Tilburg University

Cooperative Situations

Kleppe, J.; Borm, P.E.M.; Hendrickx, R.L.P.; Reijnierse, J.H.

Publication date:
2012

[Link to publication in Tilburg University Research Portal](#)

Citation for published version (APA):

Kleppe, J., Borm, P. E. M., Hendrickx, R. L. P., & Reijnierse, J. H. (2012). *Cooperative Situations: Representations, Games and Cost Allocations*. (CentER Discussion Paper; Vol. 2012-029). Operations research.

General rights

Copyright and moral rights for the publications made accessible in the public portal are retained by the authors and/or other copyright owners and it is a condition of accessing publications that users recognise and abide by the legal requirements associated with these rights.

- Users may download and print one copy of any publication from the public portal for the purpose of private study or research.
- You may not further distribute the material or use it for any profit-making activity or commercial gain
- You may freely distribute the URL identifying the publication in the public portal

Take down policy

If you believe that this document breaches copyright please contact us providing details, and we will remove access to the work immediately and investigate your claim.

No. 2012-029

**COOPERATIVE SITUATIONS: REPRESENTATIONS,
GAMES AND COST ALLOCATIONS**

By

John Kleppe, Peter Borm,
Ruud Hendrickx, Hans Reijnierse

April 3, 2012

ISSN 0924-7815

Cooperative situations: representations, games and cost allocations

John Kleppe^{1,2} Peter Borm¹ Ruud Hendrickx³ Hans Reijnierse¹

March 28, 2012

Abstract

A cooperative situation is characterised by a group of players choosing from a set of alternatives, where each alternative results in a joint cost. In this paper we assume that an alternative with minimum total cost will be chosen and we focus on the corresponding cost allocation problem by providing a procedure to associate a given cooperative situation with an appropriate transferable utility game in a methodic way. The basis of this procedure is the introduction of an order problem representation of the cooperative situation, which we use to individualise the joint costs. For each order problem representation we also define a generalised Bird allocation which we show to be in the core of the corresponding appropriate transferable utility game.

We apply the procedure to both existing and new classes of cooperative situations: sequencing situations without initial ordering, maintenance problems, minimum cost spanning tree situations, permutation situations without initial allocation, public congestion network situations, travelling salesman problems, shared taxi problems and travelling repairman problems.

Keywords: cooperative situation, order problem representation, transferable utility game, generalised Bird allocation

JEL Classification Number: C71

1 Introduction

The aim of this paper is to investigate the issues of cooperation and allocation, with the notion of a cooperative situation as a starting point. A *cooperative situation* is defined as a group of players, a set of alternatives and a joint cost function denoting a total cost for each alternative. The set of alternatives and associated costs stem from an operations research problem in which several players have a similar objective, *e.g.*, to get their job processed on a machine, to get connected to a source, to be visited by a salesman. The resources available to attain this objective are assumed to be public. Well-known examples

¹CentER and Department of Econometrics and Operations Research, Tilburg University.

²Corresponding author: PO Box 90153, 5000 LE Tilburg, The Netherlands. E-mail: J.Kleppe@uvt.nl.

³CentER and Department of Organization and Strategy, Tilburg University.

of *classes* of cooperative situations are the class of travelling salesman problems (Potters et al. (1992)) and the class of minimum cost spanning tree situations (Claus and Kleitman (1973) and Bird (1976)). We refer to Borm et al. (2001) for a comprehensive survey of this type of problems, usually summarised under the heading of operations research games.

A cooperative situation gives rise to two main questions; which alternative should be realised and how should the costs of this alternative be divided? The literature in which (either one of) these two questions are (is) addressed is extensive, but in most papers only one specific class of cooperative situations is discussed. In this paper we focus on the cost allocation question. However, our analysis is not limited to one specific class, as by using common underlying features we introduce a procedure which can in principle be applied to *all* classes of cooperative situations. We assume that there is a general consensus among the players that an alternative should be chosen with minimal total cost. Then transferable utility games are a tool designed to answer the cost allocation question. In general it is, however, not evident which transferable utility game best fits the cooperative situation under consideration. Therefore, the first issue we address is finding an appropriate transferable utility game for a class of cooperative situations. It is clear what the cost of the grand coalition in an appropriate game should be: simply the total cost of a cheapest alternative. Defining the cost of a coalition, however, is a modelling decision that is not always straightforward.

Following Von Neumann and Morgenstern (1944), it is widely accepted that the cost of a coalition should somehow reflect the cost it can guarantee itself when its members would separate from the grand coalition and decide not to cooperate with players outside their coalition. This notion may, however, be rather ambiguous as it does not specify how the other players should (re)act to this defection. Should the coalition treat the outsiders as if they were not there? Should the coalition fear the worst-case scenario in which the outsiders act as unfavourably as possible? Or can the coalition expect that the complementary coalition just ignores them and simply minimises its own costs? In this paper we categorise classes of cooperative situations on the basis of either a positive or a negative externality that players have on each other. We propose for each of the two categories (*types*) a specific transition into an appropriate transferable utility game.

A class of cooperative situations is characterised by an operations research problem, which in essence is the players' problem of creating a specific feasible structure (an alternative) with minimal total cost. A structure is feasible for a group of players if it meets all their objectives. In each minimum cost spanning tree situation, *e.g.*, the objective of each player is to get connected to a single source, possibly via other players. This implies that for this class a feasible structure is a network with a path from each player to the source, such as a spanning tree on the source and all players. It is the type of structure, with

the corresponding cost function, that differentiates one class of cooperative situations from another. Note that for many classes of cooperative situations this structure is some kind of network (*e.g.*, travelling salesman problems and minimum cost spanning tree situations), but in the case of sequencing situations (Curiel et al. (1989) and Klijn and Sánchez (2006)), *e.g.*, this structure is an ordering of the players. Since we aim for a widely applicable model we do not put any restrictions on this structure beforehand.

The procedure we present in this paper consists of three steps. We first briefly point out each of these steps. In the remainder of the introduction we elaborate on each of them. We start out from an arbitrary class of cooperative situations.

In the first step we represent each cooperative situation in the class in the same way by a *proper order problem representation*. In doing so we *individualise* the joint cost function. If each proper order problem representation is *efficient* we proceed to the second step¹.

In the second step we categorise the order problem representations. If all order problem representations of a class are of the same type we propose on the basis of that type an appropriate transferable utility game for the underlying class of cooperative situations². This step is the hallmark feature of our procedure.

In the third and final step we provide a reasonable cost allocation. Here we use the core of the transferable utility game, proposed in the second step, as a benchmark.

In the first step we represent each cooperative situation of a particular class by a proper order problem representation. An *order problem representation* consists of three elements: the player set of the underlying cooperative situation, the set of all possible orderings of the player set and an individualised cost function that describes for each ordering of the player set the corresponding cost for every player. Hence, by the use of an order problem representation we decompose the joint cost function of the underlying cooperative situation into individual parts. There are many possibilities to define an order problem representation of a cooperative situation. However, an order problem representation is only useful in our procedure if it is a proper order problem representation that is efficient.

The order problem representation forms the basis for the transferable utility game and the cost allocation defined in the second and third step of our procedure, and both of these concepts should reflect the strength/weakness of each individual player and of each group of players. Therefore, we define an order problem representation to be *proper* if the individualised cost function reflects that within this representation each player acts as a

¹If not every proper order problem representation is efficient, we conclude that this class of cooperative situations is not suited for our procedure.

²If not all order problem representations are of the same type we cannot propose an appropriate transferable utility game on the basis of our procedure.

cost minimising individual. The idea that each player acts as an individual implies that a player is unable to collaborate with either his predecessors or his followers. The result that a player is unable to collaborate with his predecessors implies that he is unable to change the structure created by them (he can only extend it). Further, since he is also unable to collaborate with his followers he needs to fulfill his objective at his turn as he cannot anticipate on a follower helping him later on. Since the players cannot collaborate with each other it is obvious that each player wants to minimise his individual cost. The contribution of the player to the total cost in this ordering is therefore exactly equal to the minimum costs that are involved in fulfilling his objective. Note that this contribution is in particular independent of the choices of his followers. Furthermore, in a proper order problem representation after the turn of each player in an ordering, the players' individual choices have led to a feasible structure for all players up to then.

For each order problem representation, not only for a proper order problem representation, the sum of the individual costs can be determined for each ordering of the player set³. If for an optimal ordering (an ordering with minimal total cost) this sum exactly equals the total cost of a cheapest alternative in the cooperative situation, then the order problem representation is called *efficient*. Hence, efficiency implies that the costs of the grand coalition in the cooperative situation and the corresponding order problem representation are the same. If a proper order problem representation is efficient we call the order problem representation adequate.

We illustrate our idea of the first step of our procedure by means of a minimum cost spanning tree situation satisfying the triangle inequality. The objective of each player is to be connected to the source, possibly via other players. Hence, the set of alternatives consists of all networks in which there is a path from each player to the source. The total cost of such an alternative is equal to the cost to create the network. We define the corresponding proper order problem representation as follows. A player fulfills his objective at his turn, given the network created by his predecessors. Since it is by definition of a minimum cost spanning tree situation allowed to be connected via another player, at his turn a player can choose to connect his node to any of the nodes that are already connected to the source. In this way he fulfills his objective and since his followers cannot change the created structure, this connection is final. The question that remains is which connection to choose. Following the cost minimisation principle, he chooses a cheapest connection

³Note that for a proper order problem representation for each ordering the sum of all the individual costs is greater than or equal to the cost of the grand coalition, which is given by the total cost of the cheapest alternative. However, if the order problem representation is not proper, then the individualised cost function is not restricted, and therefore the sum of all the individual costs is not necessarily greater than or equal to the cost of the grand coalition.

possible for his connection to the source, and the corresponding costs are then assigned to him.

Note that this set-up also ensures that for each ordering of the player set there arises a feasible structure for all the players after the turn of the final player. Furthermore, by the minimum cost spanning tree algorithm of Prim (1957) this proper order problem representation is also efficient, and therefore adequate. Throughout the paper we see more examples connecting adequateness of an order problem representation with an algorithm (from the literature). Minimum cost spanning tree situations are extensively discussed in Section 5.

Once we have defined an adequate order problem representation for each cooperative situation within a certain class we use it in the second step of our procedure to propose an appropriate transferable utility game for the class of cooperative situations. We differentiate between two types of order problem representations. In a positive externality order problem representation the minimum cost for each group of players is obtained for an ordering in which they are ordered last, while in a negative externality order problem representation the minimum cost for each group of players is obtained for an ordering in which they are ordered first. We argue (in Section 2) that each positive externality order problem representation is appropriately modelled by the so called direct cost game. In determining the cost of a specific coalition in this game, it is assumed that this coalition is ordered first. Furthermore, we argue that each negative externality order problem representation is appropriately modelled by the dual of the direct cost game, called the marginal cost game. Consequently, our arguments lead to the result that if for a class of cooperative situations one can define a proper order problem representation for each cooperative situation and every order problem representation is an efficient *positive (negative)* externality order problem representation, then the *direct (marginal)* cost game is an appropriate transferable utility game for the class of cooperative situations.

For each cooperative situation the ultimate goal is to find a reasonable cost allocation of the chosen alternative. The third and final step of our procedure is therefore the systematic determination of a specific suitable cost allocation. To determine whether a cost allocation proposal is reasonable we use the core of the corresponding appropriate transferable utility game (obtained in Step 2) of the cooperative situation as a benchmark. We introduce for each order problem representation a generalised Bird allocation. This allocation is inspired by Bird's tree allocation (Bird (1976)) for the class of minimum cost spanning tree situations, in the sense that each player contributes his individual cost in an optimal ordering for the grand coalition. As a further evaluation tool we associate with

each order problem representation a minimal and a maximal cost game. In the minimal cost game the cost of a coalition equals the cost of a cheapest ordering for the players of that coalition according to the individualised cost function of the order problem representation. The maximal cost game is the dual of the minimal cost game. We show that for each order problem representation a generalised Bird allocation is an element of the core of the maximal cost game.

We use this result to show that for each *negative* externality order problem representation a generalised Bird allocation is a core element of the *marginal* cost game. And for each *positive* externality order problem representation that satisfies predecessor order independence, which means that individual costs only depend on the set of predecessors and not on their exact ordering, each generalised Bird allocation is an element of the core of the *direct* cost game. Note that in both cases the transferable utility game under consideration is appropriate (by Step 2) for the underlying cooperative situation. Therefore, the generalised Bird allocation is reasonable in those cases, which concludes (Step 3 of) our procedure.

The above described third step of our procedure, finding a specific reasonable cost allocation, can be generalised by moving from order problem representations, discussed so far, to alternative problem representations. An *alternative problem representation* consists of three elements: the player set of the underlying cooperative situation, the set of alternatives (feasible structures for all players) of the underlying cooperative situation and an individualised cost function that describes for each alternative the corresponding cost for every player. An alternative problem representation stays closer to the underlying cooperative situation, as it individualises the total costs based on the alternatives directly, instead of via orderings of the player set as for order problem representations. However, that also means that alternative problem representations can not be used to find an appropriate transferable utility game for a class of cooperative situations, as determining the type of externality in Step 2 is explicitly based upon orderings of the player set. Nevertheless, alternative problem representations can be used to find an element of the core of a given transferable utility game. For this reason we define a generalised Bird allocation and a minimal and maximal cost game within this context. We show that for each alternative problem representation a generalised Bird allocation is an element of the core of the maximal cost game. We use this result, *e.g.*, for providing an alternative proof of the fact that each Bird's tree allocation is an element of the core of the cost game proposed by Bird (1976) for the class of minimum cost spanning tree situations.

We apply our three-step procedure to several classes of cooperative situations. We first

discuss the class of sequencing situations without initial order, introduced by Klijn and Sánchez (2006), who propose two transferable utility games. Interestingly, by our procedure we derive an appropriate transferable utility game that is different from those two games, and we show that the core of our transferable utility game is a subset of the core of the two games of Klijn and Sánchez (2006). Moreover, as a result of our general theorem it follows that a generalised Bird allocation is a core element.

Secondly, we discuss the class of maintenance problems (Megiddo (1978) and Koster (1999)) in which a group of players has to maintain a tree network. Koster (1999) defines the cost game in which the cost of a coalition is equal to the cost to maintain the smallest trunk in which all the players of the coalition are contained. Our three-step procedure supports this transferable utility game, and we derive that a generalised Bird allocation is an element of its core.

Thirdly, we consider the class of minimum cost spanning tree situations. Claus and Kleitman (1973) and Bird (1976) implicitly assume that the edge between any two players can only be used if the two players cooperate with each other, and within this context Bird (1976) proposes the transferable utility game in which the cost for each coalition equals the cost of a minimum cost spanning tree on all players of the coalition plus the source. We, on the other hand, assume that all edges are publicly available. Nevertheless, by our three-step procedure the same transferable utility game is obtained. This enables us to provide two alternative proofs, one by the use of order problem representations and one by the use of alternative problem representations, of the fact that Bird's tree allocation is in the core of this game.

Fourthly, we introduce and analyse the class of permutation situations without initial allocation. Permutation situations are introduced by Tijs et al. (1984), who assume that there is an initial allocation of machines to the players and that a coalition of players can interchange their machines between them. We assume on the contrary that there is no initial allocation of machinery. For this class of cooperative situations we derive by our procedure an appropriate transferable utility game of which a generalised Bird allocation is an element of its core.

Fifthly, we consider the class of public congestion network situations, introduced by Kleppe et al. (2010). We show that our procedure supports the use of the marginal congestion cost game, as proposed by Kleppe et al. (2010), for public congestion network situations with *convex* cost functions. For the class of public congestion network situations with *concave* cost functions, however, we show that not every situation has an adequate order problem representation. Therefore, we cannot support the use of the direct congestion cost game, as proposed by Kleppe et al. (2010), by the use of the three-step procedure.

Sixthly, we discuss the class of travelling salesman problems. Also for this class we

come to the conclusion that an adequate order problem representation does not exist for every situation. Consequently, on the basis of our procedure we cannot derive an appropriate transferable utility game for this class either. A different type of modelling problem arises for the closely related class of shared taxi problems. We introduce this class of cooperative situations in which all players are brought to their individual destinations by a single shared taxi for which the cost depends on the distance from its starting point to the final destination. Here we find an adequate order problem representation for all situations. However, not all of them are either positive or negative externality order problem representations. This implies that we cannot categorise the order problem representations and argue in favour of either the direct or the marginal cost game to model these kind of situations. Finally, we discuss the class of travelling repairman problems. The corresponding optimisation problem is considered by Afrati et al. (1986), here we consider the associated cost allocation problem. A travelling repairman is looking for a way to visit all players such that the total waiting time of the players is minimised. By the use of the three-step procedure we derive an appropriate transferable utility game of which a generalised Bird allocation is an element of its core.

The structure of this paper is as follows. In Section 2 we present our procedure to find an appropriate transferable utility game, and a reasonable cost allocation of a chosen alternative for a class of cooperative situations. In the remainder of this paper we apply this three-step procedure to several classes of cooperative situations: sequencing situations without initial order (Section 3), maintenance problems (Section 4), minimum cost spanning tree situations (Section 5), permutation situations without initial allocation (Section 6), public congestion network situations (Section 7), and travelling salesman, shared taxi and travelling repairman problems (Section 8).

2 A three-step procedure for cooperative situations

2.1 Appropriate transferable utility games via order problem representations

In this subsection we present the first two steps of our procedure, which lead to the proposal of an appropriate transferable utility game for a specific class of cooperative situations under consideration. A cooperative situation is defined by $(N, \mathcal{A}, K)$, with N the set of players, $\mathcal{A}$ the finite set of alternatives and $K : \mathcal{A} \rightarrow \mathbb{R}$ a cost function that assigns to each alternative $\alpha \in \mathcal{A}$ a certain cost $K(\alpha)$. We assume that a cooperative situation has an underlying operations research problem in which various decision makers (players) with similar objectives are involved in a joint cost minimisation problem. We say that

two cooperative situations are in the same class if their operations research problems are analogous.

A transferable utility or TU-game is a pair (N, c) , where N denotes the finite set of players and $c : 2^N \rightarrow \mathbb{R}$ is the characteristic function, assigning to every coalition $S \subseteq N$ of players a cost, $c(S)$, representing the total joint cost of this coalition of players when they cooperate. By convention, $c(\emptyset) = 0$.

Solving the cost minimisation problem of a cooperative situation results in the cost for the grand coalition, $c(N) = K(\alpha^*)$, with $\alpha^* \in \arg \min_{\alpha \in \mathcal{A}} K(\alpha)$. To determine the costs of subcoalitions in an appropriate way we first introduce the notion of an order problem representation. This is Step 1 of our procedure.

An *order problem representation (opr)* of a cooperative situation is given by (N, Π, k) , with N the finite player set (of the cooperative situation) and Π the set of orderings of the player set. Here an ordering is a bijection $\pi : \{1, \dots, |N|\} \rightarrow N$. Finally, $k : \Pi \rightarrow \mathbb{R}^N$ is an *individualised cost function*, denoting for each ordering $\pi \in \Pi$ an individual cost $k^i(\pi)$ for each player $i \in N$.

When defining an *opr* we have to consider the operations research problem on which the cooperative situation is based. In this operations research problem each player has an objective and it is the players' joint task to create a structure in which all objectives are met with minimum total cost. A structure is called *feasible for* $S \subseteq N$ if it meets the objective of all the players in S . Since we use the *opr* in the second step of our procedure to find an appropriate TU-game for the underlying class of cooperative situations, the *opr* should be adequate. We call an *opr* *adequate* if it is both *proper* and *efficient*.

Our interpretation of an *opr* of a cooperative situation is that players “enter” (the structure created by the predecessors) according to an ordering $\pi \in \Pi$. An *opr* is then *proper* if the individualised cost function reflects that all players act as a cost minimising individual. This notion boils down to the following four properties.

1. At his turn a player is unable to change the structure created by his predecessors, he can only extend it.
2. A player extends the structure such that it becomes feasible for him (and all his predecessors).
3. The individual cost of a player exactly corresponds to the cost to extend the structure created by his predecessors to the new feasible structure.
4. A player's sole objective is to minimise his individual cost.

Hence, for a proper *opr*, for each ordering $\pi \in \Pi$, a feasible structure for N is constructed by the end of the turn of player $\pi(|N|)$. Generically each structure has a different cost,

which implies that each feasible structure for a coalition $S \subseteq N$ has a unique cheapest way of being extended to a feasible structure for $S \cup \{i\}$ for all $i \in N \setminus S$. Consequently, a generic cooperative situation has a unique proper *opr*. For non-generic cooperative situations one needs a secondary objective⁴ in property (4) to make sure that each ordering results in a unique final structure.

Note that each player in a proper *opr* acts as a cost minimising (properties (3) and (4)) individual, as he is unable to collaborate with either his predecessors (property (1)) or his followers (property (2)). Note also that the cost of a player may depend on (the order of) his predecessors, but does not depend on (the order of) his followers.

An *opr* is called *efficient* if an optimal ordering of N results in a feasible structure for N with minimal total cost. Hence, let there be a cooperative situation $(N, \mathcal{A}, K)$ with *opr* (N, Π, k) . We say that (N, Π, k) is *efficient* if $\min_{\pi \in \Pi} \sum_{i \in N} k^i(\pi) = K(\alpha^*) (= c(N))$, with $\alpha^* \in \operatorname{argmin}_{\alpha \in \mathcal{A}} K(\alpha)$. Efficiency ensures that in the *opr* and the underlying cooperative situation the resulting costs for the grand coalition are the same, which implies that the same costs are to be allocated among the players.

Note that if (N, Π, k) is proper, then $\sum_{i \in N} k^i(\pi) \geq K(\alpha^*)$ for all $\pi \in \Pi$ as $\sum_{i \in N} k^i(\pi)$ corresponds to the cost of a feasible, but not necessarily optimal, structure (alternative) for N : $\sum_{i \in N} k^i(\pi) = K(\alpha)$, with α the alternative that results from the proper *opr* for ordering π . Therefore, to check whether a proper *opr* is efficient one needs to find *an* ordering of N that results in a feasible structure for N with minimal total cost. If we come to the conclusion that for each cooperative situation within the class the proper *opr* is efficient, then we proceed to Step 2 of our procedure, otherwise we unfortunately have to conclude that the class of cooperative situations under consideration is not suited for our procedure.

In the second step of our procedure we divide the *oprs* into two types. On the basis of these types we propose an appropriate TU-game for the whole class. Given an *opr* (N, Π, k) we denote by

$$\pi_S^* \in \operatorname{argmin}_{\pi \in \Pi} \sum_{i \in S} k^i(\pi)$$

a cheapest or optimal ordering⁵ for coalition $S \subseteq N$. By $\Pi_S \subseteq \Pi$ we denote the set of all orderings $\pi \in \Pi$ such that the players in $S \subseteq N$ are placed on the first $|S|$ positions, *i.e.*, $\pi^{-1}(i) < \pi^{-1}(j)$ for all $i \in S, j \in N \setminus S$. Note that $\Pi_N = \Pi$.

⁴The precise objective might depend on the class of cooperative situations under consideration, but one could consider altruism in case of a tie.

⁵Given an *opr* (N, Π, k) an optimal ordering π_S^* for coalition $S \subseteq N$ need not be unique, even if the resulting alternative is.

The $opr(N, \Pi, k)$ is a *negative externality order problem representation (neopr)* if for all $S \subseteq N$ there exists an optimal ordering π_S^* such that

$$\pi_S^* \in \Pi_S.$$

The $opr(N, \Pi, k)$ is a *positive externality order problem representation (peopr)* if for all $S \subseteq N$ there exists an optimal ordering π_S^* such that

$$\pi_S^* \in \Pi_{N \setminus S}.$$

Hence, in a *neopr* each group of players prefers to be ordered first, while for a *peopr* each group of players wants to be ordered last. An *opr* may not belong to any of the above two types, in which case our model is not suited to suggest an appropriate TU-game for the underlying class of cooperative situations.

Next we introduce two TU-games. For an $opr(N, \Pi, k)$ we define the *direct cost game* (N, c_d) by

$$c_d(S) = \min_{\pi \in \Pi_S} \sum_{i \in S} k^i(\pi) \quad (2.1)$$

for all $S \subseteq N$. Hence, in calculating the cost of a coalition in the direct cost game it is assumed that this coalition is ordered first, *i.e.*, before $N \setminus S$. However, the coalition S can optimise using the first $|S|$ positions of the ordering as slots for the players. Furthermore, we define the *marginal cost game* (N, c_m) as the dual of the direct cost game, *i.e.*,

$$c_m(S) = c_d(N) - c_d(N \setminus S) \quad (2.2)$$

for all $S \subseteq N$. The cost of coalition S in the marginal cost game reflects its marginal cost to the grand coalition.

Generally speaking, given a cooperative situation the coalitional cost of an associated TU-game, $c(S)$, should be based on what this coalition, $S \subseteq N$, can *guarantee* itself when its members would separate from the grand coalition and decide not to cooperate with players outside their coalition.

Since in a *peopr* each group of players prefers to be ordered last, the cost of a coalition is minimal for an ordering in which all players *outside* the coalition are ordered first. However, the possible positive effect of the presence of other players should not be incorporated in the coalitional cost. Hence, based on the above principle of guaranteeing it is reasonable to model a *peopr* by the direct cost game. Consequently, we obtain the following observation.

Observation: Consider a class of cooperative situations. If one can define a proper *opr* for each cooperative situation and every proper *opr* is an efficient *peopr*, then the direct cost game is an appropriate TU-game for the class of cooperative situations.

In a *neopr* each group of players prefers to be ordered first. The direct cost game, based on orderings in which the players in the coalition are ordered first, would therefore be too optimistic in the sense that the coalitional cost would reflect the minimal attainable cost. Hence, for a *neopr* we argue that one has to take account of the possibility that the players outside the coalition are ordered first. This, however, still leaves many options open. The most pessimistic view is that the other players will try to frustrate the coalition as much as possible. This, however, seems to be an empty threat, as these others want to minimise their own costs as well. (Note that this viewpoint implicitly assumes collaboration between the players outside the coalition, but since full cooperation is to be expected in the first place, this is a reasonable assumption in order to determine the cost of a coalition.) Building on the assumption that the objective of the other players is to minimise their own cost, one could further argue that the players in the coalition can alter the ordering jointly preferred by the other players (an ordering where the other players are on the first positions) as long as the coalition compensates the others for the additional costs. Following this reasoning a coalition has to decide on an ordering that minimises the sum of the cost of its own players plus the possible compensation to the players outside the coalition for reordering them with respect to their optimal ordering. Exactly this idea is reflected by the marginal cost game.

We can also motivate the use of the marginal cost game for a *neopr* from a different viewpoint. If the players in S are not present (from the perspective of the costs of $N \setminus S$, equivalent with the players in S being on the last $|S|$ positions of an ordering) it is clear that $N \setminus S$ has a cost of $c_d(N \setminus S)$. The presence of the players in S (equivalent to the players in S not necessarily on the last $|S|$ positions) can by definition of a *neopr* only increase the cost of $N \setminus S$. Therefore, $c_d(N \setminus S)$ is the minimum amount coalition $N \setminus S$ ever has to pay in an ordering and is therefore a reasonable minimum contribution to $c(N)$. This implies that a reasonable maximum contribution for S to the total cost equals $c_m(S) = c_d(N) - c_d(N \setminus S)$. This results in the following observation⁶.

Observation: Consider a class of cooperative situations. If one can define a proper *opr* for each cooperative situation and every proper *opr* is an efficient *neopr*, then the marginal

⁶If the adequate *opr* of one cooperative situation of a certain class is a *neopr*, while the adequate *opr* of another cooperative situation of that class is a *peopr*, our procedure is not suited to propose an appropriate TU-game for that class of cooperative situations.

cost game is an appropriate TU-game for the class of cooperative situations.

Our procedure to obtain appropriate coalitional costs for a *neopr* is well imbedded in the game theoretic literature; think, *e.g.*, of bankruptcy situations (O'Neill (1982), Aumann and Maschler (1985) and Thomson (2003)) in which an estate has to be divided among a number of players, each with his own claim on (part of) the estate. In order to determine a coalitional value in the corresponding bankruptcy game one first treats the players outside a coalition in an optimistic way (by giving them their full claim if possible), putting the value of a coalition equal to the remainder of the estate. Our TU-game for an *neopr* is based on the same idea. One could see a bankruptcy situation as a cooperative situation with revenues instead of costs. The idea of a proper *opr* for a bankruptcy situations boils down to the following: for each ordering of the player set the players obtain their full claim one by one until there is no money left. Note that this *opr* is efficient as for each ordering the estate is exactly divided. Hence, the *opr* is adequate. Furthermore, in this adequate *opr* each group of players prefers to be ordered first and hence this *opr* is a *neopr*. Consequently, our model suggests the marginal cost game to model bankruptcy situations. The coalitional value of the direct cost game is the amount this coalition can get if they are ordered first, which is the minimum of the sum of their claims and the estate. Hence, the coalitional value according to the marginal cost game is either the estate minus the claims of the players outside the coalition, or zero. This exactly boils down to the bankruptcy game used in Aumann and Maschler (1985).

2.2 Core elements via generalised Bird allocations

In this subsection we focus on the third and final step of our procedure: finding specific core elements of an appropriate TU-game. The core elements we introduce generalise Bird allocations for minimum cost spanning tree situations. We first proceed with the procedure on the basis of *opr*s. In the second part of this subsection we discuss alternative problem representations, which are only to be used in this particular step of the procedure.

2.2.1 Order problem representations

In this part we consider *opr*s. For analytical purposes we start by introducing two auxiliary TU-games. For an *opr* (N, Π, k) we define the *minimal cost game* (N, c_-) by

$$c_-(S) = \min_{\pi \in \Pi} \sum_{i \in S} k^i(\pi)$$

for all $S \subseteq N$. Equivalently, $c_-(S) = \sum_{i \in S} k^i(\pi_S^*)$ for some π_S^* . The *maximal cost game* (N, c_+) is defined as the dual of the minimal cost game, *i.e.*,

$$c_+(S) = c_-(N) - c_-(N \setminus S)$$

for all $S \subseteq N$. Obviously, we obtain the following proposition.

Proposition 2.1 *Let (N, Π, k) be a neopr. Then the associated direct cost game (N, c_d) and minimal cost game (N, c_-) coincide. As a consequence, also the marginal cost game (N, c_m) and maximal cost game (N, c_+) coincide.*

We define a *generalised Bird allocation* $\beta(N, \Pi, k)$ associated with an *opr* (N, Π, k) such that each player contributes his individual cost according to an optimal ordering π_N^* for the grand coalition. Hence, such an allocation⁷ is defined by

$$\beta^i(N, \Pi, k) = k^i(\pi_N^*) \quad (2.3)$$

for all $i \in N$. This type of allocation β is called a generalised Bird allocation, because it is a generalisation of a Bird allocation for minimum cost spanning tree situations (Bird (1976)).

The core $C(N, c)$ (Gillies (1959)) of a TU-game (N, c) consists of those cost allocations for which no coalition would be better off if it would separate itself and would pay its coalitional cost. It is given by

$$C(N, c) = \{x \in \mathbb{R}^N \mid x(N) = c(N), x(S) \leq c(S) \text{ for all } S \subseteq N\},$$

with $x(S) = \sum_{i \in S} x^i(S)$ for all $S \subseteq N$.

Theorem 2.2 Let (N, Π, k) be an *opr* with associated maximal cost game (N, c_+) . Then $\beta(N, \Pi, k) \in C(N, c_+)$.

Proof: By definition $\sum_{i \in N} \beta^i(N, \Pi, k) = \sum_{i \in N} k^i(\pi_N^*) = c_+(N)$. Furthermore, for all $S \subseteq N$,

$$\begin{aligned} c_+(S) &= c_-(N) - c_-(N \setminus S) \\ &= \sum_{i \in N} k^i(\pi_N^*) - \sum_{i \in N \setminus S} k^i(\pi_N^*) \\ &\geq \sum_{i \in N} k^i(\pi_N^*) - \sum_{i \in N \setminus S} k^i(\pi_N^*) \\ &= \sum_{i \in S} k^i(\pi_N^*) \\ &= \sum_{i \in S} \beta^i(N, \Pi, k). \end{aligned}$$

⁷Since π_N^* is not unique one should formally denote $\beta^i(N, \Pi, k; \pi_N^*)$, but following Bird (1976) we omit π_N^* in the expression.

□

Using Proposition 2.1, we immediately obtain the next result.

Theorem 2.3 Let (N, Π, k) be a *neopr* with associated marginal cost game (N, c_m) . Then $\beta(N, \Pi, k) \in C(N, c_m)$.

Hence, if for a class of cooperative situations one can define a proper *opr* for each cooperative situation, and each proper *opr* is an efficient *neopr*, then β is a reasonable allocation, because β is an element of the core of the appropriate TU-game (N, c_m) .

A similar result, placing a generalised Bird allocation in the core of the direct cost game associated with a *peopr*, is not immediate and requires an additional condition. This condition, which is called *predecessor order independence*, boils down to the idea that the *ordering* of the predecessors of each player is irrelevant for his individual cost. Let (N, Π, k) be an *opr* and let $V^i(\pi) = \{j \in N \mid \pi^{-1}(j) < \pi^{-1}(i)\}$ denote the set of predecessors of player i given ordering $\pi \in \Pi$. Then (N, Π, k) satisfies *predecessor order independence* (*poi*) if, for all $i \in N$,

$$k^i(\pi) = k^i(\bar{\pi})$$

for all $\pi, \bar{\pi} \in \Pi$ such that $V^i(\pi) = V^i(\bar{\pi})$. Note that this condition also implies that the cost of a player is not influenced by the ordering of his followers.

Theorem 2.4 Let (N, Π, k) be a *peopr* that satisfies *poi* and let (N, c_d) be the associated direct cost game. Then $\beta(N, \Pi, k) \in C(N, c_d)$.

Proof: Note that $\sum_{i \in N} \beta^i(N, \Pi, k) = \sum_{i \in N} k^i(\pi_N^*) = c_d(N)$. Let $S \subseteq N$ and let $\hat{\pi}_S^* \in \Pi_S$ be such that $\sum_{i \in S} k^i(\hat{\pi}_S^*) = c_d(S)$. Recall that $\pi_{N \setminus S}^*$ is an optimal ordering for $N \setminus S$. Let $\pi_{N \setminus S}^*$ be such that the players in $N \setminus S$ are ordered in the last positions of $\pi_{N \setminus S}^*$. Since (N, Π, k) is a *peopr* such a $\pi_{N \setminus S}^*$ exists. Next define $\tilde{\pi}_N$ such that $\tilde{\pi}_N(t) = \hat{\pi}_S^*(t)$ for all

$t \in \{1, \dots, |S|\}$ and $\tilde{\pi}_N(t) = \pi_{N \setminus S}^*(t)$ for all $t \in \{|S| + 1, \dots, n\}$. Then, for all $S \subseteq N$,

$$\begin{aligned}
c_d(S) &= \sum_{i \in S} k^i(\hat{\pi}_S^*) \\
&\geq \sum_{i \in S} k^i(\hat{\pi}_S^*) + \sum_{i \in N \setminus S} k^i(\pi_{N \setminus S}^*) - \sum_{i \in N \setminus S} k^i(\pi_N^*) \\
&= \sum_{i \in N} k^i(\tilde{\pi}_N) - \sum_{i \in N \setminus S} k^i(\pi_N^*) \\
&\geq \sum_{i \in N} k^i(\pi_N^*) - \sum_{i \in N \setminus S} k^i(\pi_N^*) \\
&= \sum_{i \in S} k^i(\pi_N^*) \\
&= \sum_{i \in S} \beta^i(N, \Pi, k),
\end{aligned}$$

where the second equality follows from the fact that (N, Π, k) satisfies *poi*. $\square$

Hence, if for a class of cooperative situations one can define a proper *opr* for each cooperative situation, and each proper *opr* is an efficient *peopr* that satisfies *poi*, then β is a reasonable allocation, because β is an element of the core of the appropriate TU-game (N, c_d) .

2.2.2 Alternative problem representations

An *alternative problem representation (apr)* of a cooperative situation is given by $(N, \mathcal{A}, k)$, with N the finite player set, $\mathcal{A}$ the finite set of alternatives (feasible structures for N) and $k : \mathcal{A} \rightarrow \mathbb{R}^N$ an individualised cost function denoting for each alternative $\alpha \in \mathcal{A}$ a corresponding cost $k^i(\alpha)$ for every player $i \in N$. Hence, where in a cooperative situation only the total cost for each alternative is defined, in the *apr* we define a cost per player for each alternative. The main difference between an *opr* and an *apr* is that for an *opr* a final feasible structure is the result of a sequence of individual choices, while for an *apr* a feasible structure, *i.e.*, an alternative, is immediately chosen. Note that an *opr* is therefore not a special case of an *apr*, but a different representation altogether.

The definitions of a *neopr* and a *peopr*, as well as the definitions of the direct and the marginal cost game are based upon orderings of the player set and can therefore not be used in this set-up. Consequently, we cannot define in a similar fashion when an *apr* is adequate and then use it to find an appropriate TU-game for a class of cooperative situations (Steps 1 and 2 of our procedure). This set-up can, however, help us to find a core element of a TU-game (Step 3) once we have determined (*e.g.*, by the first two steps of the procedure) that this game is appropriate. For that reason we define the generalised Bird

allocation and the minimal and maximal cost game within this framework.

For an *apr* $(N, \mathcal{A}, k)$ we define the *minimal cost game* (N, c_-) by

$$c_-(S) = \min_{\alpha \in \mathcal{A}} \sum_{i \in S} k^i(\alpha) \quad (2.4)$$

for all $S \subseteq N$. The *maximal cost game* (N, c_+) is defined as the dual of the minimal cost game, *i.e.*,

$$c_+(S) = c_-(N) - c_-(N \setminus S) \quad (2.5)$$

for all $S \subseteq N$.

By $\alpha_S^* \in \operatorname{argmin}_{\alpha \in \mathcal{A}} \sum_{i \in S} k^i(\pi)$ we denote a cheapest or optimal alternative⁸ for coalition $S \subseteq N$. A generalised Bird allocation for an optimal alternative α_N^* is defined by

$$\beta^i(N, \mathcal{A}, k) = k^i(\alpha_N^*) \quad (2.6)$$

for all $i \in N$. The proof of the following theorem is similar to the proof of Theorem 2.2 and is therefore omitted.

Theorem 2.5 Let $(N, \mathcal{A}, k)$ be an alternative problem with associated maximal cost game (N, c_+) . Then $\beta(N, \Pi, k) \in C(N, c_+)$.

3 Sequencing situations without initial order

In this section we discuss the class of sequencing situations without initial order. The cost allocation problem associated with sequencing situations *with* initial order has been introduced by Curiel et al. (1989). Note that an initial order gives each (group of) player(s) an ex ante position and cost. As a consequence, determining the cost for a subcoalition boils down to the same problem as for the grand coalition. In case there is no initial allocation the problem for a subcoalition is essentially different from the problem for the grand coalition, as the role of the outsiders is undetermined. Klijn and Sánchez (2006) initiated the analysis of sequencing situations without initial order.

In this section we define the adequate *opr* of a sequencing situation without initial order and show that it is a *neopr*. Therefore, we suggest to model this class of cooperative situations by the associated marginal cost game. Furthermore, we observe by the use of Theorem 2.3 that each associated generalised Bird allocation is an element of the core of

⁸Given an alternative problem $(N, \mathcal{A}, k)$ an optimal alternative α_S^* for coalition $S \subseteq N$ need not be unique.

this game. We also compare the marginal cost game with the two cost games proposed by Klijn and Sánchez (2006).

A *sequencing situation without initial order* is given by a triple⁹ $Q = (N, p, \delta)$, with N a finite set of players. Each player $i \in N$ owns one job that has to be processed on a single machine. The job of player i is with slight abuse of notation also denoted by i . The processing times of the jobs are given by $p = \{p^i\}_{i \in N}$ with $p^i > 0$ for all $i \in N$. Furthermore, with $\delta = \{\delta^i\}_{i \in N}$ each player has a cost function $c^i : [0, \infty) \rightarrow \mathbb{R}$ given by $c^i(t) = \delta^i t$ for $t \in [0, \infty)$, where $\delta^i > 0$. The expression $c^i(t)$ is interpreted as the cost incurred by agent i if his job is completed at time t . The optimal ordering of jobs for the grand coalition is obtained by putting them in a non-decreasing order of their urgency indices (Smith (1956)), which are defined as $u^i = \frac{\delta^i}{p^i}$ for all $i \in N$.

Since it is assumed that only schedules are considered in which the jobs are processed without any breaks in between, the cost of player i is completely determined by an ordering of the player set $\pi \in \Pi$. Therefore, the cost function c can also be given by $c^i(\pi) = \delta^i \sum_{j \in (V^i(\pi) \cup \{i\})} p^j$ for all $i \in N$ and all $\pi \in \Pi$. Consequently, a sequencing situation without initial order $Q = (N, p, \delta)$ can be reformulated as a cooperative situation $(N, \mathcal{A}, K)$, with $\mathcal{A} = \Pi$ and¹⁰ $K(\pi) = \sum_{i \in N} c^i(\pi)$ for all $\pi \in \Pi$. Hence, the definition of the proper *opr* is straightforward: (N, Π, k) , with $k : \Pi \rightarrow \mathbb{R}^N$ defined by

$$k^i(\pi) = \delta^i \sum_{j \in (V^i(\pi) \cup \{i\})} p^j$$

for all $i \in N$ and all $\pi \in \Pi$. It is easily seen that this proper *opr* is also efficient. The associated marginal cost game (N, c_m) and a generalised Bird allocation $\beta(N, \Pi, k)$ are determined by (2.2) and (2.3), respectively. The following proposition follows from the fact that $p^i > 0$ for all $i \in N$.

Proposition 3.1 *Let Q be a sequencing situation without initial order. Then the corresponding *opr* (N, Π, k) is a *neopr*.*

Since (N, Π, k) is an adequate *neopr* of Q , we suggest to model the class of sequencing situations without initial order by the marginal cost game. Since (N, Π, k) is a *neopr* the following proposition follows by Theorem 2.3.

Proposition 3.2 *Let Q be a sequencing situation without initial order with corresponding *opr* (N, Π, k) . Then $\beta(N, \Pi, k) \in C(N, c_m)$.*

⁹Klijn and Sánchez (2006) use the notation (N, p, α) .

¹⁰Note that, due to the nature of this class of cooperative situations, the total cost of an alternative is derived from individual costs.

Note that according to π_N^* the jobs are ordered with non-decreasing urgency indices. Hence, $\beta^i(N, \Pi, k)$ equals job i 's completion time multiplied by player i 's cost per time unit.

Next we compare the marginal cost game to the cost games proposed by Klijn and Sánchez (2006) to model sequencing situations without initial order. In the tail game (N, c_{tail}) it is assumed that the players in $N \setminus S$ are ordered first and the players in S can only optimise their sequence in the tail of the ordering. This game is defined by

$$c_{tail}(S) = \min_{\pi \in \Pi_{N \setminus S}} \sum_{i \in S} k^i(\pi)$$

for all $S \subseteq N$. We obtain the following proposition.

Proposition 3.3 *Let Q be a sequencing situation without initial order with associated tail game (N, c_{tail}) . Let (N, Π, k) be the corresponding opr with associated marginal cost game (N, c_m) . Then $C(N, c_m) \subseteq C(N, c_{tail})$.*

Proof: By definition $c_m(N) = c_{tail}(N)$. Hence, it suffices to show that $c_m(S) \leq c_{tail}(S)$ for all $S \subseteq N$. Let $S \subseteq N$ and let $\hat{\pi}_{N \setminus S}^*$ be an ordering that is optimal for $N \setminus S$ such that $c_{tail}(S) = \sum_{i \in S} c^i(\hat{\pi}_{N \setminus S}^*)$. Recall that π_N^* is an optimal ordering for N . Then

$$\begin{aligned} c_{tail}(S) &= \sum_{i \in S} k^i(\hat{\pi}_{N \setminus S}^*) \\ &= \sum_{i \in N} k^i(\hat{\pi}_{N \setminus S}^*) - \sum_{i \in N \setminus S} k^i(\hat{\pi}_{N \setminus S}^*) \\ &\geq \sum_{i \in N} k^i(\pi_N^*) - \sum_{i \in N \setminus S} k^i(\hat{\pi}_{N \setminus S}^*) \\ &= c_d(N) - c_-(N \setminus S) \\ &= c_d(N) - c_d(N \setminus S) \\ &= c_m(S). \end{aligned}$$

Note that the equal sign between $c_d(N) - c_-(N \setminus S)$ and $c_d(N) - c_d(N \setminus S)$ follows from the fact that (N, Π, k) is a *neopr*. $\square$

Klijn and Sánchez (2006) also introduce a pessimistic game (N, c_{pess}) of which they show that $C(N, c_{tail}) \subseteq C(N, c_{pess})$ for each sequencing situation without initial order. Therefore, the core of the marginal cost game is also a subset of the core of the pessimistic game. This implies that any allocation stable under the marginal cost game approach is also stable under the two approaches of Klijn and Sánchez (2006). Moreover, since each generalised Bird allocation is an element of the core of the marginal cost game, such a stable allocation exists.

4 Maintenance problems

In this section we consider the class of maintenance problems (Megiddo (1978) and Koster (1999)). In these problems the players have to maintain a tree network. This class of cooperative situations contains the class of airport problems, introduced by Littlechild and Owen (1973). We derive that the proper *opr* of each maintenance problem is a *peopr*. Hence, on that basis we propose the direct cost game to model this class of cooperative situations. It turns out that this game coincides with the game from literature, *e.g.*, discussed in Koster (1999). Furthermore, we obtain by Theorem 2.4 that a generalised Bird allocation is in the core of this game.

Formally, a *maintenance problem* is given by a triple $I = (N, G, \gamma)$, with N the finite player set. The source is given by 0 and by S^0 , $S \subseteq N$, we denote the set $S \cup \{0\}$. Here $G = (N^0, E)$ is a tree with the set of nodes N^0 and edge set E , such that the source 0 has only one adjacent edge. The objective of each player is that the unique path from his node to the source is maintained. The cost to maintain an edge is given by $\gamma : E \rightarrow \mathbb{R}^+$. A maintenance problem can be reformulated as a cooperative situation $(N, \mathcal{A}, K)$, with $\mathcal{A}$ consisting of one element, the tree G , and $K(G) = \sum_{e \in E} \gamma(e)$ the cost to maintain that tree. Note that for this class of cooperative situations there is no operations research problem to solve, as there is only one alternative.

To answer the cost allocation question we define for a maintenance problem $I = (N, G, \gamma)$ the proper *opr* (N, Π, k) , where $k : \Pi \rightarrow \mathbb{R}^N$ denotes the cost for a player to maintain the part of the unique path from his node to the source that is not maintained by his predecessors. Let $\pi \in \Pi$. Let P_{ij} be the unique path from player i to player j . Define $j_i^* \in N^0$ as the last node on the unique path from i to the source 0 such that each edge of the path $P_{ij_i^*}$ is not maintained. Note that j_i^* might be equal to 0 or i . Then

$$k^i(\pi) = \sum_{e \in P_{ij_i^*}} \gamma(e)$$

for all $i \in N$ and $\pi \in \Pi$. The associated direct cost game (N, c_d) and a generalised Bird allocation $\beta(N, \Pi, k)$ are determined by (2.1) and (2.3), respectively. Since there is only one alternative it follows that this proper *opr* (N, Π, k) is efficient. We obtain the following proposition by the non-negativity of the cost function.

Proposition 4.1 *Let I be a maintenance problem. Then the corresponding *opr* (N, Π, k) is a *peopr*.*

Since (N, Π, k) is an adequate *peopr* of I , we suggest to model the class of maintenance problems by the direct cost game. This cost game coincides with the cost game in literature,

e.g., discussed in Koster (1999). Further, one can easily deduce that (N, Π, k) satisfies *poi*. Therefore, we obtain by Theorem 2.4 the following proposition.

Proposition 4.2 *Let I be a maintenance problem with corresponding opr (N, Π, k) . Then $\beta(N, \Pi, k) \in C(N, c_d)$.*

5 Minimum cost spanning tree situations

In this section we consider the class of minimum cost spanning tree, or *mcst*, situations. Claus and Kleitman (1973) and Bird (1976) assume that the edge between any two players can only be used if the players cooperate with each other, and within this context Bird (1976) proposes a transferable utility game. In this section we assume that all edges are publicly available. Interestingly however, the three-step procedure leads to the same transferable utility game. By the use of Theorems 2.2, 2.4 and 2.5 this enables us to provide two alternative proofs of the fact that Bird's tree allocation is in the core of this game. We refer to Kleppe (2010) for an alternative proof of the fact that also the P -value (Branzei et al. (2004)) is in the core of this game.

Formally, a *minimum cost spanning tree situation* is a triple $M = (N, 0, \gamma)$, with N the finite player set and 0 the source. The objective of each player is to get connected to the source. By E_T we denote the set of all edges between pairs in $T \subseteq N^0$, *i.e.*, (T, E_T) is the complete network on T . Further, $\gamma : E_{N^0} \rightarrow \mathbb{R}_+$ is a non-negative cost function specifying the cost $\gamma(e)$ to construct an edge $e \in E_{N^0}$. We assume that γ satisfies the triangle inequality, which means that $\gamma(j, \ell) \leq \gamma(j, i) + \gamma(i, \ell)$ for all $i, j, \ell \in N^0$. An edge is also denoted by (j, ℓ) , with $j, \ell \in N^0$.

An *mcst* situation can be reformulated as a cooperative situation $(N, \mathcal{A}, K)$, with $\mathcal{A}$ the set¹¹ of all spanning trees on N^0 and $K(\alpha)$ the cost of spanning tree $\alpha \in \mathcal{A}$.

Bird (1976) associates with each *mcst* situation $M = (N, 0, \gamma)$ the cooperative cost game (N, c_M) , where $c_M(S)$, $S \subseteq N$, represents the minimum cost of a tree on S^0 :

$$c_M(S) = \min \left\{ \sum_{e \in R} \gamma(e) \mid R \subseteq E_{S^0} \text{ and } (S^0, R) \text{ is a tree} \right\}$$

for all $S \subseteq N$.

Let a minimum cost spanning tree for the grand coalition N be given by (N^0, R^*) and let e^i , for all $i \in N$, be the first edge on the unique path in (N^0, R^*) from player i to the

¹¹Formally, $\mathcal{A}$ consists of all networks with a path from each player to the source. However, by limiting ourselves to spanning trees on N^0 we simplify the presentation of the analysis, without changing any of the results. In particular, the cost games and cost allocations that result from our procedure are the same.

source. Then *Bird's tree allocation*¹² $\dot{\beta}$ of M is obtained by assigning to each player $i \in N$ the cost of e^i , hence

$$\dot{\beta}^i(M) = \gamma(e^i)$$

for all $i \in N$.

5.1 Alternative problem representation

In Subsection 5.2 we derive by the three-step procedure that the game (N, c_M) is appropriate for *mcst* situations. However, we think that an approach with alternatives, in the form of spanning trees, is a more natural starting point for the analysis of *mcst* situations. Therefore, we start by providing an alternative proof, based on the third step of the procedure using the *apr* set-up, of the result that the Bird allocation is in the core of the game (N, c_M) .

For an *mcst* situation $M = (N, 0, \gamma)$ we define the *apr* $(N, \mathcal{A}, k)$, where the individualised cost function is given by $k : \mathcal{A} \rightarrow \mathbb{R}^N$, with

$$k^i(\alpha) = \gamma(e^i(\alpha)).$$

Here $e^i(\alpha)$ denotes the first edge on the unique path in spanning tree $\alpha \in \mathcal{A}$ from player i to the source. The minimal cost game (N, c_-) and the maximal cost game (N, c_+) , as well as a generalised Bird allocation $\beta(N, \mathcal{A}, k)$ are straightforwardly determined by (2.4), (2.5) and (2.6), respectively.

Lemma 5.1 *Let $(N, \mathcal{A}, k)$ be an apr of mcst situation M . Then for every Bird allocation $\dot{\beta}(M)$ there exists a generalised Bird allocation $\beta(N, \mathcal{A}, k)$ such that $\dot{\beta}(M) = \beta(N, \mathcal{A}, k)$, and vice versa.*

Proof: Let $\alpha_N^* = (N^0, R^*)$ be a minimum cost spanning tree. Then there exists a $\beta(N, \mathcal{A}, k)$ such that

$$\begin{aligned} \dot{\beta}^i(M) &= \gamma^i(e^i(N^0, R^*)) \\ &= \gamma^i(e^i(\alpha_N^*)) \\ &= k^i(\alpha_N^*) \\ &= \beta^i(N, \mathcal{A}, k) \end{aligned}$$

for all $i \in N$. □

¹²We denote this allocation by $\dot{\beta}$ instead of β to distinguish between a Bird allocation and a *generalised* Bird allocation defined in Section 2.

A careful inspection leads to the conclusion that the coalitional costs for $S \subseteq N$ according to the minimal cost game equal the additional costs for coalition S to build a tree on N^0 , given that the players in $N \setminus S$ constructed a tree on $N^0 \setminus S$. Therefore, (N, c_-) equals the “optimistic TU-game” of Bergantiños and Vidal-Puga (2007).

Example 5.2 Consider the *mcs*t situation $M = (N, 0, \gamma)$ depicted in Figure 5.1. The

Figure 5.1: A minimum cost spanning tree situation

coalitional costs for the games (N, c_M) , (N, c_-) and (N, c_+) are given in the table below.

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1, 2\}$	$\{1, 3\}$	$\{2, 3\}$	N
$c_M(S)$	4	5	6	5	6	8	7
$c_-(S)$	1	1	2	3	3	3	7
$c_+(S)$	4	4	4	5	6	6	7

By definition $c_M(S)$ denotes the cost to form a tree on S^0 . Hence, $c_M(\{1, 2\}) = 5$, because coalition $\{1, 2\}$ can use the edges $(1, 0)$ and $(2, 1)$ to form a tree on $\{0, 1, 2\}$. Further, $c_-(S)$ reflects the cost to build a tree on N^0 , given that there is a tree on $N^0 \setminus S$. Hence, coalition $\{1, 2\}$ only has a cost of 3 to connect itself to the nodes of the set $\{0, 3\}$, as players 1 and 2 can use the edges $(2, 1)$ and $(1, 3)$ for this. The individual costs for the corresponding tree $\alpha \in \mathcal{A}$ are given by $k(\alpha) = (2, 1, 6)$.

The unique minimum cost spanning tree, $\alpha_N^* \in \mathcal{A}$, consists of the edges $(1, 0)$, $(2, 1)$ and $(3, 1)$, which leads to $\beta(N, \mathcal{A}, k) = (4, 1, 2)$. $\triangleleft$

Lemma 5.3 *Let M be an $mcst$ situation with associated cost game (N, c_M) . Let $(N, \mathcal{A}, k)$ be the corresponding apr with associated maximal cost game (N, c_+) . Then $C(N, c_+) \subseteq C(N, c_M)$.*

Proof: Since $\sum_{i \in N} k^i(\alpha_N^*)$ is the cost of the minimum cost spanning tree it is clear that $c_M(N) = c_+(N)$. Hence, it suffices to show that $c_+(S) \leq c_M(S)$ for all $S \subseteq N$. Let $S \subseteq N$. Then

$$\begin{aligned} c_+(S) &= c_-(N) - c_-(N \setminus S) \\ &\leq \min_{\alpha \in \mathcal{A}} \left\{ \sum_{i \in N} k^i(\alpha) \mid (S^0, \bigcup_{i \in S} e^i(\alpha)) \text{ is a tree} \right\} - c_-(N \setminus S) \\ &= c_M(S) + c_-(N \setminus S) - c_-(N \setminus S) \\ &= c_M(S), \end{aligned}$$

where the second equality follows from the fact that $\min_{\alpha \in \mathcal{A}} \{ \sum_{i \in N} k^i(\alpha) \mid (S^0, \bigcup_{i \in S} e^i(\alpha)) \text{ is a tree} \}$ equals the minimum cost to build a tree on S^0 ($c_M(S)$) plus the minimum cost to build a tree on N^0 given that there is a tree on S^0 ($c_-(N \setminus S)$). $\square$

We are now ready to provide an alternative proof of the fact that each Bird's tree allocation is in the core of the game (N, c_M) . The original result is due to Bird (1976).

Theorem 5.4 *Let M be an $mcst$ situation. Then $\dot{\beta}(M) \in C(N, c_M)$.*

Proof: Let $(N, \mathcal{A}, k)$ be the corresponding apr of M . By Lemma 5.1 there exists a generalised Bird allocation $\beta(N, \mathcal{A}, k)$ such that $\dot{\beta}(M) = \beta(N, \mathcal{A}, k)$. Hence, by Theorem 2.5, $\dot{\beta}(M) \in C(N, c_+)$. Finally, by Lemma 5.3, $\dot{\beta}(M) \in C(N, c_M)$. $\square$

5.2 Order problem representation

In this subsection we derive by the three-step procedure that the game (N, c_M) is appropriate to model $mcst$ situations. Since orderings are not part of the underlying cooperative situation, an algorithm to solve $mcst$ situations that is based upon such orderings is a natural starting point for this representation.

For an $mcst$ situation $M = (N, 0, \gamma)$ we define the opr $(N, \Pi, \tilde{k})$, where $\tilde{k} : \Pi \rightarrow \mathbb{R}^N$ denotes the minimum cost for a player to connect his node to the source (possibly via other nodes) given the tree constructed by his predecessors. Formally,

$$\tilde{k}^i(\pi) = \min_{(i,j) \in E_{N^0}} \left\{ \gamma(i,j) \mid j \in (V^i(\pi) \cup \{0\}) \right\}$$

for all $i \in N$ and $\pi \in \Pi$.¹³ Note that the *opr* $(N, \Pi, \tilde{k})$ is proper. Furthermore, there exists a π_N^* such that this procedure corresponds to *Prim's algorithm* (Prim (1957)) to obtain a minimum cost spanning tree. Therefore, it is also efficient, which implies that $(N, \Pi, \tilde{k})$ is adequate. The associated direct cost game $(N, \tilde{c}_d)$ and a generalised Bird allocation $\beta(N, \Pi, \tilde{k})$ are determined by (2.1) and (2.3), respectively.

Lemma 5.5 *Let $(N, \Pi, \tilde{k})$ be an opr of an mcst situation M . Then for every Bird allocation $\hat{\beta}(M)$ there exists a generalised Bird allocation $\beta(N, \Pi, \tilde{k})$ such that $\hat{\beta}(M) = \beta(N, \Pi, \tilde{k})$, and vice versa.*

The proof of this lemma is similar to the proof of Lemma 5.1 and is therefore omitted.

Since the individualised cost function $\tilde{k}$ is only based upon the set of predecessors of each player, and not on their ordering we obtain the following lemma.

Lemma 5.6 *Let M be an mcst situation. Then the corresponding opr $(N, \Pi, \tilde{k})$ satisfies poi.*

Let $(N, \Pi, \tilde{k})$ be an *opr* of $M = (N, 0, \gamma)$. The definition of $\tilde{k}^i$ implies that the later player i is ordered, the more choice he has and therefore, the lower his individual cost. Hence, all players prefer to be ordered as late as possible. Since $(N, \Pi, \tilde{k})$ satisfies *poi* this implies that all groups of players want to be ordered as late as possible. Hence, $(N, \Pi, \tilde{k})$ is a *peopr*.

Proposition 5.7 *Let M be an mcst situation. Then the corresponding opr $(N, \Pi, \tilde{k})$ is a peopr.*

Since the *opr* $(N, \Pi, \tilde{k})$ is an adequate *peopr* of M , we suggest to model the class of *mcst* situations by the associated direct cost game $(N, \tilde{c}_d)$. This game coincides with the TU-game (N, c_M) .

Proposition 5.8 *Let M be an mcst situation with associated cost game (N, c_M) . Let $(N, \Pi, \tilde{k})$ be the corresponding opr with associated direct cost game $(N, \tilde{c}_d)$. Then $(N, c_M) = (N, \tilde{c}_d)$.*

¹³Note that a player may have more than one cheapest option to connect himself to the source. Therefore, the tree constructed by this procedure need not be unique for a $\pi \in \Pi$. However, the choice of one player does not influence the options or costs of any other player. This implies that $\tilde{k}(\pi)$ is unique for each $\pi \in \Pi$.

Proof: Let $S \subseteq N$. The coalitional cost for S according to (N, c_M) denotes the cost of the cheapest tree on S^0 . Consider Prim's algorithm to obtain such a minimum cost spanning tree on S^0 and let $\pi \in \Pi_S$ be the ordering in which the players make their connection according to this algorithm. Then $c_M(S) = \sum_{i \in S} \tilde{k}^i(\pi) = \tilde{c}_d(S)$. $\square$

We also provide a second alternative proof of the result that Bird's tree allocation $\dot{\beta}$ is in the core of the game (N, c_M) .

Alternative proof of Theorem 5.4: Let $(N, \Pi, \tilde{k})$ be the corresponding *opr* of *mcst* situation M , with associated direct cost game $(N, \tilde{c}_d)$ and a generalised Bird allocation $\beta(N, \Pi, \tilde{k})$ such that $\beta(N, \Pi, \tilde{k}) = \dot{\beta}(M)$ (Lemma 5.5). Since $(N, \Pi, \tilde{k})$ is a *peopr* (Proposition 5.7) that satisfies *poi* (Lemma 5.6) we obtain by Theorem 2.4 that $\dot{\beta}(M) \in C(N, \tilde{c}_d)$. Finally, by Proposition 5.8, $\dot{\beta}(M) \in C(N, c_M)$. $\square$

5.3 Group efficiency

Consider the *mcst* situation of Figure 5.1. The coalitional costs for the direct cost game $(N, \tilde{c}_d)$ are given in the table below, and coincide by Proposition 5.7 with the coalitional costs for the game (N, c_M) (see Example 5.2).

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1, 2\}$	$\{1, 3\}$	$\{2, 3\}$	N
$\tilde{c}_d(S)$	4	5	6	5	6	8	7

We focus on the cost of coalition $\{2, 3\}$. This cost of 8 is the result of the ordering in *opr* $(N, \Pi, \tilde{k})$ in which player 2 is first and connects his node to the source with a cost of 5. Then player 3 connects his node to the node of player 2 with a cost of 3. Hence, together they build the network consisting of the edges $(0, 2)$ and $(2, 3)$. However, given that all edges are publicly available, one could argue that players 2 and 3 together could build the network consisting of the edges $(0, 1)$, $(1, 2)$ and $(1, 3)$ for a total cost of only 7. Hence, in this situation $\tilde{c}_d(\{2, 3\})$ does not coincide with the cost of the cheapest structure in which all the objectives of the players in $\{2, 3\}$ are met.

For a cooperative situation $(N, \mathcal{A}, K)$ we have defined $\mathcal{A}$ as the set of alternatives, *i.e.*, the set of feasible structures for N . Furthermore, we have defined that an *opr* (N, Π, k) of $(N, \mathcal{A}, K)$ is efficient if $\min_{\pi \in \Pi} \sum_{i \in N} k^i(\pi) = \min_{\alpha \in \mathcal{A}} K(\alpha)$. In a similar way we now define $\mathcal{A}_S$ as the set of structures, which are feasible for S . Moreover, we call an *opr* (N, Π, k) of $(N, \mathcal{A}, K)$ *group efficient* if for all $S \subseteq N$, $\min_{\pi \in \Pi_S} \sum_{i \in S} k^i(\pi) = \min_{\alpha_S \in \mathcal{A}_S} K(\alpha_S)$. We call a class of cooperative situations group efficient if the proper *opr* of each cooperative situation is group efficient.

Group efficiency ensures that for each group of players there exists an ordering in the *opr* that results in the cheapest structure that is feasible for that group. Obviously, this condition is not met by the *opr* of the *mcst* situation of Figure 5.1. The reason is that in this *mcst* situation the presence of the node of player 1 is beneficial to the other players, even without the participation of player 1 himself. However, to profit from it full cooperation is needed, not only the cooperation in determining the order in which the players build on the structure, as in the *opr* set-up. Note that within the context of *mcst* situations with private edges such a benefit cannot occur, as the edges to which player 1 is connected cannot be used by a coalition of which player 1 is not a member.

If a class of cooperative situations does not satisfy group efficiency, then the innate tension between the public aspect of the cooperative situation on the one hand and the individualised nature of the *opr* set-up on the other is not fully resolved. As a consequence, the proposed cost game gives a good, but incomplete picture of the possibilities of the coalitions. However, if a class of cooperative situations satisfies group efficiency, the appropriate TU-game proposed by our three-step procedure fully captures the coalitions' possibilities in the cooperative situation.

The fact that the class of *mcst* situations does not satisfy group efficiency makes this class an exception. Analysing the notion of group efficiency for the class of sequencing situations without initial ordering, as discussed in Section 3, it is readily checked that this class of cooperative situations is group efficient, because the ordering itself determines the individual cost of the players. From a more general perspective, a *neopr* typically satisfies group efficiency by its very nature, as a *neopr* reflects that the presence of other players increases the cost of each group of players.

For the class of maintenance problems, discussed in Section 4, the proper *opr* of each cooperative situation is a *peopr*. However, also here one can easily verify that the class of maintenance problems satisfies group efficiency. Furthermore, all the classes discussed in the remainder of this paper satisfy group efficiency whenever they satisfy efficiency.

6 Permutation situations without initial allocation

In this section we introduce the class of permutation situations without initial allocation. Permutation situations are introduced by Tijs et al. (1984). A *permutation situation without initial allocation* is given by a triple $P = (N, \Theta, \Gamma)$, with N the finite set of players, $\Theta = \{1, \dots, |N|\}$ the finite set of machines and Γ an $N \times \Theta$ cost matrix. Element Γ_{ij} of this matrix denotes the cost for the use of machine $j \in \Theta$ by player $i \in N$. The idea is to allocate each machine to a different player with minimal total cost.

An optimal allocation for permutation situation without initial allocation P is denoted by $\alpha^*(P)$ with $(i, j) \in \alpha^*(P)$ an assignment of machine j to player i . One can find an optimal allocation by the *Hungarian method* (Kuhn (1955)). In the paper by Tijs et al. (1984) it is assumed that there is an initial allocation of machines to players and that a coalition of players can interchange their machines between them. In the current framework, however, we assume there is no initial allocation. Therefore, a permutation situation without initial allocation P can be reformulated as a cooperative situation $(N, \mathcal{A}, K)$, with $\mathcal{A}$ the set of allocations of machines to players, such that each player has one machine and $K(\alpha)$ the sum of the individual costs of allocation $\alpha \in \mathcal{A}$.

Since we want to keep focus on the main ideas we assume in this section that each element in Γ has a different value. The results of Propositions 6.2, 6.3 and 6.4 are, however, valid for the general case. Then for a permutation situation without initial allocation $P = (N, \Theta, \Gamma)$ we define the proper *opr* (N, Π, k) . Here the individualised cost function $k : \Pi \rightarrow \mathbb{R}^N$ is defined such that $k^i(\pi)$ denotes the cost of the cheapest possible machine player i can choose at his turn in π . Let $\Theta_t(\pi)$ denote the set of available machines given that players $\pi(1), \dots, \pi(t-1)$ have made their choice. Then

$$k^i(\pi) = \min_{j \in \Theta_{\pi^{-1}(i)}(\pi)} \Gamma_{ij}$$

for all $i \in N$. In Proposition 6.2 we show that this proper *opr* is also efficient. Let for all $i \in N$, $j_i^* = \operatorname{argmin}_{j \in \Theta} \Gamma_{ij}$ be the cheapest machine for player i .

Lemma 6.1 *Let P be a permutation situation without initial allocation. In an optimal allocation of machines to players there is at least one player i that is assigned j_i^* .*

Proof: Suppose that the optimal allocation is such that there does not exist any player $i \in N$ that is assigned j_i^* . Then there exists at least one sequence of players such that the cheapest machine for each player is assigned to his follower in this sequence, with the final player of the sequence being assigned the cheapest machine of the first player. Let us then give each player the machine of his follower in the sequence. By this procedure the cost of the original allocation is decreased, which means that the original allocation was not optimal. This is a contradiction. $\square$

Proposition 6.2 *Let P be a permutation situation without initial allocation. Then the corresponding *opr* (N, Π, k) is efficient.*

Proof: Let P be given by $P = (N, \Theta, \Gamma)$. We prove efficiency by providing an algorithm that gives an optimal ordering π_N^* for the *opr* (N, Π, k) .

Algorithm

Input: a permutation situation without initial allocation $P = (N, \Theta, \Gamma)$ and an optimal assignment $\alpha^*(P)$.

Output: an ordering π_N^* for the opr (N, Π, k) .

1. Set $q = 1$ and $P_q = (N_q, \Theta_q, \Gamma_q) = (N, \Theta, \Gamma)$.
2. Let $\alpha^*(P_q)$ be an optimal assignment of P_q and let $t \in N_q$ be such that $(t, j_t^*) \in \alpha^*(P_q)$.
3. Set $\pi(q) = t$.
4. Set $N_{q+1} = N_q \setminus \{t\}$, $\Theta_{q+1} = \Theta_q \setminus \{j_t^*\}$ and $\Gamma_{q+1} = N_{q+1} \times \Theta_{q+1}$.
5. If $q = n$ stop, otherwise set $q = q + 1$ and go to Stage 2.

The above algorithm orders the players such that each player gets at his turn his cheapest machine in the reduced problem (the problem without his predecessors and the machines chosen by them). Note that this algorithm does not necessarily result in a unique ordering, as in Stage 2 there may be several optimal assignments $\alpha^*(P_q)$ with several players t such that $(t, j_t^*) \in \alpha^*(P_q)$. However, any choice results in an ordering and it follows by Lemma 6.1 that this ordering is optimal and leads to an optimal allocation. Therefore, the opr (N, Π, k) is efficient. $\square$

Since the number of available machines decreases (and therefore the cost increases) with the position of a player in the ordering we have the following proposition.

Proposition 6.3 *Let P be a permutation situation without initial allocation. Then the corresponding opr (N, Π, k) is a neopr.*

Since (N, Π, k) is an adequate neopr, we suggest to model this class of cooperative situations by the marginal cost game (N, c_m) , which is determined by (2.2). An associated generalised Bird allocation is determined by (2.3). The following proposition follows from Proposition 6.3 and Theorem 2.3.

Proposition 6.4 *Let P be a permutation situation without initial allocation with corresponding opr (N, Π, k) . Then $\beta(N, \Pi, k) \in C(N, c_m)$.*

7 Public congestion network situations

In this section we consider the class of public congestion network situations, as introduced by Kleppe et al. (2010). In such a situation all players have to connect themselves to the root, while the cost of each arc depends on the number of its users. A *public congestion network situation*, or congestion network situation as we call it from here, is formally given by a triple $G = (N, 0, \gamma)$, where N is the finite set of players that has to be connected to the root 0. By A_S we denote the set of all arcs between pairs in $S \subseteq N^0$, i.e., (S, A_S) is the complete digraph on S . For each arc $a \in A_{N^0}$ the function $\gamma_a : \{0, 1, \dots, |N|\} \rightarrow \mathbb{R}_+$ associates each number of users of arc a with a corresponding cost. The function γ_a is non-negative and (weakly) increasing. We assume that $\gamma_a(0) = 0$ for all $a \in A_{N^0}$. Elements of A_{N^0} are also denoted by (i, j) , where $i, j \in N^0$. The arc (i, j) then denotes the connection between i and j in the direction from i to j . The cost function of an arc (i, j) is denoted by $\gamma_{i,j}$. A congestion network situation is called symmetric if $\gamma_{i,j} = \gamma_{j,i}$ for all $i, j \in N^0$.

In a congestion network situation each player chooses a path from his initial node to the root. A path between any two nodes i and j is denoted by $P(i, j)$ and is a sequence of arcs $((i_0, i_1), (i_1, i_2), \dots, (i_{p-1}, i_p))$, such that $i_0 = i$, $i_p = j$ and $i_r \neq i_s$ for all $r, s \in \{0, \dots, p-1\}$, $r \neq s$. Furthermore, instead of $P(i, 0)$ we also write P^i .

A *network* is defined by an integer valued function $f : A_{N^0} \rightarrow \{0, 1, \dots, |N|\}$, such that f assigns to each arc a number of users. The netdegree for a node $i \in N^0$ with respect to network f is defined by $\text{netdegree}^f(i) = \sum_{j \in N^0 \setminus \{i\}} f(i, j) - \sum_{j \in N^0 \setminus \{i\}} f(j, i)$. For a coalition $S \subseteq N$ the collection of all feasible networks connecting the members of S to the root is given by

$$F_S = \left\{ f : A_{N^0} \rightarrow \{0, \dots, n\} \mid \begin{array}{l} \text{netdegree}^f(i) = 1 \text{ for all } i \in S, \\ \text{netdegree}^f(i) = 0 \text{ for all } i \in N \setminus S, \\ f(a) \in \{0, \dots, |S|\} \text{ for all } a \in A_{N^0} \end{array} \right\}.$$

An element of F_S is denoted by f_S . Note that in a feasible network for S each player of S is connected to the root by some path. However, as all arcs are publicly available these paths may consist of arcs between any two nodes in N^0 . Each network f induces a digraph (N^0, A_f) , where A_f consists of all used arcs: $A_f = \{a \in A_{N^0} \mid f(a) > 0\}$. The cost of a network f is defined by

$$\gamma(f) = \sum_{a \in A_f} \gamma_a(f(a)).$$

A network for $S \subseteq N$ is called optimal if it is feasible with minimum total cost. Such

an optimal network is denoted by $f_S^* \in F_S$. A congestion network situation can be reformulated as a cooperative situation $(N, \mathcal{A}, K)$, with $\mathcal{A} = F_N$ and $K(f_N) = \gamma(f_N)$, $f_N \in F_N$.

A *convex congestion network situation* $G = (N, 0, \gamma)$ is a congestion network situation in which all γ_a are convex. A cost function γ_a , $a \in A_{N^0}$, is convex if $\gamma_a(r+1) - \gamma_a(r) \geq \gamma_a(r) - \gamma_a(r-1)$ for all $r \in \{1, \dots, |N| - 1\}$.

In Kleppe et al. (2010) each convex congestion network situation is associated with a direct and a marginal cost game. To distinguish between these two games and the games defined in Section 2 of this paper, the direct cost game of Kleppe et al. (2010) is called the *direct congestion cost game* and we denote it by (N, c_{dG}) , with

$$c_{dG}(S) = \min_{f_S \in F_S} \gamma(f_S)$$

for all $S \subseteq N$. Similarly, the marginal cost game of Kleppe et al. (2010) is called the *marginal congestion cost game*, denoted by (N, c_{mG}) , with

$$c_{mG}(S) = c_{dG}(N) - c_{dG}(S)$$

for all $S \subseteq N$.

Example 7.1 An example of a symmetric convex congestion network situation $G = (N, 0, \gamma)$ is given in Figure 7.1. In this situation there are three players, which are denoted by 1, 2 and 3, and the root, which is denoted by 0. The numbers on the arcs represent the total usage costs for each number of users. All coalitional costs for the direct and marginal congestion cost game associated with this congestion network situation are given in the next table.

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1, 2\}$	$\{1, 3\}$	$\{2, 3\}$	N
$c_{dG}(S)$	1	3	2	7	6	8	14
$c_{mG}(S)$	6	8	7	12	11	13	14

The coalitional costs for the direct congestion cost game are determined by the cost of an optimal network for S in absence of $N \setminus S$. The optimal network for $S = \{2, 3\}$, *e.g.*, is given by f_S^* with $f_S^*(2, 3) = 1$, $f_S^*(3, 1) = f_S^*(1, 0) = 2$ with $\gamma(f_S^*) = 8$. By duality we obtain the marginal congestion cost game. $\triangleleft$

Let $G = (N, 0, \gamma)$ be a convex congestion network situation and let $S \subseteq N$ with feasible network f_S . Let $D_{f_S} = \{D_{f_S}^i\}_{i \in S}$ be a *decomposition* of f_S into $|S|$ parts such that $D_{f_S}^i$

Figure 7.1: A symmetric convex congestion network situation

corresponds to a specific path P^i , $i \in S$, from player i to the root. Given a network such a decomposition need not be unique, in the sense that we cannot distinguish which arcs are used by which players.

Example 7.2 Consider the convex congestion network situation of Figure 7.1, with the optimal network f_N^* given by $f_N^*(2,3) = f_N^*(3,1) = f_N^*(3,0) = 1$ and $f_N^*(1,0) = 2$. This network does not have a unique decomposition. $D_{f_N^*}$ is a decomposition of f_N^* , where $D_{f_N^*}^1$ corresponds to the path $((1,0))$, $D_{f_N^*}^2$ corresponds to $((2,3), (3,0))$ and $D_{f_N^*}^3$ corresponds to $((3,1), (1,0))$. Another decomposition is $\hat{D}_{f_N^*}$, where $\hat{D}_{f_N^*}^1$ corresponds to the path $((1,0))$, $\hat{D}_{f_N^*}^2$ corresponds to $((2,3), (3,1), (1,0))$ and $\hat{D}_{f_N^*}^3$ corresponds to $((3,0))$. Note that $D_{f_N^*}$ and $\hat{D}_{f_N^*}$ are the only possible decompositions. $\triangleleft$

In this section we first show that our procedure supports the use of the marginal congestion cost game to model convex congestion network situations. This procedure does, however, not support (nor reject) the use of the *direct* congestion cost game for *concave* congestion network situations as we show in the second part.

7.1 Convex congestion network situations

For a convex congestion network situation $G = (N, 0, \gamma)$ we define the proper *opr* (N, Π, k) , with k such that, given the current network constructed by his predecessors, player i at his turn in π chooses the cheapest path to the root. For the formal definition of the *opr* we assume for expositional purposes that each feasible network has a different cost, hence

$\gamma(f_S) \neq \gamma(f_T)$ for all $f_S \in \mathcal{F}_S$, $f_T \in \mathcal{F}_T$ and all $S, T \subseteq N$. This implies that each player has a unique cheapest path to the root given a feasible network for his predecessors. The results of Theorem 7.6 and Propositions 7.5, 7.7 and 7.8 are, however, also valid in the general case.

Let $\mathcal{P}^i$ denote the set of all paths from player i to the root. Let f^t be a feasible network constructed by the players $\pi(1), \dots, \pi(t-1)$ given $\pi \in \Pi$. Then

$$k^i(\pi) = \min_{P^i \in \mathcal{P}^i} \sum_{a \in P^i} \gamma(f^{\pi^{-1}(i)+1}(a)) - \gamma(f^{\pi^{-1}(i)}(a))$$

for all $i \in N$.

Example 7.3 Reconsider the convex congestion network situation G of Figure 7.1. The next table gives for each ordering $\pi \in \Pi$ the corresponding cost allocation $k(\pi)$ for the *opr* (N, Π, k) .

π	$k^1(\pi)$	$k^2(\pi)$	$k^3(\pi)$
(1, 2, 3)	1	6	7
(1, 3, 2)	1	8	5
(2, 1, 3)	4	3	7
(2, 3, 1)	8	3	5
(3, 1, 2)	4	8	2
(3, 2, 1)	8	6	2

Consider ordering $\pi = (2, 1, 3)$. The shortest path from player 2 to the root, given that no others formed a network, is the path $((2, 3), (3, 1), (1, 0))$ with a cost of 3. Then player 1 takes path $((1, 0))$ with a cost of 4. Finally, player 3 takes path $((3, 0))$ with a cost of 7. $\triangleleft$

We show in Corollary 7.6 that the proper *opr* (N, Π, k) is efficient. The associated direct (N, c_d) and marginal cost game (N, c_m) are determined by (2.1) and (2.2), respectively.

Example 7.4 Reconsider the convex congestion network situation G of Figure 7.1. All coalitional costs for the direct and marginal cost game associated with the corresponding *opr* (N, Π, k) are given in the next table.

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1, 2\}$	$\{1, 3\}$	$\{2, 3\}$	N
$c_d(S)$	1	3	2	7	6	8	14
$c_m(S)$	6	8	7	12	11	13	14

Note that they exactly coincide with the costs of the direct congestion cost game (N, c_{dG}) and the marginal congestion cost game (N, c_{mG}) , respectively. $\triangleleft$

Proposition 7.5 *Let G be a convex congestion network situation with associated marginal congestion cost game (N, c_{mG}) . Let (N, Π, k) be the corresponding opr with associated marginal cost game (N, c_m) . Then $(N, c_{mG}) = (N, c_m)$.*

Proof: We show that $c_{dG}(S) = c_d(S)$ for all $S \subseteq N$, which by duality leads to the desired result. To prove the equivalence of the direct congestion cost game and the direct cost game we provide an algorithm that produces for all $S \subseteq N$ an optimal ordering π_S^* for opr (N, Π, k) . It then follows that this ordering π_S^* results in an optimal network for S in absence of $N \setminus S$, f_S^* .

Let $f \setminus D_f^i$ be the network that results if you take out of a network f , with decomposition D_f , the path-network D_f^i of player $i \in N$.

Algorithm

Input: a convex congestion network situation $G = (N, 0, \gamma)$, a coalition $S \subseteq N$ and an optimal network f_S^ .*

Output: an optimal ordering π_S^ for the opr (N, Π, k) for coalition $S \subseteq N$.*

1. Set $q = 1$ and set network $\bar{f}^q = f_S^*$.
2. Let $D_{\bar{f}^q}$ be a path decomposition of $\bar{f}^q$.
3. Let player $i \in N$ be such that $D_{\bar{f}^q}^i$ corresponds to the shortest path from i to 0, given network $\bar{f}^q \setminus D_{\bar{f}^q}^i$.
4. Set $\pi_S^*(|S| + 1 - q) = i$.
5. Set $\bar{f}^{q+1} = \bar{f}^q \setminus D_{\bar{f}^q}^i$.
6. If $q = |S|$ stop, otherwise set $q = q + 1$ and return to Stage 2.

Note that in Stage 2 there may be more than one path decomposition. Any choice here, however, results in an optimal ordering. It is furthermore important to realise that in Stage 3 there always exists such a player.

We conclude that for $S \subseteq N$ with optimal network f_S^* the outcome π_S^* of the above algorithm results in an optimal network for S , i.e., $\sum_{i \in S} k^i(\pi_S^*) = \gamma(f_S^*)$. $\square$

Proposition 7.5 leads to the following corollary.

Corollary 7.6 *Let G be a convex congestion network situation. Then the corresponding $\text{opr } (N, \Pi, k)$ is efficient.*

The following proposition follows by the convexity of the cost function γ .

Proposition 7.7 *Let G be a convex congestion network situation. Then the corresponding $\text{opr } (N, \Pi, k)$ is a neopr .*

Since (N, Π, k) is an adequate neopr , we suggest to model the class of convex congestion network situations by the associated marginal cost game, which coincides by Proposition 7.5 with the marginal congestion cost game (N, c_{mG}) , used in Kleppe et al. (2010) to model these situations. Hence, the three-step procedure supports the use of the marginal congestion cost game for the class of convex congestion network situations. Finally, we also obtain a new element of its core. Note that $\beta(N, \Pi, k)$ is defined by (2.3) and corresponds to a specific decomposition of f_N^* .

Proposition 7.8 *Let G be a convex congestion network situation with corresponding $\text{opr } (N, \Pi, k)$. Then $\beta(N, \Pi, k) \in C(N, c_{mG})$.*

7.2 Concave congestion network situations

In this subsection we consider *concave* congestion network situations. A concave congestion network situation¹⁴ $G' = (N, 0, \gamma)$ is a congestion network situation in which all cost functions γ_a are concave. A cost function γ_a , $a \in A_{N^0}$, is concave if $\gamma_a(r+1) - \gamma_a(r) \leq \gamma_a(r) - \gamma_a(r-1)$ for all $r \in \{1, \dots, |N| - 1\}$.

For a concave congestion network situation the direct congestion cost game is denoted by $(N, c_{dG'})$. Kleppe et al. (2010) propose to use the direct congestion cost game to model concave congestion network situations. In this subsection we show that our procedure does not support (nor reject) this choice.

For a concave congestion network situation $G' = (N, 0, \gamma)$ we define the $\text{opr } (N, \Pi, \bar{k})$ such that

$$\bar{k}^i(\pi) = \min_{P^i \in \mathcal{P}^i} \sum_{a \in P^i} \gamma(f^{\pi^{-1}(i)+1}(a)) - \gamma(f^{\pi^{-1}(i)}(a))$$

¹⁴We denote concave congestion network situations by G' to distinguish between concave and convex congestion network situations.

for all $i \in N$.¹⁵ Note that this definition is equal to the definition of the *opr* for the class of convex congestion network situations. Therefore, $(N, \Pi, \bar{k})$ is proper.

By concavity of the cost function γ we obtain the following proposition.

Proposition 7.9 *Let G' be a concave congestion network situation. Then the corresponding *opr* $(N, \Pi, \bar{k})$ is a peopr.*

Hence, each *opr* $(N, \Pi, \bar{k})$ is appropriately modelled by the associated direct cost game. The *opr* $(N, \Pi, \bar{k})$ is, however, not efficient, which we show in the following example.

Example 7.10 Consider the symmetric concave congestion network situation $G' = (N, 0, \gamma)$ of Figure 7.2. The optimal network f_N^* is given by $f_N^*(1, 2) = 1$ and $f_N^*(2, 0) = 2$,

Figure 7.2: A concave congestion network situation

with a cost of 6. The corresponding *opr* is given by (N, Π, k) . However, both orderings $(1, 2)$ and $(2, 1)$ result in network f_N given by $f_N(2, 1) = 1$ and $f_N(1, 0) = 2$, with a cost of 7. $\triangleleft$

Since the *opr* $(N, \Pi, \bar{k})$ is not efficient, the concave congestion network situation G' and the corresponding *opr* $(N, \Pi, \bar{k})$ do not describe the same cost allocation problem. Consequently, our procedure cannot be used to find an appropriate TU-game for the class of concave congestion network situations, in particular it cannot be used to support the use of the direct congestion cost game.

8 Travelling salesman and related problems

In this section we consider three closely related classes of cooperative situations; next to the well-known travelling salesman problems, we introduce shared taxi problems and travelling repairman problems.

¹⁵Also here we assume for expositional purposes that $\gamma(f_S) \neq \gamma(f_T)$ for all $f_S \in \mathcal{F}_S$, $f_T \in \mathcal{F}_T$ and all $S, T \subseteq N$.

8.1 Travelling salesman problems

In this subsection we discuss the class of travelling salesman problems (Potters et al. (1992)). In a travelling salesman problem, or *tsp*, the objective of each player is to be visited by a single salesman. Formally, a *travelling salesman problem* can be given by a triple¹⁶ $L = (N, 0, \gamma)$, with N the finite set of players that has to be visited by the salesman, who starts and ends at node 0, called *home*. The function $\gamma : E_{N^0} \rightarrow \mathbb{R}_+$ is a non-negative cost function, which can be viewed as the travel time of the salesman from one node to another. The cost for the group of players is given by the total travel time of the salesman. We assume that all links are publicly available and therefore we can impose without loss of generality the triangle inequality. A *tsp* can be reformulated as a cooperative situation $(N, \mathcal{A}, K)$, with $\mathcal{A}$ all tours starting at 0, where a tour is defined as a path from a node to itself visiting all other nodes at least once. $K(\alpha)$ is the total travel time of the salesman in tour $\alpha \in \mathcal{A}$.

Let $L = (N, 0, \gamma)$ be a *tsp*. A proper *opr* must be such that each player builds a feasible structure at minimum cost without changing the structure created by his predecessors. For the first player the feasible structure with minimal cost is a tour from home to himself and back to home. Based on the properties of a proper *opr* the second player cannot change this structure. Therefore, he extends this structure with a tour from home to himself and back. This process continues for all other players. Hence, the proper *opr* (N, Π, k) of a *tsp* $L = (N, 0, \gamma)$ is such that $k^i(\pi) = 2 * \gamma(i, 0)$ for all $\pi \in \Pi$ and $i \in N$, *i.e.*, each player pays for a “private” visit by the salesman. The *opr* (N, Π, k) is clearly not efficient. Therefore, our procedure is unfortunately of no use in finding an appropriate TU-game for the class of *tsp*s.

One might get the idea that the link from the last player in a tour back home, which leads to the fact that the costs of $|N| + 1$ edges are to be shared by $|N|$ players, is the reason our procedure is not suited for the class of *tsp*s. To study the effect of this extra edge we analyse a closely related class of cooperative situations in the next subsection.

8.2 Shared taxi problems

In this subsection we introduce and analyse the class of *shared taxi problems*. The idea is as follows. A group of players is at a particular location and shares a single taxi in order to reach their individual destinations. The cost of the taxi only depends on the distance from the starting point to the final individual destination. The group of players wants to minimise the total cost.

¹⁶Potters et al. (1992) use N^0 to indicate the player set including home. Furthermore, they denote the $(|N| + 1) \times (|N| + 1)$ cost matrix by K .

Formally, a *shared taxi problem*, or *stp*, is given by a triple $H = (N, 0, \gamma)$, where N is the finite set of players that shares the taxi, which starts at node 0. The function $\gamma : E_{N^0} \rightarrow \mathbb{R}_+$ is a non-negative cost function, which can be viewed as the taxi cost from one node to another. We consider publicly available networks, which implies that the taxi is able to use the edge between any two nodes of the network. As a result, we can impose without loss of generality that the function γ satisfies the triangle inequality. An *stp* can be reformulated as a cooperative situation $(N, \mathcal{A}, K)$, with $\mathcal{A}$ the set of routes starting from 0, where a route is a path from one node to another node visiting all remaining nodes at least once. $K(\alpha)$ is the cost for the taxi given route $\alpha \in \mathcal{A}$.

The cheapest way to drop off all the players is with a trip, where a trip is a path from node 0 to a node $i \in N$ such that all players $j \in N$ are visited *exactly once*. The ordering of the players in this trip is given by $\pi \in \Pi$. It follows that the total cost for a group of players $S \subseteq N$ to use the taxi equals $\gamma(0, \pi(1)) + \sum_{t=2}^{|S|} \gamma(\pi(t-1), \pi(t))$, with $\pi \in \Pi_S$.

Note that the class of *stps* is closely related to the class of *tsps*, in the sense that the only difference with the latter is that for an *stp* the link from the last player back to node 0 is excluded from the structure (and the costs).

We analyse the associated cost allocation problem by the use of the three-step procedure. For an *stp* $H = (N, 0, \gamma)$ we define the proper *opr* (N, Π, k) such that the individualised cost function $k : \Pi \rightarrow \mathbb{R}^N$ is given by

$$k^i(\pi) = \gamma(\pi(\pi^{-1}(i) - 1), i)$$

for all $i \in N$ and $\pi \in \Pi$. Hence, the individual cost of a player denotes the cost for the taxi to travel from the node of his direct predecessor to his own node.

It is easily seen that the proper *opr* (N, Π, k) is efficient. Hence, the change from a *tsp* to an *stp* by not including the edge back to node 0 in the structure is an important difference for our procedure. The following example, however, shows that there exists an *stp* H for which the corresponding *opr* (N, Π, k) is neither a *neopr* nor a *peopr*.

Example 8.1 Let Figure 8.1 represent the *stp* $H = (N, 0, \gamma)$. The numbers on the edges denote the cost for the taxi to travel from one node to another. The individual costs for a given ordering according to the corresponding *opr* (N, Π, k) are given in the next table.

Figure 8.1: A shared taxi problem

π	$k^1(\pi)$	$k^2(\pi)$	$k^3(\pi)$
(1, 2, 3)	1	2	3
(1, 3, 2)	1	3	4
(2, 1, 3)	2	2	4
(2, 3, 1)	4	2	3
(3, 1, 2)	4	2	5
(3, 2, 1)	2	3	5

Each player pays for the edge directly leading to him. Therefore, $\pi = (1, 2, 3)$ leads to the individualised costs $k^1(\pi) = 1$, $k^2(\pi) = 2$ and $k^3(\pi) = 3$. Since an optimal ordering for player 3 is either $(1, 2, 3)$ or $(2, 3, 1)$ it is not beneficial for all coalitions to be ordered first. Hence, the *opr* (N, Π, k) is no *neopr*. Further, as an optimal ordering for player 1 is either $(1, 2, 3)$ or $(1, 3, 2)$ it is also not beneficial for all coalitions to be ordered last. Therefore, the *opr* (N, Π, k) is no *peopr* either. $\triangleleft$

Example 8.1 demonstrates that the *opr* (N, Π, k) is neither a *neopr* nor a *peopr*, which implies that we cannot argue in favour of either the direct cost game (N, c_d) , determined by (2.1) or the marginal cost game (N, c_m) , determined by (2.2), for the class of *tsps*.

We can therefore also conclude that the fact that the three-step procedure cannot be applied to *tsps* is not (only) due to the link from the last player back home. Part of the reason also lays in the fact that the presence of other players can have either a positive or a negative effect on the cost of a player.

8.3 Travelling repairman problems

In this subsection we consider situations in which several players need to be visited by a single repairman. These players, as well as the repairman, are not located at the same place and therefore, the repairman has to decide on a specific way to visit all the players. The cost for each player depends on the time he has to wait for the arrival of the repairman. As a result we obtain the problem of finding a route that minimises the total waiting time of the players. This operations research problem is known as a travelling repairman problem (Afrati et al. (1986)). A natural example of the above situation is to think of the players as factories with broken machinery that needs to be repaired. In this case the costs reflect opportunity costs of production.

Note that a travelling repairman problem can be seen as a special type of sequencing situation in which the processing time of a player depends on his predecessor, *e.g.*, due to changeover costs. The difference between the class of travelling repairman problems on the one hand and the classes of travelling salesman and shared taxi problems on the other is that for the latter two classes the goal is to minimise the travel time of the salesman or taxi, while for the class of travelling repairman problems the players' waiting time is to be minimised. As a result, the players have individual costs.

A *travelling repairman problem*, or *trp*, is given by a triple $T = (N, 0, \gamma)$, where N is the finite set of players (nodes) that has to be visited by the repairman. He starts at node 0, called *home*. The function $\gamma : E_{N^0} \rightarrow \mathbb{R}_+$ is a non-negative cost function, which can be viewed as the travel time of the repairman from one node to another.

In a *trp* the single repairman has to visit all players. We consider publicly available networks, which means that the repairman is able to use the edge between any two nodes of the network. As a result, we can impose without loss of generality that the function γ satisfies the triangle inequality. Hence, a *trp* can be reformulated as a cooperative situation $(N, \mathcal{A}, K)$, with $\mathcal{A}$ the set of routes starting from 0, where a route is a path from one node to another node visiting all remaining nodes at least once. $K(\alpha)$ is the sum of the individual waiting times given route $\alpha \in \mathcal{A}$.

The optimal way for the repairman to visit all the players is to make a trip in which he visits all the players according to some ordering $\pi \in \Pi$. The individual cost of a player i for a particular ordering π equals the total waiting time of this player, and is given by $c^i(\pi) = \gamma(0, \pi(1)) + \sum_{t=2}^{\pi^{-1}(i)} \gamma(\pi(t-1), \pi(t))$.

We focus here on the cost allocation problem by the three-step procedure. For a more elaborate discussion of *trps* we refer to Kleppe (2010). The proper *opr* of a *trp* is straight-

forward. For a *trp* $T = (N, 0, \gamma)$ we define the *opr* (N, Π, k) with

$$k^i(\pi) = \gamma(0, \pi(1)) + \sum_{t=2}^{\pi^{-1}(i)} \gamma(\pi(t-1), \pi(t)).$$

It follows immediately that (N, Π, k) is also efficient. The corresponding marginal cost game (N, c_m) and a generalised Bird allocation $\beta(N, \Pi, k)$ are determined by (2.2) and (2.3), respectively.

From the fact that γ satisfies the triangle inequality we immediately obtain the next proposition.

Proposition 8.2 *Let T be a trp. Then the corresponding opr (N, Π, k) is a neopr.*

Since (N, Π, k) is an adequate *neopr*, we suggest to model the class of *trps* by the marginal cost game (N, c_m) . Therefore, we also have the following proposition.

Proposition 8.3 *Let T be a trp with corresponding opr (N, Π, k) . Then $\beta(N, \Pi, k) \in C(N, c_m)$.*

References

- Afrati, F., S. Cosmadakis, C. Papadimitriou, G. Papageorgiu, and N. Papakostantinou (1986). The complexity of the traveling repairman problem. *RAIRO Theoretical Informatics and Applications*, **20**, 79–87.
- Aumann, R. and M. Maschler (1985). Game theoretic analysis of a bankruptcy problem from the Talmud. *Journal of Economic Theory*, **36**, 195–213.
- Bergantiños, G. and J. Vidal-Puga (2007). The optimistic TU game in minimum cost spanning tree problems. *International Journal of Game Theory*, **36**, 223–239.
- Bird, C. (1976). On cost allocation for a spanning tree: a game theoretic approach. *Networks*, **6**, 335–350.
- Borm, P., H. Hamers, and R. Hendrickx (2001). Operations research games: a survey. *TOP*, **9**, 139–216.
- Branzei, R., S. Moretti, H. Norde, and S. Tijs (2004). The P -value for cost sharing in minimum cost spanning tree situations. *Theory and Decision*, **56**, 47–61.
- Claus, A. and D. Kleitman (1973). Cost allocation for a spanning tree. *Networks*, **3**, 289–304.

- Curiel, I., G. Pederzoli, and S. Tijs (1989). Sequencing games. *European Journal of Operational Research*, **40**, 344–351.
- Gillies, D. (1959). Solutions to general non-zero-sum games. *Contributions to the Theory of Games*, **4**, 47–85.
- Kleppe, J. (2010). *Modelling Interactive Behaviour, and Solution Concepts*. Ph. D. thesis, Tilburg University, Tilburg.
- Kleppe, J., M. Quant, and H. Reijnierse (2010). Public congestion network situations and related games. *Networks*, **55**, 368–378.
- Klijn, F. and E. Sánchez (2006). Sequencing games without initial order. *Mathematical Methods of Operations Research*, **63**, 53–62.
- Koster, M. (1999). *Cost Sharing in Production Situations and Network Exploitation*. Ph. D. thesis, Tilburg University, Tilburg.
- Kuhn, H. (1955). The Hungarian method for the assignment problem. *Naval Research Quarterly*, **2**, 83.
- Littlechild, S. and G. Owen (1973). A simple expression for the Shapley value in a special case. *Management Science*, **20**, 370–372.
- Megiddo, N. (1978). Computational complexity of the game theory approach to cost allocation for a tree. *Mathematics of Operations Research*, **3**, 189–196.
- Neumann, J. von and O. Morgenstern (1944). *Theory of Games and Economic Behavior*. Princeton: Princeton University Press.
- O’Neill, B. (1982). A problem of rights arbitration from the Talmud. *Mathematical Social Sciences*, **2**, 345–371.
- Potters, J., I. Curiel, and S. Tijs (1992). Traveling salesman games. *Mathematical Programming*, **53**, 199–211.
- Prim, R. (1957). Shortest connection networks and some generalizations. *Bell Systems Technical Journal*, **36**, 1389–1401.
- Smith, W. (1956). Various optimizers for single-stage production. *Naval Research Logistics Quarterly*, **3**, 59–66.
- Thomson, W. (2003). Axiomatic and game theoretic analysis of bankruptcy and taxation problems: a survey. *Mathematical Social Sciences*, **45**, 249–297.
- Tijs, S., T. Parthasarathy, J. Potters, and V. Rajendra Prasad (1984). Permutation games: another class of totally balanced games. *OR Spektrum*, **6**, 119–123.