(X_i(T) - \chi_i(T) - (1 + r)^T \cdot p_i(T)) \right] = \mathbb{E}_{\mathbb{P}_i} \left[ u_i(X_i(T)) \right], \text{ for } i \in N.
$$

We report the value of the zero-utility premium for firm $i$ relative to the date-zero best
estimate value of the firm \(i\)'s prior risk profile, i.e.,

\[
ZU_i(T) = \frac{p_i(T)}{BEL_i}, \quad \text{for } i \in N.
\]  

(38)

Table 1 summarizes the simulated gains from redistributing as a function of the time horizon \(T\). The upper (lower) panel corresponds to the case where the firms have homogeneous (heterogeneous) beliefs regarding the underlying probability distribution. In each case, the first two columns display the percentage reduction in the date-zero best estimate value of the liabilities, as defined in (36). The last two columns display the relative zero-utility premium corresponding to the risk distribution, as defined in (38).

<table>
<thead>
<tr>
<th>(T)</th>
<th>(%\text{RedBEL}_i(T))</th>
<th>(ZU_i(T))</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Pension fund</td>
<td>Insurer</td>
</tr>
<tr>
<td>1</td>
<td>1.7%</td>
<td>-8.2%</td>
</tr>
<tr>
<td>5</td>
<td>3.3%</td>
<td>-16.3%</td>
</tr>
<tr>
<td>10</td>
<td>3.7%</td>
<td>-18.5%</td>
</tr>
<tr>
<td>(T_{\text{max}})</td>
<td>5.1%</td>
<td>-25.6%</td>
</tr>
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<table>
<thead>
<tr>
<th>(T)</th>
<th>(%\text{RedBEL}_i(T))</th>
<th>(ZU_i(T))</th>
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<tbody>
<tr>
<td></td>
<td>Pension fund</td>
<td>Insurer</td>
</tr>
<tr>
<td>1</td>
<td>5.1%</td>
<td>3.1%</td>
</tr>
<tr>
<td>5</td>
<td>5.5%</td>
<td>4.5%</td>
</tr>
<tr>
<td>10</td>
<td>6.5%</td>
<td>5.7%</td>
</tr>
<tr>
<td>(T_{\text{max}})</td>
<td>6.6%</td>
<td>5.9%</td>
</tr>
</tbody>
</table>

Table 1: The simulated gains of the risk redistribution according to the Nash bargaining solution from (21), as a function of the horizon \(T\). Here, \(\%\text{RedBEL}_i(T)\) is defined in (36) and \(ZU_i(T)\) is defined in (38). The upper panel corresponds to the case where the firms have homogeneous beliefs, namely LC(1977-2009) and in the lower panel, the pension fund has again beliefs LC(1977-2009) while the insurer has beliefs LC(1987-2009).

First consider the case with homogeneous beliefs. Remember that the redistribution occurs according to a swap contract in which, in each state of the world, either the pension fund makes a net payment to the death benefit insurer on date \(T\), or the death benefit insurer makes a net payment to the pension fund on date \(T\). As can be seen from Table 1, the risk redistribution is such that in expectation, the death benefit insurer makes a net payment to the pension fund, i.e., the expected value of \(CL_1(T)\) decreases, and, hence, the expected value of \(CL_2(T)\) increases. However, even though the death benefit insurer loses in expectation and the pension fund gains in expectation, both the pension fund and the insurer benefit from the redistribution in expected utility terms. The relative zero-utility premium of the death benefit insurer is relatively high as compared to the relative zero-utility premium of the pension fund. For example, the relative zero-utility premium according to the risk redistribution with horizon \(T = 1\) equals 7.2% of the best-estimate value of the liabilities for the death benefit insurer, and 3.3% of the best-estimate value of the liabilities for the pension fund. This difference is due to the fact that the death benefit insurer has a significantly more risky prior risk profile, and hence benefits more from redistribution. As \(T\) increases, this effect becomes stronger. However, the benefits from choosing a horizon longer than ten years are relatively small. Given the potentially important drawbacks of choosing a long horizon,
this suggests that parties can benefit most from redistributing their risks over a relatively short horizon.

In contrast to the case with homogeneous beliefs, the percentage increase in the expected Net Asset Value is positive for both parties in case of heterogeneous beliefs. Thus, they both believe that through redistribution, they can increase their expected Net Asset Value at horizon date. Moreover, for both parties, the relative zero-utility premium is significantly higher than in the case with homogeneous beliefs regarding the underlying probability distribution. This is also what we expected because if a firm believes that a state has a relatively high probability to occur, this firm will have a relatively low posterior pay-off in this state (see, e.g., (13)).

4.2.3 Effect of relative size of insurer and pension fund

In this subsection we investigate the effect of the relative size of the pension fund and the death benefit insurer. We quantify the relative size via the ratio of the best-estimate value of their current liabilities, based on LC(1977-2009), i.e.,

\[
\gamma = \frac{E_{LC(1977-2009)} \left( \sum_{\tau > 0} \frac{L_{1,\tau}}{(1+r)^\tau} \right)}{E_{LC(1977-2009)} \left( \sum_{\tau > 0} \frac{L_{2,\tau}}{(1+r)^\tau} \right)}.
\]  

(39)

We consider various values of \( \gamma \in \left\{ \frac{k}{100} : k = 1, \ldots, 100 \right\} \) by adjusting the number of insureds of the death benefit insurer. All other characteristics of the portfolios are as given in Appendix B.

In Figure 4, we display the effect of \( \gamma \) on the benefits from redistribution for the case where \( T = 1 \). We focus on the benefits for the pension fund, and distinguish the case where the firms have homogeneous beliefs (solid lines) and the case where they have heterogeneous beliefs (dashed lines). The top panel displays the percentage reduction in the best-estimate value of the liabilities for the pension fund. The bottom panel displays the relative zero-utility premium corresponding to the optimal redistribution.

First consider the case of homogeneous beliefs (the solid lines). Figure 4 shows that whereas the percentage increase in the expected Net Asset Value of the pension fund is monotonically increasing in \( \gamma \), the relative zero-utility premium is concave and reaches a maximum approximately at \( \gamma = 0.3 \). This can be understood as follows. By construction, the Nash bargaining solution that we consider corresponds to a Pareto optimal redistribution of risk. As can be seen from (13) with \( \mathbb{P}_1 = \mathbb{P}_2 \), this implies that the aggregate risk is redistributed proportionally with deterministic side-payments. This in turn implies that all Pareto optimal posterior risk profiles for a given \( \gamma \) have the same variance because they only differ in the deterministic side-payments. Moreover, both for the pension fund and for the insurer, the variance of the posterior risk profile is a quadratic convex function of \( \gamma \) that reaches a global minimum.\(^9\) Therefore, there is a value of \( \gamma \) for which the hedge benefits are maximized. In

\[\text{var}(X_{i,\text{post}}) = \frac{\lambda^2}{N_i} \text{var}(X_1(T)+\gamma \cdot X_2(T)) = \frac{\lambda^2}{N_i} \text{var}(X_1(T)) + \frac{\lambda^2}{N_i} \gamma^2 \cdot \text{var}(X_2(T)) - \frac{\lambda^2}{N_i} \gamma \cdot \text{cov}(X_1(T), X_2(T)),\]

for all \( i \in N \) and \( (X_{i,\text{post}})_{i \in N} \in \mathcal{P}(\mathcal{O}(N)) \).

\(^9\) Let the prior risk profiles corresponding to \( \gamma = 1 \) be given by \( X_1(T) \) and \( X_2(T) \). Then, the prior risk profiles corresponding to a \( \gamma \in \{ \frac{k}{100} : k = 1, \ldots, 100 \} \) are given by \( X_1(T) \) and \( \gamma \cdot X_2(T) \), respectively. Then, it follows from (13) that for the posterior risk profiles corresponding to \( \gamma \), it holds that
Figure 4: The effect of $\gamma$ on the benefits from redistribution for the pension fund, for the case where $T = 1$. The solid lines correspond to the case of homogeneous beliefs, i.e., both the pension fund and the insurer use LC(1977-2009); the dashed lines correspond the case of heterogeneous beliefs, i.e., the pension fund uses LC(1977-2009), and the insurer uses LC(1987-2009). The upper panel displays the percentage increase in expected value (i.e., $\%\text{RedBEL}_1(1)$) and the lower panel displays the relative zero-utility premium (i.e., $\text{ZU}_1(1)$).

In this case, it would be most beneficial for a pension fund to find a death benefit insurer with a value of $\gamma$ around 0.3.

Now, consider the effect of heterogeneity in the beliefs on the benefits from redistribution. Comparing the solid and the dashed lines in the bottom panel of Figure 4 shows that, for all values of $\gamma$, the benefits from redistribution are larger in the case of heterogeneous beliefs. Moreover, with heterogeneous beliefs, the benefits are monotonically increasing in $\gamma$. However, whereas the increase in the benefits is rather steep when $\gamma$ increases from 0 to 0.4, a further increase in $\gamma$ has a very small effect on the benefits from redistribution. These results suggest that, while the capacity for hedging longevity risk in life annuities via death benefits may indeed be small, most of the benefits already occur with a relatively small death benefit insurer.

5 Conclusion

In this paper we introduce a game-theoretic model to determine redistributions of risk that are considered fair by all parties. We then use the model to investigate the potential benefits from Over-the-Counter (OTC) redistributions of longevity risk between pension funds and life insurance companies. The redistribution takes the form of a swap contract with a prespecified horizon, i.e., depending on the realized mortality rates at horizon date, a net payment occurs
from either the pension fund to the insurer, or vice versa. We allow for the case where the involved parties may have heterogeneous beliefs regarding the underlying probability distribution, arising for example from the use of different models to forecast future mortality rates. A swap contract will be acceptable to all parties only if they all believe that they (weakly) benefit from it. Our results suggest that the benefits from redistributing risk can be substantial, even on a one-year horizon. Our results also suggest that heterogeneous beliefs regarding the underlying probability distribution can make redistribution more attractive.

References


### A Proofs

**Proof of Proposition 1.** (i) The fact that Stability implies No Pareto Improvement follows immediately from (5) and (6). To see that Stability implies Individual Rationality, note that for all $i \in N$, it holds that $X_i \in \mathcal{F}([i])$ and $\Delta U_i(X_i) = 0$. Moreover, it follows from (6) that if $(X_i^{\text{post}})_{i \in N}$ satisfies Stability, then $X_i^{\text{post}} \in \mathcal{N}(i)$ for all $i \in N$. Combined
with (4), this implies that if \((X_i^{\text{post}})_{i \in N}\) satisfies Stability, then \(\Delta U_i(X_i^{\text{post}}) \geq \Delta U_i(X_i) = 0\) for all \(i \in N\).

(ii) it suffices to show that No Pareto Improvement and Individual Rationality implies Stability. This follows immediately from (5) and (6), and the fact that Individual Rationality implies that \(X_i^{\text{post}} \in \mathcal{N}(i)\) for all \(i \in N\).

**Proof of Theorem 2.** (i) Riddell (1981) shows this result for the case of two firms, and \(D_i = \mathbb{R}\), for \(i \in N\). It can be verified that the assumptions \(\lim_{x \to \inf D_i} u_i'(x) = \infty\), and \(\lim_{x \to \sup D_i} u_i'(x) = 0\), for all \(i \in N\), imply that the constraints \(X_i^{\text{post}}(w) \in D_i\), for \(w \in \Omega\) and \(i \in N\) are not binding. The first order optimality conditions therefore follow immediately from Riddell (1981) after straightforward generalization to \(|N|\) firms.

(ii) Let \(k \in \mathbb{R}_+^N\) be given. First note that because the state space is finite, (9) and (10) is a finite set of constraints. Existence of a solution follows immediately from the fact that \(\lim_{x \to \inf D_i} u_i'(x) = \infty\), and \(\lim_{x \to \sup D_i} u_i'(x) = 0\), for all \(i \in N\). Uniqueness of the solution follows from the fact that, for all \(i \in N\), strict concavity of \(u_i\) implies strict concavity of \(\Delta U_i\).

**Proof of Proposition 4.** We start with the “if” part.

Suppose there exists a \(k \in \mathbb{R}_+^N\), such that \(\frac{u_i'(X_i(\omega))F_i(\omega)}{u_j'(X_j(\omega))F_j(\omega)} = k_j\) for all \(\omega \in \Omega\) and all \(j \in N \setminus \{1\}\). Then, the prior risk profiles \((X_i)_{i \in N}\) satisfy (9) and (10). Hence, according to Theorem 2, \((X_i)_{i \in N} \in \mathcal{P}(N)\). It then follows from (7) that there do not exist feasible posterior risk profiles \((X_i^{\text{post}})_{i \in N} \in \mathcal{F}(N)\) such that \((\Delta U_i(X_i^{\text{post}}))_{i \in N} \geq (\Delta U_i(X_i))_{i \in N} = 0\).

Next, we show the “only if” part. Suppose that there does not exist an \((X_i^{\text{post}})_{i \in N} \in \mathcal{P}(N)\) with \((\Delta U_i(X_i^{\text{post}}))_{i \in N} \geq 0_N\). Then, it holds that \((X_i)_{i \in N} \in \mathcal{P}(N)\), and, hence, \((X_i)_{i \in N}\) should satisfy (9) and (10) for some \(k \in \mathbb{R}_+^N\) (Theorem 2). This implies that \(u_i'(X_i(\omega))F_i(\omega) = k_j u_j'(X_j(\omega))F_j(\omega)\) for all \(\omega \in \Omega\) and all \(j \in N \setminus \{1\}\). This concludes the proof.

In order to prove Theorem 6, we introduce the correspondence \(V : 2^N \to \mathbb{R}\) that assigns to each set of firms \(S \subseteq N\) the set of potential expected utility gains from feasible Redistributions of risk, allowing for “free disposal”, i.e., for all \(S \subseteq N\) :

\[
V(S) = \{ a \in \mathbb{R}^S \mid \exists (X_i^{\text{post}})_{i \in S} \in \mathcal{F}(S) \text{ such that } a \leq (\Delta U_i(X_i^{\text{post}}))_{i \in S} \}.
\] (41)

Moreover, for any \(S \subseteq N\), we let \(\mathcal{P}(S)\) be the set of Pareto optimal Redistributions of risk if and only the firms in \(S\) redistribute their risk, i.e., \(\mathcal{P}(S)\) is given by (7) with \(N\) replaced by \(S\). Then we have the following proposition.

**Proposition 9** For every \(S \subseteq N\), it holds that:

(i) \(V(S)\) is convex;

(ii) \(\partial V(S) = \{ (\Delta U_i(X_i^{\text{post}}))_{i \in S} : (X_i^{\text{post}})_{i \in S} \in \mathcal{P}(S) \}\); 

(iii) for every \(x \in \partial V(S)\), there exists a unique \((X_i^{\text{post}})_{i \in S} \in \mathcal{P}(S)\) such that \(x = (\Delta U_i(X_i^{\text{post}}))_{i \in S}\).

**Proof.** (i) The proof is a straightforward generalization of the proof of Riddell (1981), who showed this result in case of two firms. Let \(S \subseteq N\), \(a, b \in V(S)\) and \(\gamma \in (0, 1)\). Then,
there exist \((X^\text{post}, i)_{i \in S}\) and \((X_i^\text{post}, b)_{i \in S}\) such that \(\sum_{i \in S} X^\text{post, a}_i = \sum_{i \in S} X_i\), \(\sum_{i \in S} X^\text{post, b}_i = \sum_{i \in S} X_i\), \(a \leq (\Delta U_i(X^\text{post, a}_i))_{i \in S}\) and \(b \leq (\Delta U_i(X^\text{post, b}_i))_{i \in S}\). Clearly, we have
\[
\sum_{i \in S} \left(\gamma X^\text{post, a}_i + (1 - \gamma) X^\text{post, b}_i\right) = \sum_{i \in S} X_i.
\]

Moreover, by concavity of \(u_i(\cdot)\), it follows that
\[
\Delta U_i(\gamma X^\text{post, a}_i + (1 - \gamma) X^\text{post, b}_i) \geq \gamma \Delta U_i(X^\text{post, a}_i) + (1 - \gamma) \Delta U_i(X^\text{post, b}_i) \geq \gamma a_i + (1 - \gamma) b_i,
\]
for all \(i \in S\). Hence, \(\gamma a + (1 - \gamma) b \in V(S)\). This concludes the proof.

(ii) It holds that
\[
\partial V(S) = \{x \in V(S) : \exists y \in V(S) : y \geq x\}.
\]
First, note that \(a \in \partial V(S)\) if and only if there does not exist an \((\tilde{X}^\text{post})_{i \in S} \in F(S)\) such that \((\Delta U_i(\tilde{X}^\text{post, a}_i))_{i \in S} \not\geq a\). This, in turn, implies that for every \(a \in \partial V(S)\), there exist feasible posterior risk profiles \((X^\text{post, a}_i)_{i \in S} \in F(S)\) such that \(a = (\Delta U_i(\tilde{X}^\text{post, a}_i))_{i \in S}\). Moreover, it is verified immediately that \((X^\text{post, a}_i)_{i \in S} \in PO(S)\) implies that \(\Delta U_i(\tilde{X}^\text{post, a}_i))_{i \in S} \in \partial V(S)\).

Hence, (ii) holds true.

(iii) Let \(x \in \partial V(S)\) be given, and suppose that there exist \((X^\text{post, a}_i)_{i \in S}, (X^\text{post, b}_i)_{i \in S} \in PO(S)\) with \((X^\text{post, a}_i)_{i \in S} \neq (X^\text{post, b}_i)_{i \in S}\) and \((\Delta U_i(\tilde{X}^\text{post, a}_i))_{i \in S} = (\Delta U_i(\tilde{X}^\text{post, b}_i))_{i \in S} = x\). Then, by strict concavity of \(u_i(\cdot)\), for \(i \in S\), we have that \(\Delta U_i(\frac{1}{2} X^\text{post, a}_i + \frac{1}{2} X^\text{post, b}_i) \geq \frac{1}{2} \Delta U_i(X^\text{post, a}_i) + \frac{1}{2} \Delta U_i(X^\text{post, b}_i) = \Delta U_i(X^\text{post, a}_i) = \Delta U_i(X^\text{post, b}_i)\), for all \(i \in S\) with at least one strict inequality. Because \(\frac{1}{2}(X^\text{post, a}_i + \frac{1}{2}(X^\text{post, b}_i))_{i \in S} \in F(S)\), this contradicts the fact that \((X^\text{post, a}_i)_{i \in S}, (X^\text{post, b}_i)_{i \in S} \in PO(S)\). Hence, for every \(x \in \partial V(S)\), there exists a unique \((X^\text{post, a}_i)_{i \in S} \in PO(S)\) such that \(x = (\Delta U_i(X^\text{post, a}_i))_{i \in S}\).

**Proof of Theorem 6.** Scarf (1967) considers the correspondence \(\hat{V}\) defined as
\[
\hat{V}(S) = \{a \in \mathbb{R}^S \mid \exists (X^\text{post}_i)_{i \in S} \in \hat{F}(S) \text{ such that } a \leq (\hat{u}_i(X^\text{post}_i))_{i \in S}\},
\]
where \(\hat{F}(S) = \{(X^\text{post, a}_i)_{i \in S} \in \mathbb{R}^S : \sum_{i \in S} X^\text{post}_i = \sum_{i \in S} X_i\}\), and for each \(i \in N\), \(\hat{u}_i : \mathbb{R} \to \mathbb{R}\) is concave. He shows that the core of the corresponding NTU-game, i.e., the set
\[
C(N, \hat{V}) = \{x \in \hat{V}(N) : \exists S \subseteq N, (x_i)_{i \in S} \in \hat{V}(S) \\setminus \partial \hat{V}(S)\},
\]
is non-empty. First, note that the correspondence \(V\) defined in (41) follows from (43) by setting \(\hat{u}_i = \Delta U_i\), for all \(i \in N\), and by replacing \(\hat{F}(S)\) by \(F(S)\) as defined in (3), i.e., by allowing the domain \(D_i\), \(i \in N\) to be a convex subset of \(\mathbb{R}\). Using the fact that, for all \(i \in N\), concavity of \(u_i(\cdot)\) implies concavity of \(\Delta U_i\), that \(\lim_{x \to \inf D_i} u'_i(x) = \infty\), \(\lim_{x \to \sup D_i} u'_i(x) = 0\), and that \(u'_i(\cdot) < 0\), it is verified immediately that the proof in Scarf (1967) extends to the correspondence \(V\). Hence, it follows that the core of the corresponding NTU-game, which is given by
\[
C(N, V) = \{x \in V(N) : \exists S \subseteq N, (x_i)_{i \in S} \in V(S) \setminus \partial V(S)\},
\]
is non-empty. Next, we show that
\[
C(N, V) \subseteq \{(\Delta U_i(X^\text{post, a}_i))_{i \in N} : (X^\text{post}_i)_{i \in N} \in S(N)\}.
\]
Let \( a \in C(N, V) \) be given. We will show that \( a \in \{(\Delta U_i(X_{i}^{\text{post}}))_{i \in N} : (X_{i}^{\text{post}})_{i \in N} \in \mathcal{S}(N)\} \).

First, \( a \in C(N, V) \) implies that \( a \in \partial V(N) \). It therefore follows from Proposition 9(iii) that there exists an \((X_{i}^{\text{post}})_{i \in N} \in \mathcal{P}O(N)\) such that \( a = (\Delta U_i(X_{i}^{\text{post}}))_{i \in N} \). It remains to show that \((X_{i}^{\text{post}})_{i \in N} \in \mathcal{S}(N)\).

To show that \((X_{i}^{\text{post}})_{i \in N} \in \mathcal{S}(N)\), We show that if there exist \( S \subseteq N \) and \((\tilde{X}_i)_{i \in S} \in F(S)\) such that \((\Delta U_i(X_{i}^{\text{post}}))_{i \in S} \leq (\Delta U_i(\tilde{X}_i))_{i \in S}\), then \((\Delta U_i(X_{i}^{\text{post}}))_{i \in S} = (\Delta U_i(X_{i}^{\text{post}}))_{i \in S}\). Suppose there exist \( S \subseteq N \) and \((\tilde{X}_i)_{i \in S} \in F(S)\) such that

\[
(\Delta U_i(X_{i}^{\text{post}}))_{i \in S} \leq (\Delta U_i(\tilde{X}_i))_{i \in S}.
\]

This implies that \((\Delta U_i(X_{i}^{\text{post}}))_{i \in S} \in V(S)\). Because \((\Delta U_i(X_{i}^{\text{post}}))_{i \in N} = a \in C(N, V)\), it follows from (45) that \((\Delta U_i(X_{i}^{\text{post}}))_{i \in S} \in \partial V(S)\). It then follows from Proposition 9(iii) that there exists a \((\tilde{X}_i)_{i \in S} \in \mathcal{P}O(S)\) such that \((\Delta U_i(X_{i}^{\text{post}}))_{i \in S} = (\Delta U_i(\tilde{X}_i))_{i \in S}\). Because \((\tilde{X}_i)_{i \in S} \in \mathcal{T}(S)\), \((\tilde{X}_i)_{i \in S} \in F(S)\), and \((\Delta U_i(X_{i}^{\text{post}}))_{i \in S} = (\Delta U_i(\tilde{X}_i))_{i \in S}\), it follows from (4) that \((\Delta U_i(X_{i}^{\text{post}}))_{i \in S} = (\Delta U_i(\tilde{X}_i))_{i \in S}\). Hence, we can conclude that

\[
(\Delta U_i(X_{i}^{\text{post}}))_{i \in S} = (\Delta U_i(X_{i}^{\text{post}}))_{i \in S}.
\]

Hence, we have shown that there do not exist \( S \subseteq N \) and \((\tilde{X}_i)_{i \in S} \in F(S)\) such that \((\Delta U_i(X_{i}^{\text{post}}))_{i \in S} \leq (\Delta U_i(\tilde{X}_i))_{i \in S}\). This implies that \((X_{i}^{\text{post}})_{i \in N} \in \mathcal{S}(N)\). Because \( a = (\Delta U_i(X_{i}^{\text{post}}))_{i \in N} \), we can conclude that the inclusion in (46) holds true. Because \( C(N, V) \) is non-empty, this concludes the proof.

**Proof of Proposition 7.** Consider

\[
\hat{\mathcal{N}}B = \arg\max_{x \in V(N), x \geq 0} \prod_{i \in N} x_i,
\]

where \( V(N) \) is as defined in (41). We know from Proposition 9 that \( V(N) \) is convex. Moreover, it is easily verified that \( V(N) \) is comprehensive, and that \( V(N) \cap \mathbb{R}^N_+ \) is non-empty and bounded. It therefore follows from Nash (1950) that \( \hat{\mathcal{N}}B \) is non-empty, single-valued, and satisfies \( \hat{\mathcal{N}}B \subseteq \partial V(N) \cap \mathbb{R}^N_+ \). It then follows from Proposition 9(ii) that for every \( x \in \hat{\mathcal{N}}B \), there exists a \((X_{i}^{\text{post}})_{i \in N} \in \mathcal{P}O(N)\) such that \( x = \Delta U_i(X_{i}^{\text{post}}))_{i \in N} \geq 0 \), i.e.,

\[
\hat{\mathcal{N}}B \subseteq \{(\Delta U_i(X_{i}^{\text{post}}))_{i \in N} : (X_{i}^{\text{post}})_{i \in S} \in \mathcal{P}O(N) \cap \mathcal{T}(N)\}.
\]

This implies that

\[
\hat{\mathcal{N}}B = \left\{ \Delta U_i(X_{i}^{\text{post}}) : (X_{i}^{\text{post}})_{i \in N} \in \arg\max_{(X_i)_{i \in N} \in \mathcal{P}O(N) \cap \mathcal{T}(N)} \prod_{i \in N} \Delta U_i(X_i) \right\} = \left\{ (\Delta U_i(X_{i}^{\text{post}}))_{i \in N} \in \hat{\mathcal{N}}B \right\}.
\]

Because \( \hat{\mathcal{N}}B \) is non-empty, it follows that \( \mathcal{N}B \) is non-empty. Moreover, because \( \hat{\mathcal{N}}B \) is single-valued, it follows from Proposition 9(iii), that \( \mathcal{N}B \) is single-valued.

**Proof of Proposition 8.** It is sufficient to show that \( \mathcal{S}(N) \) is compact. This follows directly from that \( V(N) \cap \mathbb{R}^N_+ \) is non-empty and bounded and \( V(S) \cap \partial V(S) \) is open.
B Portfolio characteristics and Data

In this appendix, we provide the portfolio characteristics and the data that we use in Section 4.2. The characteristics of the portfolios are as follows:

1. The pension fund has 50,000 male participants. The participants have accrued rights for a (deferred) single life annuity that yields a nominal yearly payment, with a first payment at the beginning of the year in which the insured reaches age 65, and a last payment at the beginning of the year in which the insured dies. The accrued right depends on age, and is normalized to 1 for a 65 year old. The age composition as well as the accrued rights as a function of age are displayed in Figure 5. The average age of the participants is approximately 60.

2. The death benefit insurer has a portfolio of death benefit insurance contracts that pay a lump sum at the end of the year in which the insured dies, in case of decease of the insured before age 65. The age composition is displayed in Figure 6. The average age of the death benefit policyholder is approximately 42. Each policyholder has a normalized insured right in case of decease of 10 (10 times the annual annuity payment of a 65 year old).

3. In Subsection 2.4.2, the number of insureds is 19420. This implies that the relative size as defined in (39) equals 0.2, i.e., based on LC(1977-2009), the date-0 best estimate value of the liabilities of the insurer is 20% of the date-0 best estimate value of the liabilities of the pension fund. In Subsection 2.4.3, we investigate the effect of the relative size of the pension fund and the insurer by considering the case where the number of insureds is $19420 \cdot \frac{1}{100}$, for $\gamma \in \{ \frac{k}{100} : k = 1, \ldots, 100 \}$. Then, $\gamma = 1$ corresponds to the case where the date-0 best estimate value of the liabilities of the pension fund is equal to the best estimate value of the liabilities of the death benefit insurer in case of homogeneous beliefs regarding the underlying probability distribution with LC(1977-2009).

4. The return on assets equals $r = 3\%$.

The age composition of the pension fund and the accrued rights of the participants as a function of their age are displayed in Figure 5. These characteristics are based on data from a large Dutch pension fund. The age composition of the death benefit insurer’s portfolio is displayed in Figure 6.

C Lee-Carter model

In this appendix, we describe the Lee-Carter model (1992). The probability that an individual of age $x$ at time $t$ survive the next year is modeled as

$$p_{x,t} = \exp(-m_{x,t}),$$

(48)

where $m_{x,t}$ represents the central death rate of a men with age $x$ at time $t$ (see, e.g., Pitacco et al., 2009). The central death rate is given by $m_{x,t} = \frac{D_{x,t}}{E_{x,t}}$, where $D_{x,t}$ is the observed number of deaths in year $t$ in the cohort aged $x$ at the beginning of year $t$, and $E_{x,t}$ is the

10The rationale for young participants of a death benefit insurer is that death benefit insurance is often obliged if individuals buy a mortgage.
Figure 5: The age composition and the accrued pension rights ($\delta_j$) of the participants of a pension fund, where we normalize the pension right at retirement to one.

Figure 6: The age composition and the insured rights ($\delta_j$) of the participants of a death benefit insurer.
corresponding number of persons. Lee and Carter (1992) propose the following log-bilinear relationship:

\[
\log(m_{x,t}) = a_x + b_x \kappa_t + \epsilon_{x,t}, \quad \epsilon_{x,t} \sim_{i.i.d.} N(0, \sigma^2_x), \tag{49}
\]

for all \( t = t_0, \ldots, 0 \) and \( x = 1, 2, \ldots, 100 \), where \( \kappa_t = \{ \kappa_{\bar{t}} : \bar{t} = t_0, \ldots, t \} \) and \( t_0 < 0 \). Here, \( t_0 \) is the first date in the data that is used. The following normalizations are imposed:

\[
\sum_{x=1}^{100} b_x = 1 \quad \text{and} \quad \sum_{t=t_0}^{0} \kappa_{\bar{t}} = 0.
\]

Then, the dynamics of the distribution of macro longevity are captured by \( \kappa_t \). The estimates of \( a_x, b_x \) and \( \kappa_t \) are obtained via singular value decomposition.

Future values of \( \kappa_t \) are forecasted using an ARIMA(0,1,1) model:

\[
\kappa_t = \kappa_{t-1} + c + e_t + \theta e_{t-1}, \tag{50}
\]

for all \( t \geq 1 \), where we impose the following distribution of the errors:

\[
e_t : \mathcal{K}_{t-1} \sim N(0, \sigma^2). \tag{51}
\]

We include parameter uncertainty in our simulations. So, for every forecast, we take into account that the estimates of \( a_x \) and \( b_x \) at \( t = 0 \) are not fixed. Including parameter uncertainty strengthens the impact of longevity risk. For a discussion, we refer to Hāri et al. (2008). In the next tables, we show the parameter estimates for male mortality rates.

<table>
<thead>
<tr>
<th></th>
<th>( c )</th>
<th>( \theta )</th>
<th>( \sigma )</th>
</tr>
</thead>
<tbody>
<tr>
<td>LC(1977-2009)</td>
<td>-2.00</td>
<td>-0.27</td>
<td>2.29</td>
</tr>
<tr>
<td>LC(1987-2009)</td>
<td>-2.13</td>
<td>-0.11</td>
<td>2.65</td>
</tr>
</tbody>
</table>

Table 2: Estimates of the ARIMA(0,1,1) model for male mortality rates, corresponding to (50) and (51), using HMD data from 1977 to 2009 (first row), and using HMD data from 1987 to 2009 (second row).

D The probability distribution of \( CL_i(T) \) and summary statistics

In this appendix, we describe the simulation of the probability distribution of \( CL_i(T) \) and its summary statistics. Recall that

\[
CL_i(T) = \sum_{\tau=1}^{T} \tilde{L}_{i,\tau}(1 + r)^{-\tau} + \mathbb{E} \left[ \sum_{\tau=T+1}^{T_{\max}} \tilde{L}_{i,\tau}(1 + r)^{-\tau} \middle| \mathcal{F}_T \right].
\]

To simulate the probability distribution of \( CL_i(T) \), we use the following procedure:

- We simulate \( S_1 \) trajectories for \( (\tau p_{x,0})_{\tau=1,\ldots,T} \) using (48)-(51).

- For every simulated trajectory of \( (\tau p_{x,0})_{\tau=1,\ldots,T} \),
  - we determine the corresponding value of \( \sum_{\tau=1}^{T} \tilde{L}_{i,\tau}(1 + r)^{-\tau} \) using (33),
  - we re-estimate the model and simulate \( S_2 \) trajectories for \( (\tau p_{x,T})_{\tau\geq1} \) to determine the corresponding value of \( \mathbb{E} \left[ \sum_{\tau=T+1}^{T_{\max}} \tilde{L}_{i,\tau}(1 + r)^{-\tau} \middle| \mathcal{F}_T \right] \).
Throughout this paper, we set $S_2 = 1000$.

When firms have heterogeneous beliefs regarding the probability distribution of mortality rates, we discretize the range of $\sum_{i \in N} CL_i(T)$ by imposing a partition. On every interval of this partition, we use the mid-point of the interval as realization of $\sum_{i \in N} CL_i(T)$. A partition is determined such that the approximated probability for every realization is positive. The error we make is decreasing in the size of this interval. Every interval of this partition and its corresponding outcome corresponds with a state $\omega \in \Omega$. So, one loses some accuracy by discretization, but note that a model itself is also an approximation. Moreover, simulations are generally needed, which in itself is also an approximation.

Next, we describe the summary statistic of $CL_i(T)$ as a function of the horizon $T$. These are shown in Table 3. Panels A and B yields summary statistics of $CL_i(T)$ for the pension fund and for the death benefit insurer, for the case where the probability distribution of future mortality is estimated based on the Lee-Carter model with parameters estimated based on the historical period 1977-2009. Panels C and D presents summary statistics of $CL_i(T)$ for the pension fund and for the death benefit insurer for the case where the probability distribution of future mortality is estimated based on the Lee-Carter model with parameters estimated based on the historical period 1987-2009.

<table>
<thead>
<tr>
<th>Panel A: pension fund LC(1977-2009)</th>
<th>$T = 1$</th>
<th>$T = 5$</th>
<th>$T = 10$</th>
<th>$T = T^{\max}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E[CL_i(T)]$</td>
<td>$3.424 \times 10^4$</td>
<td>$3.433 \times 10^4$</td>
<td>$3.427 \times 10^4$</td>
<td>$3.424 \times 10^4$</td>
</tr>
<tr>
<td>$\sigma[CL_i(T)]$</td>
<td>$3.32 \times 10^3$</td>
<td>$5.78 \times 10^3$</td>
<td>$6.12 \times 10^3$</td>
<td>$6.75 \times 10^3$</td>
</tr>
<tr>
<td>$\sigma[CL_i(T)]/E[CL_i(T)]$</td>
<td>0.010</td>
<td>0.017</td>
<td>0.017</td>
<td>0.020</td>
</tr>
<tr>
<td>buffer</td>
<td>1.98%</td>
<td>3.14%</td>
<td>3.32%</td>
<td>3.69%</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Panel B: death benefit insurer LC(1977-2009)</th>
<th>$T = 1$</th>
<th>$T = 5$</th>
<th>$T = 10$</th>
<th>$T = T^{\max}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E[CL_i(T)]$</td>
<td>$\gamma \cdot 3.424 \times 10^4$</td>
<td>$\gamma \cdot 3.433 \times 10^4$</td>
<td>$\gamma \cdot 3.427 \times 10^4$</td>
<td>$\gamma \cdot 3.424 \times 10^4$</td>
</tr>
<tr>
<td>$\sigma[CL_i(T)]$</td>
<td>$\gamma \cdot 1.58 \times 10^4$</td>
<td>$\gamma \cdot 2.86 \times 10^4$</td>
<td>$\gamma \cdot 3.32 \times 10^4$</td>
<td>$\gamma \cdot 3.73 \times 10^4$</td>
</tr>
<tr>
<td>$\sigma[CL_i(T)]/E[CL_i(T)]$</td>
<td>0.046</td>
<td>0.084</td>
<td>0.097</td>
<td>0.109</td>
</tr>
<tr>
<td>buffer</td>
<td>9.25%</td>
<td>16.16%</td>
<td>19.61%</td>
<td>21.02%</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Panel C: pension fund LC(1977-2009)</th>
<th>$T = 1$</th>
<th>$T = 5$</th>
<th>$T = 10$</th>
<th>$T = T^{\max}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E[CL_i(T)]$</td>
<td>$3.496 \times 10^7$</td>
<td>$3.498 \times 10^7$</td>
<td>$3.494 \times 10^7$</td>
<td>$3.499 \times 10^7$</td>
</tr>
<tr>
<td>$\sigma[CL_i(T)]$</td>
<td>$4.78 \times 10^3$</td>
<td>$7.77 \times 10^3$</td>
<td>$8.73 \times 10^3$</td>
<td>$9.45 \times 10^3$</td>
</tr>
<tr>
<td>$\sigma[CL_i(T)]/E[CL_i(T)]$</td>
<td>0.014</td>
<td>0.022</td>
<td>0.025</td>
<td>0.027</td>
</tr>
<tr>
<td>buffer</td>
<td>2.64%</td>
<td>4.35%</td>
<td>4.65%</td>
<td>5.31%</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Panel D: death benefit insurer LC(1987-2009)</th>
<th>$T = 1$</th>
<th>$T = 5$</th>
<th>$T = 10$</th>
<th>$T = T^{\max}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E[CL_i(T)]$</td>
<td>$\gamma \cdot 3.366 \times 10^5$</td>
<td>$\gamma \cdot 3.368 \times 10^5$</td>
<td>$\gamma \cdot 3.361 \times 10^5$</td>
<td>$\gamma \cdot 3.366 \times 10^5$</td>
</tr>
<tr>
<td>$\sigma[CL_i(T)]$</td>
<td>$\gamma \cdot 3.62 \times 10^4$</td>
<td>$\gamma \cdot 3.89 \times 10^4$</td>
<td>$\gamma \cdot 4.03 \times 10^4$</td>
<td>$\gamma \cdot 4.10 \times 10^4$</td>
</tr>
<tr>
<td>$\sigma[CL_i(T)]/E[CL_i(T)]$</td>
<td>0.108</td>
<td>0.116</td>
<td>0.117</td>
<td>0.120</td>
</tr>
<tr>
<td>buffer</td>
<td>22.9%</td>
<td>24.1%</td>
<td>24.5%</td>
<td>24.8%</td>
</tr>
</tbody>
</table>

Table 3: Summary statistics of $CL_i(T)$. The buffer is defined given by $Q_{0.975,P_i}(CL_i(T))/E[CL_i(T)]$, where $Q_{0.975,P_i}(CL_i(T))$ is the 97.5%-quantile with respect to firm $i$’s subjective probability measure.
We note that the expected value of $CL_i(T)$ is independent of the horizon $T$. This follows immediately from (34), which implies that for all horizons $T$, the expected value of $CL_i(T)$ equals the date-zero best estimate value of the liabilities, i.e.,

$$E_{P_i}[CL_i(T)] = E_{P_i} \left[ \sum_{\tau \geq 1} \frac{L_{i,\tau}}{(1+r)^\tau} \right] = L_i(0), \text{ for } i \in \{1, 2\}.$$ 

Table 4 displays the correlation between $CL_1(T)$ and $CL_2(T)$, as a function of the horizon $T$. We find strong negative correlations. This is also illustrated in the scatter plot in Figure 2.

<table>
<thead>
<tr>
<th></th>
<th>$T = 1$</th>
<th>$T = 5$</th>
<th>$T = 10$</th>
<th>$T = T_{\text{max}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>LC(1977-2009)</td>
<td>-0.91</td>
<td>-0.97</td>
<td>-0.97</td>
<td>-0.97</td>
</tr>
<tr>
<td>LC(1987-2009)</td>
<td>-0.47</td>
<td>-0.75</td>
<td>-0.88</td>
<td>-0.97</td>
</tr>
</tbody>
</table>

Table 4: Correlation coefficient of $CL_1(T)$ and $CL_2(T)$, where Firm 1 is a pension fund and Firm 2 a death benefit insurer. The value of $\gamma$ is irrelevant.