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INTERACTIVE PURCHASING SITUATIONS

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Interactive purchasing situations

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Abstract

This paper introduces a new class of interactive cooperative purchasing situations and provides an explicit alternative characterization of the nucleolus of cooperative games, which offers an alternative to Kohlberg (1971). In our cooperative purchasing situation, the unit price of a commodity depends on the largest order quantity within a cooperating group of players. Due to a decreasing unit price function, players can obtain cost savings by purchasing cooperatively. However, to establish fruitful cooperation a decision has to be made about an adequate allocation of the corresponding cost savings. We analyze Maximum Cooperative Purchasing (MCP)-situation from the perspective of allocation by defining corresponding cooperative MCP-games. It turns out that in an MCP-game all coalitional values can be determined from the values of two-player coalitions. Moreover, it is shown that a decreasing unit price function is a sufficient condition for a non-empty core: the Direct Price solution is both a core element and a marginal vector. It is seen that the nucleolus of an MCP-game can be derived in polynomial time from the Direct Price solution, using a so-called nucleolus determinant. To show this result, the explicit alternative characterization of the nucleolus is used. Using the decomposition of an MCP-game into unanimity games we find an explicit expression for the Shapley value. Interestingly, the Shapley value can be interpreted as a specific tax and subsidize system. Finally, the behavior of the three solution concepts is compared numerically.

Keywords: cooperative purchasing, direct pricing, nucleolus, Shapley value.

JEL classification number: C71.

1 Introduction

A purchasing cooperative consists of organizations that collaborate in their purchasing process — either in sharing information, bundling order quantities, sharing transportation services or other forms of cooperation — in order to obtain benefits. If we focus on bundling order quantities, cooperative purchasing³ seems an easy and rational solution for a group of buyers facing quantity discounts. However, it is not as simple as that. In this paper we analyze interactive purchasing situations where the unit price depends on the largest order quantity within a group of players. According to Tella and Virolainen (2005) the main motives of organizations to become member of a cooperative are the obtained information and — on the long run — the obtained cost savings, due to increasing returns to scale. Increasing returns to scale are analyzed in many micro-economic situations, e.g. in the production industry (Hsu and Li, 2009) and health care industry (Schneider et al., 2008). In purchasing practise the increasing returns to scale of production often translates in a quantity discount for the buyers or customers (e.g. Monahan, 1984). Naturally buyers want to exploit these discounts, but they cannot simply

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³Cooperative purchasing is also referred to as group buying or group purchasing.

increase their order quantities. So organizations try to cooperate in purchasing. But why do all hospitals in the US *not* unite themselves to suppress pharmacists' prices of medicines? How come that all high-schools in the Netherlands do *not* purchase computers cooperatively to obtain a substantial price reduction?

The reasons are simple. Firstly, it is quite hard to manage and combine the purchasing processes of a large group of organizations. But even if the groups would be manageable, they would not easily form because of a second reason: the group members should agree on how to divide the obtained cost savings beforehand. Too simple cost savings allocation schemes may cause some members of the cooperative to feel better off alone, to be better off in sub-cooperatives or simply think that the allocation method is not fair. According to Schotanus (2007) the fair allocation of cost savings is one of the main critical success factors for the stability of purchasing groups. Also in a large-scale survey among logistic service providers in Flanders, Cruijsen et al. (2007) found that organizations believe in the potential of horizontal cooperation, but they consider the allocation of the actual cost savings as one of the most important impediments of the cooperative.

We are not the first to investigate Cooperative Purchasing (CP)-situations. Anand and Aron (2003) are pioneers in studying cooperative purchasing using analytical models. Amongst others, they derive optimal pricing schedules for the supplier facing purchasing cooperatives. Also, from the buyers perspective analytical models have been developed, for example on the development of purchasing groups in the health care industry (Nollet and Beaulieu, 2003) and the formation of coalitions by means of the internet (Granot and Sošić, 2005). There have appeared qualitative considerations in the literature linking game theory and cooperative purchasing, but more in an explanatory sense than in analyzing its exact implications, e.g. Blomqvist et al. (2002) or Tella and Virolainen (2005). Only recently, CP-situations have been formally modeled as a game. Keskinocak and Savasaneril (2008) analyze the situation where buyers are possible competitors. In Heijboer (2003), Schotanus (2007) and Nagarajan et al. (2010) a purchasing cooperative is modeled as a cooperative transferable utility game. The unit price depends on the sum of the order quantities and the higher the total order quantity, the lower the unit price. The main and common goal of the authors investigating CP-situations and related cooperative CP-games is to find fair and stable allocations of the obtained cost savings.

In this paper we analyze CP-situations that are not covered by the cooperative models considered in the literature above. Consider the situation where a general practitioner needs daily supplies like sterile needles, bandages, compresses and drugs. He can buy these supplies at a pharmaceutical company. The pharmaceutical industry and its prices are not transparent, so one can imagine that the general practitioner ends up with a high unit price. A large hospital buys its supplies at the same pharmaceutical companies, but it has far more knowledge of the market and a better bargaining position towards its possible suppliers. As a result the hospital can pull down the unit prices. The general practitioner could try to set up some cooperative with the hospital to suppress his own unit prices. However adding the two order quantities will not result in a lower unit price for both organizations. If the single practitioner can use the terms and contract of the hospital, he would be willing to pay a (small) fee for this 'riding along'. Both the hospital and the general practitioner would then be better off in this small cooperative.

This form of cooperative purchasing fits in a framework developed by Schotanus (2007): a typology of organizational forms of cooperative purchasing. One of these types involves *piggy-backing* groups, informal purchasing cooperatives that wish to keep cooperation as simple as possible. Mostly it enhances a relatively large organization that negotiates with the supplier on its own and the resulting contract may be used by some smaller organizations. The example the author

provides is a consortium of local governmental institutions in the North of the Netherlands. These institutions have a piggy-backing group that has existed for more than 20 years.

The CP-situations we will investigate in this paper deal with horizontal cooperation between two or more organizations that find themselves at the same position in the supply chain. We consider a group of organizations all having individual order quantities with respect to a certain commodity. The involved organizations might be competitors in the end market, but we assume that the fact that the organizations are possible competitors is not likely to influence the cooperation, since they are better off within a cooperative. The Dutch *Superunie* is a good example. It is a purchasing cooperative consisting of small competing supermarket chains, who must cooperate to remain competitive towards large organizations like *Albert Heijn*.

Like in previous work, we will consider the bundling and sharing of purchasing volumes. We focus on the main motive on the long run: quantity discounts. These discounts imply direct cost savings for the members of a cooperative. Contrary to Schotanus (2007) and Nagarajan et al. (2010) we will consider purchasing situations where the unit price does not depend on the sum of those order quantities, but on the maximum of the individual purchasing volumes. The organizations can negotiate for their own terms and unit prices separately. The outcome of each negotiation depends on the size (in terms of order quantity) of a buyer in each cooperative and is independent of the other order quantities. As an individual, the largest organization will obtain the lowest unit price. By cooperating the organizations can obtain a smaller unit price and obtain cost savings. Within a group of purchasing organizations, the smaller organizations simply let the largest organization add their order quantities to the total order. They use the terms and contract of the larger organization and his individually negotiated unit price. As explained by Schotanus (2007), the coordination costs for this form of cooperative purchasing can be assumed to be relatively low.

We will explicitly address the problem of finding suitable allocation methods for the cost savings in this class of cooperative purchasing situations. To this end we define a cooperative transferable utility game corresponding to a Maximum Cooperative Purchasing (MCP)-situation; i.e., a CP-situation with a max unit price function. An associated MCP-game possesses a rather particular property: the value of each coalition is the sum of the values of two-player coalitions. This is not the case for the CP-games as considered by Heijboer (2003). Also it is not necessarily a k -game with $k = 2$ as defined by van den Nouweland et al. (1996). In general a decreasing unit price function is a sufficient condition for a nonempty core: we can always find a stable allocation of the cost savings. One of the core-elements is a marginal vector of the MCP-game and can be obtained via the Direct Price solution method. Every player pays the most optimistic unit price, the price that follows from the grand coalition. For the largest player this implies that he receives no price reductions at all. This allocation method is population monotonic, but in terms of piggy-backing: this method leads to a cooperation fee equal to zero. Hence, the big player will not easily agree with the Direct Price solution as allocation method. Therefore we propose the two following allocation methods: the nucleolus (Schmeidler, 1969) and the Shapley value (Shapley, 1953) of an MCP-game.

In general, finding the nucleolus of a cooperative game is a hard task. Kohlberg (1971) developed a criterium to check whether an allocation equals the nucleolus of that game. Based on this criterium, several algorithms have been developed to compute the nucleolus, all not of polynomial time. For a compact overview see Leng and Parlar (2010). Reijnierse and Potters (1998) explain which collections of coalitions are essential to determine the nucleolus and show that these collections may differ from the ones of Kohlberg. Inspired by the results of Potters et al. (1996) and Reijnierse and Potters (1998), in this paper we will derive an alternative and

explicit characterization of the nucleolus for general cooperative games with a non-empty core. To its advantage this characterization is more constructive in nature than the Kohlberg criterium.

Using this new criterium, the nucleolus of an MCP-game can be found via a so-called nucleolus-determinant: a collection of disjoint coalitions and corresponding excesses. These excesses can be interpreted as the fee the players in the coalition have to pay. It is shown that according to the nucleolus in principle every player obtains the Direct Price solution with a correction according to these fees. Moreover we show how to find a nucleolus-determinant recursively with an algorithm of polynomial time. If there is more than one largest player, we will see that the cooperation fee nullifies, and therefore the nucleolus equals the Direct Price solution.

If we decompose an MCP-game as a linear combination of unanimity games, we find alternatingly positive and negative coefficients. Using this decomposition we derive an explicit expression for the Shapley value, which can be interpreted as stemming from a tax and subsidize system. An individual player receives or pays a certain percentage of the cost savings of all two-player coalitions.

Both the nucleolus and the Shapley value are attractive solution concepts from a general game-theoretic point of view. We conclude the paper with a numeric comparison between the behavior of the Shapley value and the nucleolus in MCP-situations and for illustrative reasons we also compare the two game theoretic solution concepts with the Direct Price solution. We will see however, that for MCP-situations, the Shapley value and the nucleolus prescribe similar allocation proposals and that the differences between these two allocations are relatively small. Players with order quantities close to the order quantity of the largest player are expected to be better off in the Shapley value, while players with smaller order quantities are expected to be better off in the nucleolus. In contrast the difference between the two game theoretic solutions on the one hand and the Direct Price solution on the other hand is relatively large.

The organization of this paper is as follows. In Section 2 we will formally introduce MCP-situations, define corresponding MCP-games, discuss some appealing properties of these games and explain their specific structure. Then, in Section 3 we analyze the Direct Price solution. In Section 4 an explicit alternative characterization is provided for the nucleolus of a general cooperative game with a non-empty core, and in section 5 we calculate the nucleolus of an MCP-game, based on this alternative characterization. Section 6 focusses on the Shapley value of an MCP-game and section 7 provides a numerical comparison between the various allocation proposals.

2 MCP-situations and corresponding games

This section provides the formal description of Maximum Cooperative Purchasing (MCP)-situations and defines corresponding cooperative MCP-games.

Formally, we have a player set $N = \{1, \dots, n\}$, $n \geq 2$, with order quantities $q \in \mathbb{R}_+^N$. There is a commonly known unit price function $p : [0, \infty) \rightarrow [0, \infty)$ that maps an order quantity to some unit price. We assume the unit price function to be weakly decreasing. For the remainder of this paper we assume, without loss of generality, that the order quantities are arranged in nondecreasing order, i.e. $0 < q_1 \leq q_2 \leq \dots \leq q_n$. The class of all MCP-situations is denoted by $\mathcal{M}$ and a single MCP-situation is given by the triple $(N, q, p) \in \mathcal{M}$. Note that n , the number

of players, is variable.

To analyze the allocation aspects of an MCP-situation we will construct a corresponding cooperative TU-game. A cooperative game (N, v) is defined by a finite set N of players and a function v on the set 2^N of all subsets (coalitions) of N . This function assigns to each coalition $S \in 2^N$ a value $v(S) \in \mathbb{R}$ such that $v(\emptyset) = 0$. The value $v(S)$ represents what a coalition S can accomplish or realize by optimal cooperation among themselves.

Without loss of generality we assume that a TU-game is zero-normalized.

Consider a subgroup of purchasing organizations $S \in 2^N$. The unit price corresponding to that coalition of purchasers is determined by its member with maximal order quantity. By assumption this is the player with highest index. Hence S pays $p(q_s)$ per unit, with $s = \max\{i : i \in S\}$. Without cooperation every player $i \in S$ would have paid $p(q_i)q_i$ per unit. Looking at the corresponding cost savings as monetary revenues, the characteristic function of the cooperative MCP-game (N, w) corresponding to an MCP-situation $(N, q, p) \in \mathcal{M}$ is defined for all $S \in 2^N$ by

$$w(S) = \sum_{j \in S} [p(q_i)q_i] - p(q_s) \sum_{j \in S} q_j \quad (1)$$

and it reflects the maximum cost savings a coalitions S can establish. For ease of notation we set $p(q_i) = p_i$ for all $i \in N$. Specifically, from (1) it readily follows that the value of an arbitrary two-player coalition with $i < j$ equals

$$w(\{i, j\}) = (p_i - p_j)q_i. \quad (2)$$

Since we assume the unit price function to be weakly decreasing we find a specific order in the two-player coalitions. If $j < k$ then for all $i < j$ it holds that

$$w(\{i, j\}) \leq w(\{i, k\}). \quad (3)$$

By the nature of cost savings, each MCP-game is nonnegative with all single player coalitions having value 0. The following example illustrates an MCP-game.

Example 2.1. Consider an MCP-situation $(N, q, p) \in \mathcal{M}$ with $N = \{1, 2, 3, 4\}$, $q = (2, 4, 8, 12)$ and unit price function $p : [0, \infty) \rightarrow [0, \infty)$ with $p(d) = 10 + \frac{12}{d}$. Hence the individual ordering costs of player 1 are $p(2) \cdot 2 = 16 \cdot 2 = 32$. For player 2, 3 and 4, the individual ordering costs are 52, 92 and 132 respectively. The following table represents the corresponding MCP-game.

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{4\}$	$\{1, 2\}$	$\{1, 3\}$	$\{1, 4\}$	$\{2, 3\}$
$w(S)$	0	0	0	0	6	9	10	6
S	$\{2, 4\}$	$\{3, 4\}$	$\{1, 2, 3\}$	$\{1, 2, 4\}$	$\{1, 3, 4\}$	$\{2, 3, 4\}$	$\{1, 2, 3, 4\}$	
$w(S)$	8	4	15	18	14	12	22	

Note that $w(\{1, 4\}) + w(\{2, 4\}) + w(\{3, 4\}) = w(\{1, 2, 3, 4\})$, while $w(\{1, 3\}) + w(\{2, 3\}) = w(\{1, 2, 3\})$. $\blacklozenge$

The final observations we made in Example 2.1 are true in general. Since the price depends on the maximum order quantity, the largest player s in a coalition S solely determines the unit price. Every other player in coalition S profits from the unit price decrease, independent of the order quantities of the players in $S \setminus \{s\}$. Thus the value of a coalition S consists of what every individual player in S can accomplish with the largest player in S . This leads to the following theorem.

Theorem 2.2. Let (N, w) be the MCP-game corresponding to an MCP-situation $(N, q, p) \in \mathcal{M}$. Let $S \subset N$ with $|S| \geq 2$ and $s \in S$ with $s = \max\{i : i \in S\}$. Then,

$$w(S) = \sum_{j \in S} w(\{j, s\}). \quad (4)$$

Proof. By (1) and (2) we have

$$w(S) = \sum_{j \in S} p_j q_j - \sum_{j \in S} p_s q_j = \sum_{j \in S} (p_j - p_s) q_j = \sum_{j \in S} w(\{j, s\}). \quad \square$$

Using Theorem 2.2, it is straightforward to verify that each MCP-game is monotonic, i.e. for all $S, T \in 2^N$, $S \subset T$ with $w(S) \leq w(T)$ and it can also be derived that each MCP-game is superadditive: for all $S, T \in 2^N$ with $S \cap T = \emptyset$, $w(S) + w(T) \leq w(S \cup T)$.

3 The core and the Direct Price solution

Within the game theoretic literature one can find several ways or policies to allocate the value of the grand coalition. The coalitional values $v(S)$ in a game (N, v) form a natural benchmark to evaluate an allocation method. There are two basic evaluation properties. The first is efficiency. An allocation $x \in \mathbb{R}^N$ is efficient if $\sum_{i \in N} x_i = v(N)$. The second is stability. An allocation x is stable if for all $S \in 2^N$, $\sum_{i \in S} x_i \geq v(S)$. The set of all efficient and stable allocation vectors is called the core of the game and is denoted by $\mathcal{C}(v)$. In general, this set can be empty.

A frequently studied property for a game is convexity. Convexity requires that for all $j \in N$ and $S, T \in 2^N$ with $S \subset T \subset N \setminus \{j\}$, $v(S \cup \{j\}) - v(S) \leq v(T \cup \{j\}) - v(T)$. If (N, v) is convex, then $\mathcal{C}(v) \neq \emptyset$ (Shapley, 1967).

Example 3.1. (Example 2.1 continued) Reconsider the game of Example 2.1. This game is not convex. If player 3 joins player 1, they gain additional cost savings of $w(\{1, 3\}) - w(\{1\}) = 9$. If, however, player 3 joins the larger coalition $\{1, 4\}$, they gain extra cost savings of $w(\{1, 3, 4\}) - w(\{1, 4\}) = 4$, which is lower than 9, contradicting convexity. The core of this game is non-empty and is given by

$$\begin{aligned} \mathcal{C}(w) = \text{conv}\{ & (10, 8, 4, 0), (10, 8, 0, 4), (10, 6, 0, 6), (10, 2, 4, 6), \\ & (9, 8, 0, 5), (5, 8, 4, 5), (9, 2, 4, 7), (5, 6, 4, 7), (9, 6, 0, 7) \}, \end{aligned}$$

where ‘conv’ denotes the convex hull. A special core element is the allocation $(10, 8, 4, 0)$. This allocation is a result from applying a direct pricing principle: every player pays the price that can be negotiated for the grand coalition. In this case that price is 11. Thus player 1 obtains cost savings for his own order quantity of $q_1(p_1 - p_4) = 2(16 - 11) = 10$, player 2 of $4(13 - 11) = 8$, player 3 of $8(11.5 - 11) = 4$ and player 4 of 0. $\blacklozenge$

The direct pricing principle illustrated in Example 3.1 can be formalized.

Definition 1. The *Direct Price solution* δ on $\mathcal{M}$ is such that for all $(N, q, p) \in \mathcal{M}$ and for all $i \in N$

$$\delta_i(N, q, p) = (p_i - p_n)q_i.$$

Note that according to (2), $\delta_i(N, q, p) = w(\{i, n\})$ for all $i \in N$ and all $(N, q, p) \in \mathcal{M}$, with (N, w) the corresponding MCP-game. For an arbitrary MCP-situation, the allocation resulting from the Direct Price solution method belongs to the core of the corresponding MCP-game.

Theorem 3.2. *Let (N, w) be the MCP-game corresponding to an MCP-situation $(N, q, p) \in \mathcal{M}$. Then, $\delta(N, q, p) \in \mathcal{C}(w)$.*

Proof. Using Theorem 2.2

$$\sum_{i \in N} \delta_i(N, q, p) = \sum_{i \in N} w(\{i, n\}) = w(N).$$

Let $S \in 2^N$. Using (3) we find that

$$w(S) = \sum_{i \in S} w(\{i, s\}) \leq \sum_{i \in S} w(\{i, n\}) = \sum_{i \in S} \delta_i(N, q, p),$$

where $s = \max\{i : i \in S\}$. □

For a game (N, v) denote by $\Pi(N)$ the set of all bijections from $\{1, \dots, |N|\}$ to N . Given $\sigma \in \Pi(N)$, $\sigma(k)$ is interpreted as the player in position k . For $\sigma \in \Pi(N)$, the marginal vector corresponding to $m^\sigma(v) \in \mathbb{R}^N$ is defined by

$$m_{\sigma(k)}^\sigma(v) = v(\{\sigma(1), \dots, \sigma(k)\}) - v(\{\sigma(1), \dots, \sigma(k-1)\})$$

for all $k \in \{1, \dots, |N|\}$. A game (N, v) is convex if and only if all marginal vectors of that game belong to the core. An MCP-game is in general not convex, but we will point out several specific marginal vectors of an MCP-game (N, w) that are elements of $\mathcal{C}(w)$.

Lemma 3.3. *Let (N, w) be the MCP-game corresponding to an MCP-situation $(N, q, p) \in \mathcal{M}$.*

- (i) *The marginal vector corresponding to any $\sigma^1 \in \Pi(N)$ with $\sigma^1(1) = n$, is a core element.*
- (ii) *The marginal vector corresponding to any $\sigma^2 \in \Pi(N)$ with $\sigma^2(1) = n-1$ and $\sigma^2(2) = n$, is a core element.*

Proof. (i) Using (4) the marginal vector corresponding to σ^1 is given by

$$m^{\sigma^1}(w) = (w(\{1, n\}), w(\{2, n\}), \dots, w(\{n-1, n\}), 0),$$

which corresponds to the Direct Price solution.

(ii) Using (4) the marginal vector corresponding to σ^2 is given by

$$m^{\sigma^2}(w) = (w(\{1, n\}), w(\{2, n\}), \dots, w(\{n-2, n\}), 0, w(\{n-1, n\})).$$

Thus for all $S \in 2^N$

$$\sum_{i \in S} m_i^{\sigma^2}(w) = \begin{cases} w(\{n-1, n\}) + \sum_{i \in S \setminus \{n-1\}} w(\{i, n\}) & \text{if } n \in S, \\ \sum_{i \in S \setminus \{n-1\}} w(\{i, n\}) & \text{otherwise.} \end{cases}$$

If $n \in S$, then $w(\{n-1, n\}) + \sum_{i \in S \setminus \{n-1\}} w(i, n) \geq \sum_{i \in S} w(i, n) \geq w(S)$. If $n \notin S$, then $\sum_{i \in S \setminus \{n-1\}} w(\{i, n\}) \geq \sum_{i \in S \setminus \{n-1\}} w(\{i, s\}) = w(S)$, with $s = \max\{i : i \in S\}$. Hence, we have that $\sum_{i \in S} m_i^{\sigma^2}(w) \geq w(S)$ for all $S \in 2^N$. □

The next Theorem shows that if there are two players with lowest unit price, the core of the MCP-game consists of one single point, the Direct Price solution. Moreover this is the only class of MCP-situations for which the core of the MCP-game equals the Direct Price solution.

Theorem 3.4. Let $(N, q, p) \in \mathcal{M}$ and let (N, w) be the corresponding MCP-game. Then

$$\mathcal{C}(w) = \{\delta(N, q, p)\} \text{ if and only if } p_{n-1} = p_n.$$

Proof. Let $\mathcal{C}(w) = \{\delta(N, q, p)\}$. Then, according to Lemma 3.3, $m^{\sigma_1}(w) = m^{\sigma_2}(w)$ with σ_1 and σ_2 as defined there. Hence $0 = w(\{n-1, n\}) = (p_{n-1} - p_n)q_{n-1}$ and thus $p_{n-1} = p_n$.

Let $p_{n-1} = p_n$. This implies that for all $i \in \{1, \dots, n-2\}$, $w(\{i, n-1\}) = w(\{i, n\})$ and $w(\{n-1, n\}) = 0$. By Theorem 3.2 it is sufficient to show that $x \in \mathcal{C}(w)$ implies that $x = \delta(N, q, p)$. Let $x \in \mathcal{C}(w)$. Then

$$\begin{aligned} 0 = w(\{n\}) &\leq x_n \leq w(N) - w(N \setminus \{n\}) = \sum_{j=1}^{n-1} w(\{j, n\}) - \sum_{j=1}^{n-2} w(\{j, n-1\}) \\ &= w(\{n-1, n\}) \\ &= 0. \end{aligned}$$

Hence $x_n = 0 = \delta_n(N, q, p)$.

Similarly we have for all $i \in N \setminus \{n\}$,

$$\begin{aligned} x_i &\leq w(N) - w(N \setminus \{i\}) = \sum_{j \in N} w(\{j, n\}) - \sum_{j \in N \setminus \{i\}} w(\{j, n\}) \\ &= w(\{i, n\}) \end{aligned}$$

and hence

$$w(N) = \sum_{i=1}^{n-1} x_i \leq \sum_{i=1}^{n-1} w(\{i, n\}) = w(N).$$

Hence for all $i \in N \setminus \{n\}$, $x_i = w(\{i, n\}) = \delta_i(N, q, p)$. □

The Direct Price solution has the appealing property that the players in N do not have a monetary incentive to reject a new agent j , if he wants to join N .

Theorem 3.5. Let $(N, q, p) \in \mathcal{M}$ and consider a player $j \notin N$ with demand q_j . Then for all $i \in N$

$$\delta_i(N, q, p) \leq \delta_i(N \cup \{j\}, \bar{q}, p),$$

where $\bar{q} = \{q_\ell\}_{\ell \in N \cup \{j\}}$.

Proof. Let $i \in N$. There are two possibilities: either $q_j > q_n$ or $q_j \leq q_n$. In the first case $p_j \leq p_n$ and $\delta_i(N \cup \{j\}, \bar{q}, p) = (p_i - p_j)q_i \geq (p_i - p_n)q_i = \delta_i(N, q, p)$. In the second case, obviously $\delta_i(N \cup \{j\}, \bar{q}, p) = \delta_i(N, q, p)$. □

In fact, from Theorem 3.5 and Theorem 3.4, one readily derives that the Direct Price solution leads to a population monotonic allocation scheme (Sprumont, 1990).

Example 3.6. Reconsider Example 2.1 with $\delta(N, q, p) = (10, 8, 4, 0)$. This allocation is quite extreme as all bilateral profits go to the ‘smaller’ players. A less extreme allocation x would be to let the two players i and n share the obtained cost savings equally, leading to

$$\begin{aligned} x &= \left(\frac{1}{2}w(\{1, 4\}), \frac{1}{2}w(\{2, 4\}), \frac{1}{2}w(\{3, 4\}), \frac{1}{2}w(\{1, 4\}) + \frac{1}{2}w(\{2, 4\}) + \frac{1}{2}w(\{3, 4\}) \right) \\ &= (5, 4, 2, 11). \end{aligned}$$

This allocation is not a core element. It can be seen, however, that all core elements can be obtained if we allow differentiation with respect to the sharing level among pairs of players. For example for the barycenter $y = (8\frac{5}{9}, 6, 2\frac{2}{9}, 5\frac{2}{9})$ of the core, we have that

$$y = \left(\frac{77}{90}w(\{1,4\}), \frac{3}{4}w(\{2,4\}), \frac{5}{9}w(\{3,4\}), \frac{13}{90}w(\{1,4\}) + \frac{1}{4}w(\{2,4\}) + \frac{4}{9}w(\{3,4\}) \right).$$

In the tetrahedron of Figure 1, the dotted box represents all possible sharing methods. The polyhedron inside that box is the core of the game. $\blacklozenge$

Based on the ideas of Example 3.6 we define a *sharing rule* δ^λ on $\mathcal{M}$. Define Δ^N as the set of all share vectors $\lambda \in \mathbb{R}^N$ with for all $i \in N$, $\lambda_i \in [0, 1]$. Here λ_i can be interpreted as the share player i obtains of his bilateral profits with n .

Definition 2. For all $(N, q, p) \in \mathcal{M}$ and all $\lambda \in \Delta^N$

$$\delta_i^\lambda(N, q, p) = \begin{cases} \lambda_i(p_i - p_n)q_i & \text{if } i \in N \setminus \{n\}, \\ \sum_{j \in N} (1 - \lambda_j)(p_j - p_n)q_j & \text{if } i = n. \end{cases}$$

Note that the choice of λ_n does not affect the allocation proposed by δ^λ . Also note that $\delta^{\lambda^*} = \delta$, for $\lambda^* \in \Delta^N$ such that $\lambda_i^* = 1$ for all $i \in N \setminus \{n\}$. Next, consider the *Shared Set*, defined by

$$\mathcal{S}(N, q, p) = \{\delta^\lambda(N, q, p) | \lambda \in \Delta^N\},$$

as all possible allocations generated by a sharing rule. The next proposition shows that all core elements can be generated via sharing rules.

Proposition 3.7. Let $(N, q, p) \in \mathcal{M}$ and let (N, w) be the corresponding MCP-game. Then

$$\mathcal{C}(w) \subset \mathcal{S}(N, q, p).$$

Proof. Let $x \in \mathcal{C}(w)$. Define $\lambda \in \mathbb{R}^N$ by $\lambda_i = \frac{x_i}{(p_i - p_n)q_i}$, $i \in \{1, \dots, n-1\}$ and $\lambda_n = 1$. Since, for all $i \in N \setminus \{n\}$, $0 \leq x_i \leq w(N) - w(N \setminus \{i\}) \leq w(\{i, n\})$, we have $\lambda \in \Delta^N$. Obviously, $\delta_i^\lambda(N, q, p) = x_i$ for all $i \in N$ and hence $x \in \mathcal{S}(N, q, p)$. $\square$

4 The Nucleolus of a game with a non-empty core

Let (N, v) be a TU-game. We define $C^N = 2^N \setminus \{\emptyset, N\}$ as the collection of proper subsets of N . The imputation set is given by all individually rational and efficient vectors, i.e.

$$I(v) = \left\{ x \in \mathbb{R}^N \mid \sum_{j \in N} x_j = v(N), x_i \geq v(\{i\}) \text{ for all } i \in N \right\}.$$

Let $E(S, x) = v(S) - \sum_{j \in S} x_j$ be the excess of coalition $S \in 2^N$ with respect to an imputation $x \in I(v)$. Define $\theta(x) \in \mathbb{R}^{|2^N|}$ as the vector of excesses arranged in non-increasing order. The nucleolus $nuc(v)$ (Schmeidler, 1969) of a TU-game (N, v) with $I(v) \neq \emptyset$, is the unique imputation for which θ is lexicographically the smallest. Thus for all $x \in I(v)$,

$$\theta(nuc(v)) \leq_L \theta(x),$$

where $x \leq_L y$ denotes that x is lexicographically smaller than y .⁴

The nucleolus minimizes the maximum dissatisfaction level of a coalition and it is a stable solution concept, i.e. if $\mathcal{C}(v) \neq \emptyset$, then $nuc(v) \in \mathcal{C}(v)$.

Finding the nucleolus of a TU-game is not easy, in general it takes $\mathcal{O}(|N| \times 2^{|N|})$ steps. To check whether a certain imputation $x \in I(v)$ is the nucleolus of the game, one can use the following criterium, due to Kohlberg (1971).

Let $\mathcal{B}_1(x) = \{T \in C^N \mid E(T, x) \geq E(S, x) \text{ for all } S \in C^N\}$ be the collection of coalitions with highest excess. Recursively define for $r = 2, 3, \dots$

$$\mathcal{B}_r(x) = \left\{ T \in C^N \mid T \notin \bigcup_{k=1}^{r-1} \mathcal{B}_k(x), E(T, x) \geq E(S, x) \right. \\ \left. \text{for all } S \in C^N \text{ with } S \notin \bigcup_{k=1}^{r-1} \mathcal{B}_k(x) \right\}.$$

Let $t \in \mathbb{N}$ be such that $\mathcal{B}_t(x) \neq \emptyset$ and $\mathcal{B}_{t+1}(x) = \emptyset$.

For $r \in \{1, \dots, t\}$ define

$$\overline{\mathcal{B}}_r(x) = \bigcup_{k=1}^r \mathcal{B}_k(x).$$

Theorem 4.1. (Kohlberg, 1971) *Let (N, v) be a TU-game with $\mathcal{C}(v) \neq \emptyset$ and let $x \in I(v)$. Then, $x = nuc(v)$ if and only if $\overline{\mathcal{B}}_r(x)$ is balanced for all $r \in \{1, \dots, t\}$.*

Here $\mathcal{B} \subset C^N$ is balanced if there exists a vector $\lambda \in \mathbb{R}^{C^N}$ such that $\lambda_S > 0$ for all $S \in \mathcal{B}$ and $\lambda_S = 0$ for all $S \notin \mathcal{B}$, such that $\sum_{S \in \mathcal{B}} \lambda_S \mathbf{1}^S = \mathbf{1}^N$, with $\mathbf{1}^S$ the unit vector in $\mathbb{R}^N$ of coalition $S \in C^N$.

From this theorem Kohlberg (1972) derived a procedure to calculate the nucleolus of a cooperative game, by solving one large linear program. Due to e.g. Potters et al. (1996) there is a faster method to determine the nucleolus. The rough idea is based on efficiency of the nucleolus. If one has found two disjoint coalitions S and T that belong to $\mathcal{B}_1(nuc(v))$ and one also knows

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⁴For $x, y \in \mathbb{R}^t$ we have $x \leq_L y$, if $x = y$ or there exists an $s \in \{1, \dots, t\}$ such $x_k = y_k$ for all $k \in \{1, \dots, s-1\}$ and $x_s < y_s$.

their corresponding excesses, then, one also knows the excesses that belong to $N \setminus S$, $N \setminus T$, $S \cup T$ and $N \setminus (S \cup T)$. As a follow up Reijnierse and Potters (1998) provide a sufficient condition for a collection of coalitions to determine the nucleolus of a game. This condition inspired us to formulate a new alternative criterium to check whether a certain imputation is the nucleolus of the game.

For a collection $\mathcal{B} \subset C^N$ define

$$H(\mathcal{B}) = \{S \in 2^N \mid \mathbf{1}^S \in \text{span}(\mathbf{1}^N, [\mathbf{1}^T]_{T \in \mathcal{B}})\},$$

where span denotes the linear hull of the unit vectors.

An alternative way of finding $H(\mathcal{B})$ is using $\mathcal{H}$ -closed sets. A collection of coalitions $\mathcal{W} \subset 2^N$ is $\mathcal{H}$ -closed, if

- (i) $N \in \mathcal{W}$
- (ii) for all $A \in \mathcal{W}$, $N \setminus A \in \mathcal{W}$
- (iii) for all $A, B \in \mathcal{W}$ with $A \cap B = \emptyset$, $A \cup B \in \mathcal{W}$.

Then, $H(\mathcal{B})$ is the smallest $\mathcal{H}$ -closed set containing $\mathcal{B}$.

The following theorem gives an explicit alternative characterization of the nucleolus of a game with a nonempty core.

Theorem 4.2. *Let (N, v) be a TU-game with $\mathcal{C}(v) \neq \emptyset$. Let $x \in I(v)$. Then $x = \text{nuc}(v)$ if and only if there exists a sequence $\mathcal{D}_1, \mathcal{D}_2, \dots, \mathcal{D}_\tau$ of nonempty subcollections of C^N with the following properties*

- (i) for all $r \in \{1, \dots, \tau\}$ the collection $\overline{\mathcal{D}}_r = \bigcup_{k=1}^r \mathcal{D}_k$ is balanced,
- (ii) there exists a sequence of real numbers $\gamma_1, \gamma_2, \dots, \gamma_\tau$ such that $E(S, x) = \gamma_r$ for every $S \in \mathcal{D}_r$ and $\gamma_1 \geq \gamma_2 \geq \dots \geq \gamma_\tau$.
- (iii) for all $S \in C^N \setminus \overline{\mathcal{D}}_\tau$ we have $S \in H(\{T \in \overline{\mathcal{D}}_\tau : E(T, x) \geq E(S, x)\})$.

Proof. “only if”-part: Let $x = \text{nuc}(v)$ and $t \in \mathbb{N}$ is such that $\mathcal{B}_t(\text{nuc}(v)) \neq \emptyset$ and $\mathcal{B}_{t+1}(\text{nuc}(v)) = \emptyset$. Set $\tau = t$ and $\mathcal{D}_r = \mathcal{B}_r(\text{nuc}(v))$ for all $r \in \{1, \dots, \tau\}$. Then, by Theorem 4.1 it is obvious that the sequence satisfies Properties (i) and (ii) and since $\overline{\mathcal{D}}_\tau = C^N$ also Property (iii) is satisfied.

“if”-part: Let $x \in I(v)$ and let $t \in \mathbb{N}$ be such that $\mathcal{B}_t(x) \neq \emptyset$ and $\mathcal{B}_{t+1}(x) = \emptyset$. Let the collections $\mathcal{D}_1, \dots, \mathcal{D}_\tau$ and the numbers $\gamma_1, \gamma_2, \dots, \gamma_\tau$ be such that Properties (i), (ii) and (iii) are satisfied. For all $r \in \{1, \dots, t\}$, define $\kappa_r = E(S, x)$ for $S \in \mathcal{B}_r(x)$. Obviously $\kappa_1 > \kappa_2 > \dots > \kappa_t$. We will show that x meets the Kohlberg criterium, proceeding by means of induction.

If $\kappa_1 = \gamma_\tau$, define $\ell = \tau$. Otherwise, $\gamma_r = \kappa_1 > \gamma_{r+1}$ for some $r \in \{1, \dots, \tau - 1\}$, define $\ell = r$. Then, $\kappa_1 = \gamma_1 = \dots = \gamma_\ell$ and $\overline{\mathcal{D}}_\ell \subset \mathcal{B}_1(x)$. Hence

$$\mathcal{B}_1(x) = \overline{\mathcal{D}}_\ell \cup \{S \notin \overline{\mathcal{D}}_\tau \mid E(S, x) = \kappa_1\}.$$

For any $S \in \mathcal{B}_1(x)$ with $S \notin \overline{\mathcal{D}}_\tau$

$$\{T \in \overline{\mathcal{D}}_\tau : E(T, x) \geq E(S, x)\} = \overline{\mathcal{D}}_\ell.$$

Hence, by Property (iii), $S \in H(\overline{\mathcal{D}}_\ell)$.

In order to prove that $\mathcal{B}_1(x)$ is balanced, it is sufficient to show that for any $S \in \mathcal{B}_1(x)$ with $S \notin \overline{\mathcal{D}}_\tau$, $\overline{\mathcal{D}}_\ell \cup \{S\}$ is balanced. Since $S \in H(\overline{\mathcal{D}}_\ell)$,

$$\mathbf{1}^S = \mu_N \mathbf{1}^N + \sum_{T \in \overline{\mathcal{D}}_\ell} \mu_T \mathbf{1}^T \text{ for some } \mu \in \mathbb{R}^{2^N \setminus \{\emptyset\}}.$$

Moreover, since $\overline{\mathcal{D}}_\ell$ is balanced there exists a vector $\lambda \in \mathbb{R}^{C^N}$ with $\lambda_T > 0$ for all $T \in \overline{\mathcal{D}}_\ell$ such that

$$\sum_{T \in \overline{\mathcal{D}}_\ell} \lambda_T \mathbf{1}^T = \mathbf{1}^N,$$

we have

$$\mathbf{1}^S = \sum_{T \in \overline{\mathcal{D}}_\ell} (\mu_N \lambda_T + \mu_T) \mathbf{1}^T$$

and for any $\alpha \in \mathbb{R}$,

$$\sum_{T \in \overline{\mathcal{D}}_\ell} \lambda_T \mathbf{1}^T + \alpha(\mathbf{1}^S - \sum_{T \in \overline{\mathcal{D}}_\ell} (\mu_N \lambda_T + \mu_T) \mathbf{1}^T) = \mathbf{1}^N,$$

i.e.,

$$\alpha \mathbf{1}^S + \sum_{T \in \overline{\mathcal{D}}_\ell} (\lambda_T - \alpha[\mu_N \lambda_T + \mu_T]) \mathbf{1}^T = \mathbf{1}^N.$$

Choosing $\alpha > 0$ small enough, we derive that $\overline{\mathcal{D}}_\ell \cup \{S\}$ is balanced.

Proceeding by induction we assume that $\overline{\mathcal{B}}_k(x)$ is balanced for some $k \in \{2, \dots, \tau - 1\}$. We will show that $\overline{\mathcal{B}}_{k+1}(x)$ is balanced too. If $\kappa_{k+1} \leq \gamma_\tau$, define $\ell = \tau$. Otherwise, $\gamma_r \geq \kappa_{k+1} > \gamma_{r+1}$ for some $r \in \{1, \dots, \tau - 1\}$. In this case, define $\ell = r$. Then, $\overline{\mathcal{D}}_\ell \subset \overline{\mathcal{B}}_{k+1}(x)$. Hence,

$$\overline{\mathcal{B}}_{k+1}(x) = \overline{\mathcal{D}}_\ell \cup \{S \notin \overline{\mathcal{D}}_\tau \mid E(S, x) = \kappa_{k+1}\}.$$

It is sufficient to prove that for any $S \in \mathcal{B}_{k+1}(x)$ with $S \notin \overline{\mathcal{D}}_\tau$, $\overline{\mathcal{D}}_\ell \cup \{S\}$ is balanced. Since $S \in H(\overline{\mathcal{D}}_\ell)$ and since $\overline{\mathcal{D}}_\ell$ is balanced, we can use the same argument as for the induction base. There exists a vector $\mu \in \mathbb{R}^{2^N \setminus \{\emptyset\}}$, and a vector $\lambda \in \mathbb{R}^{C^N}$ with $\lambda_T > 0$ for all $T \in \overline{\mathcal{D}}_\ell$ such that for any α

$$\alpha \mathbf{1}^S + \sum_{T \in \overline{\mathcal{D}}_\ell} (\lambda_T - \alpha[\mu_N \lambda_T + \mu_T]) \mathbf{1}^T = \mathbf{1}^N.$$

Again, choosing $\alpha > 0$ small enough, we derive that $\overline{\mathcal{D}}_\ell \cup \{S\}$ is balanced. $\square$

5 The Nucleolus of MCP-games

This section explains how one can compute the nucleolus of an MCP-game. We start with the following easy observation.

Proposition 5.1. *Let (N, w) be an MCP-game corresponding to MCP-situation $(N, q, p) \in \mathcal{M}$. Then,*

$$nuc(w) = \delta(N, q, p) \text{ if and only if } p_{n-1} = p_n.$$

Proof. If $p_{n-1} = p_n$, then by Theorem 3.4, $\mathcal{C}(w) = \{\delta(N, q, p)\}$ and hence $nuc(w) = \delta(N, q, p)$.

Let $nuc(w) = \delta(N, q, p)$. Then, for all $S \in C^N$ such that $n \in S$,

$$E(S, nuc(w)) = w(S) - \sum_{i \in S} \delta_i(N, q, p) = \sum_{i \in S} q_i(p_i - p_n) - \sum_{i \in S} q_i(p_i - p_n) = 0.$$

Since $\{S \in 2^N \setminus \{N, \emptyset\} | n \in S\}$ is not balanced, Theorem 4.1 implies that there is an $S \subset N \setminus \{n\}$ with

$$\begin{aligned} 0 = E(S, nuc(w)) &= w(S) - \sum_{j \in S} w(\{j, n\}) = \sum_{j \in S} w(\{j, s\}) - \sum_{j \in S} w(\{j, n\}) \\ &= \sum_{j \in S} q_j(p_j - p_s) - \sum_{j \in S} q_j(p_j - p_n), \end{aligned}$$

with $s = \max\{i : i \in S\}$. Hence, $p_s = p_n$ and consequently $p_{n-1} = p_n$. $\square$

Next we provide an explicit relation between the Direct Price solution of an MCP-situation and the nucleolus of the corresponding MCP-game. For this we use a so-called *nucleolus-determinant*. First, we will simply state the recursion to find such a nucleolus-determinant. This recursion is rather technical. Then, an explicit expression for the nucleolus is provided. The interpretation of this nucleolus-determinant and the nucleolus itself are given after the proof. Finally, by means of an example, we will explain how one can use this recursion and how this leads to an algorithm of polynomial time to find the nucleolus of an MCP-game.

Denote $p_S = p_s$ if $s = \max\{i : i \in S\}$.

Nucleolus-Determinant-recursion

Input: $(N, q, p) \in \mathcal{M}$ and the corresponding MCP-game (N, w) .

Initialization:

$$\begin{aligned} \text{Set} \quad \mathcal{F}_1 &= \{S | S \subset N \setminus \{n\}, S \neq \emptyset\}, \\ \tilde{\mathcal{F}}_1 &= \left\{ T \in \mathcal{F}_1 \mid \frac{(p_n - p_T) \sum_{j \in T} q_j}{1 + |T|} \geq \frac{(p_n - p_S) \sum_{j \in S} q_j}{1 + |S|} \text{ for all } S \in \mathcal{F}_1 \right\}, \\ \text{choose} \quad T_1 &\in \tilde{\mathcal{F}}_1, \\ \text{set} \quad e_1 &= \frac{(p_n - p_{T_1}) \sum_{j \in T_1} q_j}{1 + |T_1|}, \\ S_1 &= T_1 \\ \text{and} \quad r &= 2. \end{aligned}$$

Recursion:

If $\bigcup_{k=1}^{r-1} S_k \neq N \setminus \{n\}$,

define $f_r : \mathcal{F}_r \rightarrow (-\infty, 0]$ by , $f_r(S) = \frac{(p_n - p_S) \sum_{j \in S} q_j - \sum_{k=1}^{r-1} |S \cap S_k| e_k}{1 + \left| S \setminus \bigcup_{k=1}^{r-1} S_k \right|}, S \in \mathcal{F}_r,$

set

$$\mathcal{F}_r = \{S \mid S \subset N \setminus \{n\}, S \setminus \bigcup_{k=1}^{r-1} S_k \neq \emptyset\},$$

$$\tilde{\mathcal{F}}_r = \{T \in \mathcal{F}_r \mid f_r(T) \geq f_r(S) \text{ for all } S \in \mathcal{F}_r\},$$

choose

$$T_r \in \tilde{\mathcal{F}}_r,$$

set

$$e_r = f_r(T_r),$$

$$S_r = T_r \setminus \bigcup_{k=1}^{r-1} S_k$$

and

$$r = r + 1.$$

Otherwise: set

$$\tau = r - 1. \text{ STOP}$$

Output: A nucleolus-determinant $\{(S_1, e_1), (S_2, e_2), \dots, (S_\tau, e_\tau)\}$, with $(S_1, \dots, S_\tau)$ a partition of $N \setminus \{n\}$.

Theorem 5.2. Let $(N, q, p) \in \mathcal{M}$ and let (N, w) be the corresponding MCP-game.

Let $\{(S_1, e_1), (S_2, e_2), \dots, (S_\tau, e_\tau)\}$ be a nucleolus-determinant. Then, for all $r \in \{1, \dots, \tau\}$ and all $i \in S_r$

$$nuc_i(w) = q_i(p_i - p_n) + e_r$$

$$nuc_n(w) = \sum_{k=1}^{\tau} -|S_k| e_k.$$

In the proof of Theorem 5.2 we use the following lemma.

Lemma 5.3. Let $(N, q, p) \in \mathcal{M}$ and let (N, w) be the corresponding MCP-game. Let $i \in N$ and $S \in 2^N$ be such that $n \in S \subset N \setminus \{i\}$ and let $x \in I(v)$ be such that $x_i \leq w(N) - w(N \setminus \{i\})$. Then,

$$E(S \cup \{i\}, x) \geq E(S, x).$$

Proof. Since $x_i \leq w(N) - w(N \setminus \{i\})$,

$$\begin{aligned} E(S \cup \{i\}, x) - E(S, x) &= w(S \cup \{i\}) - w(S) - x_i \\ &= q_i(p_i - p_n) - x_i \\ &\geq q_i(p_i - p_n) - w(N) + w(N \setminus \{i\}) \\ &= 0. \end{aligned}$$

□

Proof of Theorem 5.2. For all $r = 1, \dots, \tau$, and for all $i \in S_r$ we set

$$\begin{cases} x_i &= q_i(p_i - p_n) + e_r \\ x_n &= \sum_{k=1}^{\tau} -|S_k| e_k. \end{cases}$$

For all $r \in \{1, \dots, \tau\}$, let $T_r \in \tilde{\mathcal{F}}_r$ be the coalition the recursion chose. Define for all $r \in \{1, \dots, \tau\}$

$$\mathcal{D}_r = T_r \cup \{N \setminus \{i\}\}_{i \in S_r}.$$

We will show that the sequence $\mathcal{D}_1, \dots, \mathcal{D}_\tau$ satisfies the three properties of Theorem 4.2.

Clearly, we have that $\overline{\mathcal{D}}_r$ is balanced for all $r \in \{1, \dots, \tau\}$. Hence the sequence $\mathcal{D}_1, \dots, \mathcal{D}_\tau$ satisfies Property (i) of Theorem 4.2.

Regarding Property (ii), we first prove that for all $r \in \{1, \dots, \tau\}$, $E(S, x) = e_r$ for all $S \in \mathcal{D}_r$. Take $r \in \{1, \dots, \tau\}$. Then for all $i \in S_r$

$$\begin{aligned} E(N \setminus \{i\}, x) &= w(N \setminus \{i\}) - \sum_{j \in N \setminus \{i\}} x_j \\ &= \sum_{j \in N \setminus \{i\}} q_j(p_j - p_n) - (w(N) - x_i) \\ &= \sum_{j \in N \setminus \{i\}} q_j(p_j - p_n) - \sum_{j \in N} q_i(p_i - p_n) + x_i \\ &= -q_i(p_i - p_n) + q_i(p_i - p_n) + e_r \\ &= e_r. \end{aligned}$$

For T_r itself it holds that

$$\begin{aligned} E(T_r, x) &= w(T_r) - \sum_{j \in T_r} x_j \\ &= \sum_{j \in T_r} q_j(p_j - p_{T_r}) - \sum_{k=1}^r \sum_{j \in T_r \cap S_k} [q_j(p_j - p_n) + e_k] \\ &= \sum_{j \in T_r} q_j(p_n - p_{T_r}) - \sum_{k=1}^r |T_r \cap S_k| e_k \\ &= f_r(T_r)(1 + |T_r \cap S_r|) - |T_r \cap S_r| e_r \\ &= e_r(1 + |S \cap S_r|) - |S \cap S_r| e_r \\ &= e_r. \end{aligned}$$

To finish the proof of Property (ii) it remains to show that $e_1 \geq e_2 \geq \dots \geq e_r$. Note that $T_{r+1} \in \mathcal{F}_r$. Suppose that for some $r \leq \tau - 1$, $e_r < e_{r+1}$. Then by definition of e_r ,

$$f_{r+1}(T_{r+1}) = e_{r+1} > e_r \geq f_r(T_{r+1}).$$

Hence

$$\frac{(p_n - p_{T_{r+1}}) \sum_{j \in T_{r+1}} q_j - \sum_{k=1}^r |T_{r+1} \cap S_k| e_k}{1 + \left| T_{r+1} \setminus \bigcup_{k=1}^r S_k \right|} > \frac{(p_n - p_{T_{r+1}}) \sum_{j \in T_{r+1}} q_j - \sum_{k=1}^{r-1} |T_{r+1} \cap S_k| e_k}{1 + \left| T_{r+1} \setminus \bigcup_{k=1}^{r-1} S_k \right|}. \quad (5)$$

For (5) to hold, we must have that $T_{r+1} \cap S_r \neq \emptyset$. Denote

$$\alpha = (p_n - p_{T_{r+1}}) \sum_{j \in T_{r+1}} q_j - \sum_{k=1}^{r-1} |T_{r+1} \cap S_k| e_k.$$

Then, however, Inequality (5) would imply

$$\begin{aligned}
\frac{\alpha - |T_{r+1} \cap S_r|e_r}{1 + \left| T_r \setminus \bigcup_{k=1}^r S_k \right|} &> \frac{\alpha}{1 + \left| T_{r+1} \setminus \bigcup_{k=1}^{r-1} S_k \right|}, \\
\left(1 + \left| T_{r+1} \setminus \bigcup_{k=1}^{r-1} S_k \right| \right) (\alpha - |T_{r+1} \cap S_r|e_r) &> \left(1 + \left| T_{r+1} \setminus \bigcup_{k=1}^r S_k \right| \right) \alpha, \\
\left(1 + \left| T_{r+1} \setminus \bigcup_{k=1}^{r-1} S_k \right| \right) \alpha - \left(1 + \left| T_{r+1} \setminus \bigcup_{k=1}^{r-1} S_k \right| \right) |T_{r+1} \cap S_r|e_r &> \left(1 + \left| T_{r+1} \setminus \bigcup_{k=1}^r S_k \right| \right) \alpha, \\
|T_{r+1} \cap S_r| \alpha - \left(1 + \left| T_{r+1} \setminus \bigcup_{k=1}^{r-1} S_k \right| \right) |T_{r+1} \cap S_r|e_r &> 0, \\
\alpha - \left(1 + \left| T_{r+1} \setminus \bigcup_{k=1}^{r-1} S_k \right| \right) e_r &> 0, \\
\frac{\alpha}{1 + \left| T_{r+1} \setminus \bigcup_{k=1}^{r-1} S_k \right|} &> e_r, \\
f_r(T_{r+1}) &> e_r,
\end{aligned}$$

establishing a contradiction.

Regarding Property (iii). Let $S \in C^N \setminus \overline{\mathcal{D}}_\tau$. We will prove that $S \in H(\{T \in \overline{\mathcal{D}}_\tau | E(T, x) \geq E(S, x)\})$.

Case 1: $n \in S$

Let $r \in \{1, \dots, \tau\}$ be such that

$$\begin{cases} N \setminus \{j\} \in \mathcal{D}_r \text{ for some } j \in N \setminus S, \\ N \setminus \{i\} \in \mathcal{D}_s \text{ with } r \geq s \text{ for all } i \in N \setminus S. \end{cases}$$

Then, $N \setminus \{i\} \in \overline{\mathcal{D}}_r$ for all $i \in N \setminus S$ and by Lemma 5.3⁵ $E(S, x) \leq E(N \setminus \{j\}, x) \leq E(N \setminus \{i\}, x)$ for all $i \in N \setminus S$. Since $\mathbf{1}^S = (|S| - n + 1)\mathbf{1}^N + \sum_{i \in N \setminus S} \mathbf{1}^{N \setminus \{i\}}$, we have that $S \in H(\{T \in \overline{\mathcal{D}}_\tau : E(T, x) \geq E(S, x)\})$.

Case 2: $n \notin S$.

Let $S \in \mathcal{F}_\tau$. Since $f_\tau(S) \leq e_\tau$, we have

$$\begin{aligned}
E(S, x) &= \sum_{j \in S} q_j(p_n - p_S) - \sum_{k=1}^\tau |S \cap S_k|e_k \\
&= f_\tau(S)(1 + |S \cap S_\tau|) - |S \cap S_\tau|e_\tau \\
&\leq e_\tau(1 + |S \cap S_\tau|) - |S \cap S_\tau|e_\tau \\
&= e_\tau.
\end{aligned}$$

---

⁵Note that since $e_1 \leq 0$, Property (ii) implies that $e_r \leq 0$ for all $r \in \{2, \dots, \tau\}$. Hence, $x_i \leq \delta_i(N, q, p)$, and thus $x_i \leq w(N) - w(N \setminus \{i\})$ for all $i \in N \setminus S$. The condition in Lemma 5.3 is satisfied.

Let $S \notin \mathcal{F}_\tau$ and let $r \in \{1, \dots, \tau - 1\}$ be such that $S \in \mathcal{F}_r \setminus \mathcal{F}_{r+1}$. Then $S \subset \bigcup_{k=1}^r S_k$ and since $f_r(S) \leq e_r$ we have

$$\begin{aligned} E(S, x) &= \sum_{j \in S} q_j(p_n - p_S) - \sum_{k=1}^r |S \cap S_k| e_k \\ &= f_r(S)(1 + |S \cap S_r|) - |S \cap S_r| e_r \\ &\leq e_r(1 + |S \cap S_r|) - |S \cap S_r| e_r \\ &= e_r. \end{aligned}$$

Let $i \in S$. Since $S \subset \bigcup_{k=1}^r S_k$, there is an S_k such that $i \in S_k$ and hence $N \setminus \{i\} \in \mathcal{D}_k$, with $k \leq r$. Moreover $\mathbf{1}^S = |S| \mathbf{1}^N - \sum_{i \in S} \mathbf{1}^{N \setminus \{i\}}$. Hence $S \in H(\{T \in \overline{\mathcal{D}}_\tau : E(T, x) \geq E(S, x)\})$. $\square$

In the proof of Theorem 5.2 we showed that $e_r \leq 0$ for all $r \in \{1, \dots, \tau\}$. Thus the nucleolus of an MCP-game modifies the Direct Price allocation. It can be interpreted in the following way. Every player in S_j should pay a fee $-e_j$ to player n for using his discounted unit price. Thus a nucleolus-determinant puts every player in a certain fee-class and determines the height of those fees. From Proposition 5.1 it is clear that if $p_{n-1} = p_n$, the fee of all players equals zero.

Example 5.4. Consider the following MCP-situation (N, q, p) , with $N = \{1, 2, 3, 4\}$, $q = (10, 45, 100, 250)$ and $p = (10, 8, 7, 5)$. Let (N, w) be the corresponding MCP-game. We are going to compute the nucleolus of the corresponding MCP-game (N, w) , using the nucleolus-determinant recursion.

Table 1: Values of $f_r(S)$

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1, 2\}$	$\{1, 3\}$	$\{2, 3\}$	$\{1, 2, 3\}$
$f_1(S)$	-25	-67.5	-100	-55	$-73\frac{1}{3}$	$-96\frac{2}{3}$	$-77\frac{1}{2}$
$f_2(S)$		$-67\frac{1}{2}$	-100	-70	$-97\frac{1}{2}$	$-96\frac{2}{3}$	-95
$f_3(S)$			-100		$-97\frac{1}{2}$	$-114\frac{1}{4}$	$-108\frac{3}{4}$

We have $\mathcal{F}_1 = \{S | S \subset \{1, 2, 3\}\}$ and the values for $f_1(S)$ are presented in Table 1, in the second row. From this table we can conclude that $\tilde{\mathcal{F}}_1 = \{1\}$, hence $T_1 = \{1\}$, $S_1 = \{1\}$ and $e_1 = -25$. From this step in the recursion one can conclude that: $nuc_1(w) = 50 - 25 = 25$ and

$$\begin{cases} E(\{1\}, nuc(w)) &= -25 \\ E(N \setminus \{1\}, nuc(w)) &= -25 \end{cases}$$

The next step in the nucleolus-determinant recursion is to find $T_2 \in \tilde{\mathcal{F}}_2$, where $\mathcal{F}_2$ equals $\{\{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$. The values for $f_2(S)$, $S \in \mathcal{F}_2$ are also given in Table 1, in the third row. Thus $T_2 = S_2 = \{2\}$ with $e_2 = -67.5$. Hence $nuc_2(w) = 67.5$ and

$$\begin{cases} E(\{2\}, nuc(w)) &= -67.5 \\ E(N \setminus \{2\}, nuc(w)) &= -67.5 \end{cases}$$

Since the nucleolus is efficient, we also know the excesses of $\{1, 2\}$ and $\{3, 4\}$,

$$\begin{cases} E(\{1, 2\}, nuc(w)) &= w(\{1, 2\}) - (nuc_1(w) + nuc_2(w)) = -72.5 \\ E(N \setminus \{1, 2\}, nuc(w)) &= w(\{3, 4\}) - (nuc_3(w) + nuc_4(w)) \\ &= w(\{3, 4\}) - (w(N) - nuc_1(w) - nuc_2(w)) = -92.5 \end{cases}$$

Hence $\mathcal{F}_3 = \{\{3\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$, and $f_3(S)$, $S \in \mathcal{F}_3$, can be found in the last row of Table 1. In this step we find that $T_3 = \{1, 3\}$ thus $S_3 = \{3\}$ with $e_3 = -97.5$. Thus for player 3, $nuc_3(w) = 102.5$. Now $S_1 \cup S_2 \cup S_3 = N \setminus \{4\}$, hence we can stop and determine $nuc_4(w) = -(-25 - 67.5 - 97.5) = 190$.

This procedure still takes quite some time. In the next paragraphs we will again calculate the nucleolus of this example, but now we will use a method that is polynomial in time.

If we look closer at the function $f_1(S)$ we can find a coalition T_1 belonging to $\tilde{\mathcal{F}}_1$ more efficiently. To do so, we need to maximize

$$\frac{\sum_{j \in S} q_j(p_4 - p_S)}{1 + |S|},$$

over all coalitions in $\mathcal{F}_1$. Since p_4 is always less than the unit prices of players 1, 2 or 3, the fraction is negative. Thus it is wise to have p_S and $|S|$ large, but $\sum_{j \in S} q_j$ low. So we need to determine the price setter of S , and then add smaller players to increase $|S|$ and keep $\sum_{j \in S} q_j$ small. If we have that player 3 is the price setter and it is beneficial to add player 2, then adding player 1 must also increase the fraction. Thus T_1 has the following structure,

$$T_1 = \{1, \dots, m, z\},$$

where z is the largest player in the coalition, the price setter, and we add smaller players 1 up to m . If $m = 0$, then $T_1 = \{z\}$. For this example we can make the following combinations.

(m, z)	$(0, 1)$	$(1, 2)$	$(1, 3)$	$(0, 2)$	$(2, 3)$	$(0, 3)$
S	$\{1\}$	$\{1, 2\}$	$\{1, 3\}$	$\{2\}$	$\{1, 2, 3\}$	$\{3\}$

In Table 1 we can see that $f_1(\{1\})$ is maximal.

Also we can speed up the search for a coalition T_2 belonging to $\tilde{\mathcal{F}}_2$. We need to maximize the following fraction

$$\frac{\sum_{j \in S} q_j(p_4 - p_S) - |S \cap S_1|e_1}{1 + |S \setminus S_1|},$$

over all $S \in \mathcal{F}_2$. Either $p_2 = p_S$ or $p_3 = p_S$, since $\{1\} \notin \mathcal{F}_2$. If adding player k not yet in S_1 increases the fraction, then also adding player $j < k$ with $j \notin S_1$ increases the fraction. We only add player 1 if $q_1(p_4 - p_S) < e_1$. Thus T_2 has the following structure

$$T_2 = \left\{ \{1, \dots, m, z\} \cap N \setminus S_1 \right\} \cup \left\{ \{1\} : q_j(p_4 - p_z) > e_1 \right\}.$$

Since $q_1(p_4 - p_3) = -20 > -25$, it is always beneficial to add player 1 to a coalition with player 3 being the price setter. Hence we start by comparing the values of $f_2(\{1, 3\})$ and $f_2(\{1, 2, 3\})$ and find that $f_2(\{1, 2, 3\})$ is the highest. Since $q_1(p_4 - p_2) = -30 < -25$ we do not want to add player 1 with to a coalition with player 2 begin the price setter. Now we need to compare $f_2(\{1, 2, 3\})$ with $f_2(\{2\})$ and we find that $f_2(\{2\})$ is the highest.

Following the same reasoning as for step 2 in the recursion we know that $p_S = p_3$ and T_3 has the following structure

$$T_3 = \{3\} \cup \left\{ \{1\} : q_1(p_4 - p_3) > e_1 \right\} \cup \left\{ \{2\} : q_2(p_4 - p_3) > e_2 \right\}.$$

We only add player 1 if $q_1(p_4 - p_3) > e_1$ and player 2 if $q_2(p_4 - p_3) > e_2$. In the previous round we found that $q_1(p_4 - p_3) > e_1$, hence we add player 1 to player 3 and since $q_2(p_4 - p_3) = -90 < -67.5$ we do not add player 2. Thus $T_3 = \{1, 3\}$. $\blacklozenge$

Example 5.4 shows that one can speed up the Nucleolus-determinant recursion at the point of determining a coalition T_r belonging to $\tilde{\mathcal{F}}_r$. T_1 has the structure $\{1, \dots, m, z\}$, thus at most $(n-1)(n-2)$ numbers need to be compared to find T_1 . Furthermore T_r for $r \in \{2, \dots, \tau\}$ has the structure

$$\{z\} \cup \left\{ \{1, \dots, m\} \cap \left\{ N \setminus \left\{ \bigcup_{k=1}^{r-1} S_k \cup \{n\} \right\} \right\} \right\} \cup \left\{ j \in S_k : q_j(p_n - p_z) - e_k > 0, j < z \right\}_{k=1}^{r-1}.$$

Fix z and look which m maximizes $f_r\left(\{z\} \cup \left\{ \{1, \dots, m\} \cap \left\{ N \setminus \left\{ \bigcup_{k=1}^{r-1} S_k \cup \{n\} \right\} \right\} \right\}\right)$. Then we add players from $\bigcup_{k=1}^{r-1} S_k$ if that further increases f_r . Hence, by comparing at most $(n-1)^2$ numbers, we can find $T_r \in \tilde{\mathcal{F}}_r$ for all $r \in \{2, \dots, \tau\}$ with $\tau \leq n-1$. From these observations we can readily derive that the nucleolus of an MCP-game can be found in polynomial time.

Proposition 5.5. *Let $(N, q, p) \in \mathcal{M}$ be an MCP-situation with (N, w) the corresponding MCP-game. Then, $\text{nuc}(w)$ can be determined in $\mathcal{O}(n^3)$ time.*

6 The Shapley value

In this section we will analyze the Shapley value (Shapley, 1953) of an MCP-game and show that it is a suitable allocation method for an MCP-situation. An explicit context-specific expression for the Shapley value is provided.

For $T \in 2^N \setminus \{\emptyset\}$ the unanimity game (N, u_T) is defined by

$$u_T(S) = \begin{cases} 1 & \text{if } T \subset S, \\ 0 & \text{otherwise,} \end{cases}$$

for all $S \in 2^N$. Every TU-game can be written in a unique way as a linear combination of unanimity games: for all (N, v) there is a unique vector $c \in \mathbb{R}^{2^N \setminus \{\emptyset\}}$, such that

$$v = \sum_{T \in 2^N \setminus \{\emptyset\}} c_T u_T.$$

In case of an MCP-game, the coefficients can be determined quite easily using Theorem 2.2. For all $T \in 2^N$ with $|T| \geq 2$ define $a_T, b_T \in T$ such that $T \cap \{1, \dots, b_T\} = \{a_T, b_T\}$, i.e., player a_T is the player within T with lowest index and player b_T is the player within T with second lowest index. The proof of Theorem 6.1 can be found in the appendix.

Theorem 6.1. *Let (N, w) be an MCP-game corresponding to an MCP-situation $(N, q, p) \in \mathcal{M}$. Let $c \in \mathbb{R}^{2^N \setminus \{\emptyset\}}$ be such that*

$$w = \sum_{T \in 2^N \setminus \{\emptyset\}} c_T u_T.$$

Then, for $|T| = 1$, $c_T = 0$ and for $|T| \geq 2$, we have that

$$c_T = (-1)^{|T|} w(\{a_T, b_T\}). \quad (6)$$

For any TU-game (N, v) with $v = \sum_{T \in 2^N \setminus \{\emptyset\}} c_T u_T$, the unique decomposition into unanimity games, the Shapley value is defined by,

$$\phi_i(v) = \sum_{T \in 2^N \setminus \{\emptyset\}, i \in T} \frac{c_T}{|T|}, \quad (7)$$

for all $i \in N$.

For all $i \in N$, define $PP(i) = \{(k, \ell) | k < \ell, \ell < i, k \in N, \ell \in N\}$ as the collection of preceding pairs, pairs of players with index smaller than i .

Theorem 6.2. *Let (N, w) be an MCP-game corresponding to $(N, q, p) \in \mathcal{M}$. Then for all $i \in N$, the Shapley value is given by*

$$\begin{aligned} \phi_i(w) = & \sum_{k=1}^{i-1} \frac{1}{(n-i+2)(n-i+1)} w(\{k, i\}) + \sum_{m=i+1}^n \frac{1}{(n-m+2)(n-m+1)} w(\{i, m\}) \\ & - \sum_{(k, \ell) \in PP(i)} \frac{2}{(n-\ell+2)(n-\ell+1)(n-\ell)} w(\{k, \ell\}). \end{aligned} \quad (8)$$

The proof of this theorem can be found in the appendix. The following example explains how the Shapley value of an MCP-game can be interpreted.

Example 6.3. Consider the following 5-player MCP-game corresponding to a 5-player MCP-situation with $q = (1, 2, 3, 4, 10)$ and $p_1 = 12, p_2 = 11, p_3 = 9.5, p_4 = 7.5$ and $p_5 = 4$. The table below provides the values of the 2-player coalitions only. The other coalitional values can be easily determined using Theorem 2.2, e.g. $w(N) = w(\{1, 5\}) + w(\{2, 5\}) + w(\{3, 5\}) + w(\{4, 5\}) = 8 + 14 + 16.5 + 14 = 52.5$.

S	$\{1,2\}$	$\{1,3\}$	$\{1,4\}$	$\{1,5\}$	$\{2,3\}$	$\{2,4\}$	$\{2,5\}$	$\{3,4\}$	$\{3,5\}$	$\{4,5\}$
$w(S)$	1	2.5	4.5	8	3	7	14	6	16.5	14

Expression (8) can be split in three parts:

- a positive part due to cost savings with players with a lower index,
- a positive part due to cost savings with players with a larger index,
- a negative part, due to paybacks to pairs of players with a smaller index (the preceding pairs).

For our example, these three parts are represented in the Tables 2, 3 and 4, respectively.

For example, Table 2 indicates that player 3 gets $\frac{1}{12}$ of the cost savings he can obtain with player 1 and also $\frac{1}{12}$ of the cost savings he can make with player 2, while player 5 gets $\frac{1}{2}$ of the cost savings he can make with any of the other players. Note that all fractions are the same per row, i.e. per player under consideration.

The second table indicates e.g. that player 3 gets $\frac{1}{6}$ of the cost savings he can make with player 4 and $\frac{1}{2}$ of the cost savings he can make with player 5. Note that in Table 3 all fractions are the same per column, i.e. per player with larger index. Moreover, the matrix in Table 3 is the transpose of the matrix in Table 2. These two tables represent what the players receive, but

Table 2: Cost savings with players with lower index

Smaller	$\{1\}$	$\{2\}$	$\{3\}$	$\{4\}$	$\{5\}$
Player 1					
Player 2	$\frac{1}{20}w(\{1, 2\})$				
Player 3	$\frac{1}{12}w(\{1, 3\})$	$\frac{1}{12}w(\{2, 3\})$			
Player 4	$\frac{1}{6}w(\{1, 4\})$	$\frac{1}{6}w(\{2, 4\})$	$\frac{1}{6}w(\{3, 4\})$		
Player 5	$\frac{1}{2}w(\{1, 5\})$	$\frac{1}{2}w(\{2, 5\})$	$\frac{1}{2}w(\{3, 5\})$	$\frac{1}{2}w(\{4, 5\})$	

Table 3: Cost savings with players with higher index

Larger	{1}	{2}	{3}	{4}	{5}
Player 1		$\frac{1}{20}w(\{1, 2\})$	$\frac{1}{12}w(\{1, 3\})$	$\frac{1}{6}w(\{1, 4\})$	$\frac{1}{2}w(\{1, 5\})$
Player 2			$\frac{1}{12}w(\{2, 3\})$	$\frac{1}{6}w(\{2, 4\})$	$\frac{1}{2}w(\{2, 5\})$
Player 3				$\frac{1}{6}w(\{3, 4\})$	$\frac{1}{2}w(\{3, 5\})$
Player 4					$\frac{1}{2}w(\{4, 5\})$
Player 5					

Table 4: Paybacks

Negative	{1,2}	{1,3}	{1,4}	{2,3}	{2,4}	{3,4}
Player 1						
Player 2						
Player 3	$-\frac{1}{30}w(\{1, 2\})$					
Player 4	$-\frac{1}{30}w(\{1, 2\})$	$-\frac{1}{12}w(\{1, 3\})$		$-\frac{1}{12}w(\{2, 3\})$		
Player 5	$-\frac{1}{30}w(\{1, 2\})$	$-\frac{1}{12}w(\{1, 3\})$	$-\frac{1}{3}w(\{1, 4\})$	$-\frac{1}{12}w(\{2, 3\})$	$-\frac{1}{3}w(\{2, 4\})$	$-\frac{1}{3}w(\{3, 4\})$

Figure 2: Composition of the Shapley value of Example 6.3

in total we will allocate too much this way. If we sum all entries in the last row in Table 2 and the last column in Table 3, we already obtain $w(N)$. So we need to get back the sum of all remaining entries, given by

$$\frac{1}{10}w(\{1, 2\}) + \frac{1}{6}w(\{1, 3\}) + \frac{1}{6}w(\{2, 3\}) + \frac{1}{3}w(\{1, 4\}) + \frac{1}{3}w(\{2, 4\}) + \frac{1}{3}w(\{3, 4\}). \quad (9)$$

According to the remaining negative part of (8), each two-player component of (9) is paid by all players with higher index than the player with the highest index in the two-person coalition at hand. One can think of this payback as a taxation on the basis of the relative size of the order quantity. E.g. for coalition $\{1, 3\}$, Table 4 indicates that $\frac{1}{6}w(\{1, 3\})$ is paid back equally by players 4 and 5. Then, the Shapley value for player 3 equals

$$\phi_3(w) = \left[\frac{1}{12}w(\{1, 3\}) + \frac{1}{12}w(\{2, 3\}) \right] + \left[\frac{1}{6}w(\{3, 4\}) + \frac{1}{2}w(\{3, 5\}) \right] - \left[\frac{1}{30}w(\{1, 2\}) \right] = 9\frac{27}{40}.$$

In fact

$$\phi(w) = \left(5\frac{1}{120}, 8\frac{7}{15}, 9\frac{27}{40}, 9\frac{17}{40}, 19\frac{37}{40}\right).$$

In Figure 2 the three building blocks of the Shapley value of this example are visualized. The white bars represent the cost savings with players with lower index, the grey bars are the cost savings with players with higher index and the black bars represent paybacks. $\blacklozenge$

In fact Theorem 6.1 and Theorem 6.2 can be generalized to all zero-normalized and nonnegative TU-games with an ordering on the players such that (3) and (4) are satisfied. A game, however, that satisfies (3) and (4) does not need to be an MCP-game. This is illustrated in the following example.

Example 6.4. Consider the 5-person TU-game (N, z) that satisfies (4) and the values of all two-player coalitions given by the following table.

S	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,4\}$	$\{2,4\}$	$\{3,4\}$	$\{1,5\}$	$\{2,5\}$	$\{3,5\}$	$\{4,5\}$
$z(S)$	4	7	5	9	7	6	11	9	7	5

Clearly, (3) is satisfied. Suppose (N, z) is an MCP-game. Thus there are vectors q and p such that (2) holds for all $i, j \in N$ with $i < j$. In particular we find that $p_2 = p_1 - \frac{4}{q_1}$, $p_3 = p_1 - \frac{7}{q_1}$ and $p_4 = p_1 - \frac{9}{q_1}$.

Then, however, $z(\{2,3\}) = q_2(p_2 - p_3) = \frac{3q_2}{q_1}$, $z(\{2,4\}) = \frac{5q_2}{q_1}$ and hence $\frac{z(\{2,4\})}{z(\{2,3\})} = \frac{5}{3}$, which contradicts the fact that $z(\{2,4\}) = 7$ and $z(\{2,3\}) = 5$. $\blacklozenge$

7 Numerical examples

This section takes a numerical look at the Direct Price solution, the nucleolus and the Shapley value as allocation rules for MCP-situations. In practice, organizations that join a purchasing cooperative might not know the exact details of their fellow cooperation members, except the quantity discount of the largest player. Naturally, they like to know what their share in total cost savings would be. Therefore we will simulate for several instances the expected allocation of cost savings to an organization that joins an MCP-situation, according to these three solution concepts.

As input for the simulation we take 5-player MCP-situations with integer-valued order quantities and with fixed q_1 and q_5 . The order quantities q_2 , q_3 and q_4 are unknown, but in between q_1 and q_5 . We restrict to cases where $q_{n-1} < q_n$ and we will use the following unit price function

$$p(d) = 10 + \frac{12}{d^x},$$

with $d \in (0, \infty)$ and $x \in [0.25, 1.25]$. This type of unit price functions adequately represents most quantity discount schemes seen in practice, as explained by Schotanus (2007). We will determine by simulation what a random player with order quantity $q_1 \leq d < q_5$ can expect as his share in the cost savings — according to the Direct Price solution, the nucleolus or the Shapley value — in such an MCP-situation.

One step of the simulation is executed as follows. For fixed q_1 , q_5 and x , the order quantities q_2 , q_3 and q_4 are randomly and simultaneously drawn from a discrete uniform distribution. Then, the Direct price solution, and the nucleolus and the Shapley value of the corresponding MCP-game are calculated. We are interested in the share in cost savings for a player with order quantity $d \in \{q_1, q_1 + 1, q_1 + 2, \dots, q_5 - 1\}$, regardless of the fact that he is player 2, 3 or 4.

Hence, we store these allocations per different value of q_i , independent of the index i . This run is repeated successively. Then, for every $d \in \{q_1, q_1 + 1, \dots, q_5 - 1\}$ we average the stored Direct Price solutions, nucleoli and Shapley values over the number of times they have appeared. This simulation is executed for twelve different instances. First, for $q_1 = 1$ and $q_5 = 20$ and for each of the discount parameters $x = 0.3$, $x = 0.5$, $x = 0.8$ and $x = 1.1$ we derived approximately 1500 MCP-situations. Second, for $q_1 = 1$, $q_5 = 40$ and for each of the discount parameters $x = 0.3$, $x = 0.5$, $x = 0.8$ and $x = 1.1$ we derived approximately 11000 MCP-situations. And third, for $q_1 = 10$ and $q_5 = 50$ we derived approximately 11000 MCP-situations, for $x = 0.3$, $x = 0.5$, $x = 0.8$ and $x = 1.1$.

In Figure 3 one can find the results of the simulations w.r.t. the MCP-situations with $q_1 = 1$, $q_5 = 20$ and the four different discount parameters x . For fixed x , we have 19 different values of d , $1, 2, \dots, 19$, each having an expected share in cost savings according to the Direct Price solution, the nucleolus and the Shapley value. The 19 points belonging to the nucleolus and the 19 points belonging to the Shapley value are connected by the light grey line and dark grey line respectively. The black line connects the 19 points belonging to the Direct Price solution. E.g. a point on the light grey line represents the expected share of cost savings for a player with order quantity d in such an MCP-situation according to the nucleolus.

In Figure 4 one can find the results of the same situation, only in this case $q_5 = 40$ and in Figure 5 one can find the results of a situation with $q_1 = 10$, $q_5 = 50$.

For the situation with $q_1 = 1$ and $q_5 = 40$ we also compared the expected shares in cost savings to players 1 and 5 for 9 different values of x , they can be found in Table 5.

Figure 3: Expected payoffs in a 5-player MCP-game with $q_1 = 1$ and $q_5 = 20$

We can make the following observations. The curve of the nucleolus always lies below the curve of the Direct Price. This is a confirmation of the results from Section 5. The Direct Price solution gives all the bilateral profits to players 1, 2, 3 and 4, while within the nucleolus players 1 up to 4 have to pay player 5 for joining the purchasing cooperative. The curve of the Shapley value does not always lie below the curve of the Direct price. This is due to the fact that the Shapley value need not lie in the core of the game.

In all situations the behavior of the Shapley value is less volatile as the nucleolus and the nucleolus is less volatile as the direct price solution. We can see this from the figures by looking at the range of the curves. The Shapley value curve has the smallest range while the Direct

Figure 4: Expected payoffs in a random 5-player MCP-game with $q_1 = 1$ and $q_5 = 40$

Figure 5: Expected payoffs in a 5-player MCP-game with $q_1 = 10$ and $q_5 = 50$

Table 5: Expected payoffs to player 1 and 5 for $q_1 = 1$ and $q_5 = 40$

x	$\bar{\phi}_1$	$n\bar{uc}_1$	$\bar{\delta}_1$	$\bar{\phi}_5$	$n\bar{uc}_5$	$\bar{\delta}_5$
0.3	6.1073	5.3565	8.0321	13.9026	14.1642	0
0.4	7.0825	6.5056	9.2562	13.3979	13.3889	0
0.5	7.7681	7.5305	10.1026	12.1327	11.7777	0
0.6	8.2865	8.3917	10.6879	10.6323	10.2442	0
0.7	8.6152	9.1919	11.0927	9.0905	8.4306	0
0.8	8.8631	9.8080	11.3726	7.7002	6.9363	0
0.9	9.0455	10.3163	11.5662	6.4195	5.5601	0
1.0	9.1663	10.7434	11.7000	5.2371	4.2668	0
1.1	9.2664	11.0067	11.7925	4.4524	3.5057	0

Price solution has the largest.

We continue by comparing the nucleolus with the Shapley value. In all three situations, the players with order quantities close to q_5 are better off with the Shapley value and the players with somewhat smaller order quantities are better off with the nucleolus. In general we can

conclude that, although the nucleolus and the Shapley value are different game theoretic solution concepts, for MCP-situations their behavior with respect to the input parameters of the model, is quite similar and the expected differences for a player are small.

If we compare Figure 3 with Figure 4 we can see that the size of player 5 does not have much effect on the differences between the allocations of the nucleolus and the Shapley value.

In all three figures we can see that increasing the discount parameter x , makes the three solution concepts come closer to each other. This is partly due to the fact that the total savings obtained decreases as x increases. Also, for larger x , the threshold for players to prefer the Shapley value over the nucleolus tends to decrease. For players with order quantities close to q_1 the nucleolus becomes more attractive as x increases.

From Table 5 we can draw a similar conclusion as above. For larger x , the smallest player has a better position than player 5. In case of $x = 1.1$ ($q_1 = 1$ and $q_5 = 40$), the expected share in total cost savings of player 1 equals 36 and 43 percent, for the Shapley value and nucleolus respectively. For player 5, these expected shares equal 16 percent (Shapley) and 12 percent (nucleolus). While for the instance with $x = 0.3$, player 1's expected shares are 12 percent (Shapley) and 11 percent (nucleolus), and for player 5 the expected shares for both the Shapley value and the nucleolus equal 26 percent.

References

- Anand, K. and Aron, R. (2003). Group buying on the web: A comparison of price-discovery mechanisms. *Management Science*, 49:1546–1562.
- Blomqvist, K., Kylheiko, K., and Virolainen, V. (2002). Filling a gap in traditional transaction cost economics:: Towards transaction benefits-based analysis. *International Journal of Production Economics*, 79:1–14.
- Cruijssen, F., Cools, M., and Dullaert, W. (2007). Horizontal cooperation in logistics: Opportunities and impediments. *Transportation Research Part E: Logistics and Transportation Review*, 43:129–142.
- Granot, D. and Sošić, G. (2005). Formation of alliances in internet-based supply exchanges. *Management Science*, 51:92–105.
- Heijboer, G. (2003). *Mathematical and statistical analysis of initial purchasing decisions*. PhD thesis, University of Twente.
- Hsu, C. and Li, H. (2009). An integrated plant capacity and production planning model for high-tech manufacturing firms with economies of scale. *International Journal of Production Economics*, 118:486–500.
- Keskinocak, P. and Savaseneril, S. (2008). Collaborative procurement among competing buyers. *Naval Research Logistics*, 55:516–540.
- Kohlberg, E. (1971). On the nucleolus of a characteristic function game. *SIAM Journal on Applied Mathematics*, 20:62–66.
- Kohlberg, E. (1972). The nucleolus as a solution of a minimization problem. *SIAM Journal on Applied Mathematics*, 23:34–39.
- Leng, M. and Parlar, M. (2010). Analytic solution for the nucleolus of a three-player cooperative game. *Naval Research Logistics*, 57:667–672.

- Monahan, J. (1984). A quantity discount pricing model to increase vendor profits. *Management Science*, 30:720–726.
- Nagarajan, M., Sošić, G., and Zhang, H. (2010). Stable group purchasing organizations. Working paper, University of British Columbia.
- Nollet, J. and Beaulieu, M. (2003). The development of group purchasing: an empirical study in the healthcare sector. *Journal of Purchasing and Supply Management*, 10:3–10.
- van den Nouweland, A., Borm, P., Golstein Brouwers van, W., Groot Bruinderink, R., and Tijs, S. (1996). A game theoretic approach to problems in telecommunication. *Management Science*, 42:294–303.
- Poblete, P., Munro, J., and Papadakis, T. (2006). The binomial transform and the analysis of skip lists. *Theoretical Computer Science*, 352:136–158.
- Potters, J., Reijnierse, H., and Ansing, M. (1996). Computing the nucleolus by solving a prolonged simplex algorithm. *Mathematics of Operations Research*, 21:757–768.
- Reijnierse, H. and Potters, J. (1998). The $\mathcal{B}$ -nucleolus of tu games. *Games and Economic Behavior*, 24:77–96.
- Schmeidler, D. (1969). The nucleolus of a characteristic function game. *SIAM Journal on Applied Mathematics*, 17:1163–1170.
- Schneider, J., Miller, T., Ohsfeldt, R., Morrissey, M., Zelner, B., and Li, P. (2008). The economics of specialty hospitals. *Medical Care Research and Review*, 65:531–553.
- Schotanus, F. (2007). *Horizontal Cooperative Purchasing*. PhD thesis, University of Twente.
- Shapley, L. (1953). A value for n -person games. *Annals of Mathematics Studies*, 28:307–317.
- Shapley, L. (1967). On balanced sets and cores. *Naval Research Logistics Quarterly*, 14:453–460.
- Sprumont, Y. (1990). Population monotonic allocation schemes for cooperative games with transferable utility. *Games and Economic Behavior*, 2:378–394.
- Tella, E. and Virolainen, V. (2005). Motives behind purchasing consortia. *International Journal of Production Economics*, 93-94:161–168.

A Proofs of Theorems 6.1 and 6.2

In the proof of Theorem 6.1, we use the following lemma.

Lemma A.1. *Let $a \in \mathbb{N}, a \geq 1$. Then*

$$(i) \sum_{j=0}^a \binom{a}{j} (-1)^j = 0,$$

$$(ii) \sum_{j=0}^{a-1} \binom{a}{j} (-1)^j = \begin{cases} -a & \text{if } a \text{ is even,} \\ a & \text{if } a \text{ is odd.} \end{cases}$$

Proof. (i) Clearly,

$$\sum_{j=0}^a \binom{a}{j} (-1)^j = \sum_{j=0}^a \binom{a}{j} (-1)^j (1)^{a-j} = (1-1)^a = 0,$$

where the second equality follows from the binomium of Newton.

(ii) This follows immediately from (i). $\square$

Proof of Theorem 6.1. First note that, for all $T \in 2^N \setminus \{\emptyset\}$ the coefficients in the decomposition of w can be determined recursively in the following way

$$c_T = w(T) - \sum_{S \in 2^N \setminus \{\emptyset\}, S \subsetneq T} c_S. \quad (10)$$

Hence, for $|T| = 1$, $c_T = w(\{i\}) = 0$. Let $|T| = 2$. According to (10)

$$c_T = w(T) = (-1)^{|T|} w(T) = (-1)^{|T|} w(\{a_T, b_T\}).$$

Thus for $|T| = 2$, (6) is valid. To prove (6), we will use induction on the number of players in T .

Assume that for all $|T| \leq k-1$

$$c_T = (-1)^{|T|} w(\{a_T, b_T\}).$$

Let $|T| = k$ and for ease of notation we set $T = \{t_1, t_2, \dots, t_k\}$ such that $q_{t_1} \leq q_{t_2} \leq \dots \leq q_{t_k}$. Clearly $a_T = t_1$ and $b_T = t_2$. Then, by (10) and by Theorem 2.2, we have

$$c_T = \sum_{h=1}^k w(\{t_h, t_k\}) - \sum_{S \in 2^N \setminus \{\emptyset\}, S \subsetneq T} c_S. \quad (11)$$

The last sum in (11) can be rewritten by counting the number of proper sub-coalitions S containing player t_h and t_i with $h < i$ such that $S \cap \{1, 2, \dots, t_i\} = \{t_h, t_i\}$. For $\{t_h, t_i\}$ with $i \geq 3$, this number is $\sum_{j=0}^{k-i} \binom{k-i}{j}$ and for $\{t_1, t_2\}$ this number is $\sum_{j=0}^{k-3} \binom{k-2}{j}$. Hence

$$\begin{aligned} \sum_{\substack{S \in 2^N \setminus \{\emptyset\} \\ S \subsetneq T}} c_S &= \sum_{i=3}^k \sum_{h=1}^{i-1} w(\{t_h, t_i\}) \sum_{j=0}^{k-i} \binom{k-i}{j} (-1)^{j+2} + w(\{t_1, t_2\}) \sum_{j=0}^{k-3} \binom{k-2}{j} (-1)^{j+2} \\ &= \sum_{h=1}^{k-1} w(\{t_h, t_k\}) + \sum_{i=3}^{k-1} \sum_{h=1}^{i-1} w(\{t_h, t_i\}) \sum_{j=0}^{k-i} \binom{k-i}{j} (-1)^j + w(\{t_1, t_2\}) \sum_{j=0}^{k-3} \binom{k-2}{j} (-1)^j \\ &= \sum_{h=1}^{k-1} w(\{t_h, t_k\}) + w(\{t_1, t_2\}) (-1)^{k-1} \end{aligned}$$

The third equality follows from Lemma A.1(i) and A.1(ii). Hence

$$\begin{aligned} c_T &= \sum_{h=1}^{k-1} w(\{t_h, t_k\}) - \sum_{h=1}^{k-1} w(\{t_h, t_k\}) - w(\{t_1, t_2\}) (-1)^{k-1} \\ &= -w(\{t_1, t_2\}) (-1)^{k-1} \\ &= w(\{a_T, b_T\}) (-1)^{|T|}. \end{aligned} \quad \square$$

In the proof of Theorem 6.2 we use the following lemma.

Lemma A.2 (Poblete et al. (2006), Table 1). *Let $a_1, a_2, \dots$ be a sequence with $a_t = \frac{1}{x+t}$ for some $x \in \mathbb{R}_+$. Then, for all $m \in \mathbb{N}$*

$$\sum_{t=0}^m \binom{m}{t} a_t (-1)^t = \frac{(x-1)!m!}{(x+m)!}.$$

Proof of Theorem 6.2. Let $i \in N$ and let $\mathcal{S}$ be the collection of all coalitions T containing i with $|T| \geq 2$. By Theorem 6.1 and (7) we get

$$\begin{aligned} \phi_i(w) &= \sum_{T \in \mathcal{S}} \frac{1}{|T|} (-1)^{|T|} w(\{a_T, b_T\}), \\ &= \sum_{T \in \mathcal{S}_1} \frac{(-1)^{|T|} w(\{a_T, b_T\})}{|T|} + \sum_{T \in \mathcal{S}_2} \frac{(-1)^{|T|} w(\{a_T, b_T\})}{|T|} + \sum_{T \in \mathcal{S}_3} \frac{(-1)^{|T|} w(\{a_T, b_T\})}{|T|}. \end{aligned} \quad (12)$$

where

- (i) $\mathcal{S}_1 = \{T \in \mathcal{S} | i = b_T\}$,
- (ii) $\mathcal{S}_2 = \{T \in \mathcal{S} | i = a_T\}$,
- (iii) $\mathcal{S}_3 = \{T \in \mathcal{S} | i > b_T\}$.

For $T \in \mathcal{S}_1$, setting $|T| = 2 + t$ with $t \in \{0, \dots, n-2\}$ and for one particular $k < i$ at place a_T we have

$$\begin{aligned} \sum_{T \in \mathcal{S}_1: k=a_T} \frac{c_T}{|T|} &= \sum_{t=0}^{n-i} \frac{1}{t+2} \binom{n-i}{t} (-1)^{t+2} w(\{k, i\}) = w(\{k, i\}) \sum_{t=0}^{n-i} \binom{n-i}{t} (-1)^t \frac{1}{t+2} \\ &= w(\{k, i\}) \frac{(2-1)!(n-i)!}{(n-i+2)!} = w(\{k, i\}) \frac{1}{(n-i+2)(n-i+1)}, \end{aligned}$$

where the third equality follows from Lemma A.2. Hence

$$\sum_{T \in \mathcal{S}_1} \frac{c_T}{|T|} = \sum_{k=1}^{i-1} \sum_{T \in \mathcal{S}_1: k=a_T} \frac{c_T}{|T|} = \sum_{k=1}^{i-1} \frac{1}{(n-i+2)(n-i+1)} w(\{k, i\}). \quad (13)$$

For $T \in \mathcal{S}_2$, setting $|T| = 2 + t$ with $t \in \{0, \dots, n-2\}$ and for one particular $m > i$ at place b_T we have

$$\sum_{T \in \mathcal{S}_2: m=b_T} \frac{c_T}{|T|} = \sum_{t=0}^{n-m} \frac{1}{t+2} \binom{n-m}{t} (-1)^{t+2} w(\{i, m\}) = w(\{i, m\}) \frac{1}{(n-m+2)(n-m+1)}.$$

Hence

$$\sum_{T \in \mathcal{S}_2} \frac{c_T}{|T|} = \sum_{m=i+1}^n \frac{1}{(n-m+2)(n-m+1)} w(\{i, m\}). \quad (14)$$

For $T \in \mathcal{S}_3$, setting $|T| = 3 + t$ with $t \in \{0, \dots, n-3\}$ and for one particular $\ell < i$ and $\ell > k$ at place b_T we have

$$\begin{aligned} \sum_{\substack{T \in \mathcal{S}_3: \\ a_T=k, b_T=\ell}} \frac{c_T}{|T|} &= \sum_{t=0}^{n-\ell-1} \frac{1}{t+3} \binom{n-\ell-1}{t} (-1)^{t+3} w(\{k, \ell\}) \\ &= (-1) w(\{k, \ell\}) \frac{(3-1)!(n-\ell)!}{(n-\ell+3)!} \\ &= w(\{k, \ell\}) \frac{-2}{(n-\ell+2)(n-\ell+1)(n-\ell)}. \end{aligned}$$

Hence

$$\sum_{T \in \mathcal{S}_3} \frac{c_T}{|T|} = \sum_{(k, \ell) \in PP(i)} \frac{-2}{(n - \ell + 2)(n - \ell + 1)(n - \ell)} w(\{k, \ell\}). \quad (15)$$

Filling (13), (14) and (15) in (12) results in (8) for all $i \in N$. □