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REWARD ALLOCATIONS IN PRODUCTION SYSTEMS

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Earnings are achieved by producing certain products which can be sold at a given market price. The resources needed to produce these products belong to various producents or groups of producents. In the classical case studied by Owen (1975) every producent owns a vector of resources and the core of the corresponding production game is not empty. Cases in which portions of the resources belong to committees with at least one veto member have also been studied (Curiel, Derks, Tijs, 1986). In these cases the core is not empty too. In this paper cases are studied where portions of the resources belong to committees which need not have a veto player. The core of the corresponding game can be empty and other methods of allocating the rewards are examined.

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1. Introduction

In linear production systems earnings are achieved by producing certain products which can be sold at a given market price. Owen [7] studied the case where the resources needed to make the products belong to various owners and proved that the core of the corresponding game in characteristic function form is not empty. In a recent paper Curiel et al. [3] consider the case where the resources are controlled by committees. They prove that if all the committees have veto players the core is not empty too.

In this paper we will study the case where the committees don't have veto players. Then the core of the linear production game may be empty and other methods have to be found to allocate the rewards. The paper is organized as follows. In section 2 the model is introduced and definitions and notations are given. In this section the proof of the non-emptiness of the core when there are veto players is given. In section 3 non-stable cases are considered, i.e., cases where the core is empty and the method of imposing multiplicative ϵ -taxes introduced by Tijs and Driessen [11] is studied. Other methods of allocating the rewards in this case are also mentioned. We conclude with some remarks in section 4.

2. Linear production games

In the classical case studied by Owen [7] there are n producers each producer $i \in \{1, \dots, n\}$ owning a vector $0 \leq b^i = (b_1^i, \dots, b_m^i)$ of resources. These resources can be used to produce r different products. To produce one unit of product $j \in \{1, \dots, r\}$ one needs a_{jk} units of the

k -th resource, where $k \in \{1, \dots, m\}$. One unit of product j can be sold at a given *market price* p_j . Every producer wants to maximize his profit by producing the bundle of resources that will fetch him most when sold. In doing this the producers may consider to work together by pooling their resources. Such a situation gives rise to a *game in characteristic function form* $\langle N, v \rangle$. Each producer is considered to be a *player*, so the set of players N is equal to $\{1, \dots, n\}$. Subsets of N are called *coalitions*. The *characteristic function* v is a real-valued function on 2^N , the family of subsets of N , with $v(\emptyset) = 0$. For every coalition S we consider $v(S)$ to denote the worth of coalition S , that is, what coalition S can achieve regardless of what the others do. In the linear production system we are considering we have

$$\begin{aligned} v(S) = \max x \cdot p \text{ subject to} \\ x \in \mathbb{R}^r, x \geq 0, xA \leq b(S). \end{aligned} \quad (*)$$

Here $b(S) = \sum_{i \in S} b^i$ is the resource vector of coalition S and

$A = [a_{jk}]_{j=1, k=1}^{r, m}$ is the production matrix. We assume that $a_{jk} \geq 0$ for all $j \in \{1, \dots, r\}$, $k \in \{1, \dots, m\}$ and that for all j there is a k such that $a_{jk} > 0$. (It is not possible to produce something with nothing.) Then the linear optimization problem given above is feasible and bounded and there exists a vector of products for which the maximum is achieved.

The question which arises now is how to divide the earnings whence the grand coalition N is formed. A distribution among the players is represented by a payoff vector $z = (z_1, \dots, z_n)$ where z_i is the amount the i -th player gets and $\sum_{i \in N} z_i = v(N)$. Such a payoff vector z is said to be *individual rational* if $z_i \geq v(\{i\})$ for all $i \in N$. This means

that z assigns to every player i at least the amount he can get when he works alone so he has no incentive to go away from the grand coalition and work on himself. But it may still be possible for a coalition S with $|S| > 1$ to earn more by leaving the grand coalition and working on its own. In this case the situation is not stable. To ensure stability it is needed that $z(S) := \sum_{i \in S} z_i \geq v(S)$ for all $S \in 2^N$.

Then no coalition can do better by splitting off. The set of payoff vectors which satisfy the above condition is called the *core* of the game v and is denoted by $C(v)$. Formally,

$$C(v) := \{z \in \mathbb{R}^N, z(N) = v(N), z(S) \geq v(S) \text{ for all } S \in 2^N\}.$$

The core of a game can be empty. Owen [7] proved that the core of the linear production game described above is not empty.

Now we will study a generalization of the linear production game given by Curiel et al. [3]. For this generalization we need the notion of a *simple game*. A game $\langle N, v \rangle$ in characteristic function form is called a simple game if

- (i) $v(S) \in \{0, 1\}$ for all $S \in 2^N$.
- (ii) $v(N) = 1$.
- (iii) $T \supset S$ implies $v(T) \geq v(S)$ for all $S, T \in 2^N$. (Monotonicity)

Such a game describes a situation where a coalition S is either all powerful, $v(S) = 1$, or completely powerless, $v(S) = 0$. Such games can be used to model voting situations and situations of committee control. A player i in a simple game is called a *veto player* if $v(S) = 1$ implies $i \in S$. Player i is called a *dictator* if $v(S) = 1$ iff $i \in S$.

In the generalization we consider the resources are available in portions. For $k \in \{1, \dots, m\}$ there are l_k portions of resource k . Such a portion we denote by B_k^q . The total amount of resource k which is available we denote by B_k , so $B_k = \sum_{q=1}^{l_k} B_k^q$. Each portion B_k^q is controlled by committees consisting of players in N . This committee control is modeled with the aid of simple games. With every B_k^q we associate a simple game $\langle N, w_k^q \rangle$. Coalition S can use B_k^q only if S is winning in w_k^q , i.e., if $w_k^q(S) = 1$. We define $B_k^q(S) = B_k^q \cdot w_k^q(S)$. The total amount of the k -th resource available to coalition S is $B_k(S) := \sum_{q=1}^{l_k} B_k^q(S)$.

The resource vector $B(S)$ of coalition S has coordinates

$B_1(S), B_2(S), \dots, B_m(S)$. The characteristic function v of the *linear production game with committee control* is again given by (*) but now with $b(S)$ replaced by $B(S)$.

It is a well known fact that the core of a simple game is not empty iff the game has at least one veto player. Let $\langle N, w \rangle$ be a simple game with V the set of veto players, $V \neq \emptyset$. Then

$$C(w) = \{z \in \mathbb{R}^n : z(V) = 1, z_i \geq 0 \text{ for all } i \in N\}.$$

The following theorem states that the non-emptiness of the core of all the simple games involved in the control of the portions is inherited by the linear production game.

Theorem 1. (Curiel, Derks, Tijs [3]).

Let $\langle N, v \rangle$ be a linear production game with committee control described by the simple games w_k^q . If every w_k^q has at least one veto player then the core of v is not empty.

Proof: We consider the dual problem of the linear optimization problem which determines $v(S)$. That is

minimize $B(S) \cdot y$ subject to

$$y \in \mathbb{R}^m, y \geq 0, Ay \geq p.$$

From the duality theorem for linear programming follows that $v(S)$ is equal to the value of this problem. Let $\hat{y} = (\hat{y}_1, \dots, \hat{y}_m)$ be a solution of the dual problem which determines $v(N)$. For every $k \in \{1, \dots, m\}$ and every $q \in \{1, \dots, l_k\}$ let z_k^q be a core element of w_k^q . We define a vector $z \in \mathbb{R}^n$ by

$$z_i = \sum_{k=1}^m \hat{y}_k \sum_{q=1}^{l_k} B_k^q(z_k^q)_i$$

$$\text{Then } \sum_{i \in N} z_i = \sum_{k=1}^m \hat{y}_k B_k = \sum_{k=1}^m \hat{y}_k \cdot B_k(N) = v(N)$$

$$\text{For all } S \in 2^N \quad \sum_{i \in S} z_i \geq \sum_{k=1}^m \hat{y}_k B_k(S) \geq v(S)$$

where the first inequality follows from the fact that

$$B_k(S) = \sum_{q=1}^{l_k} B_k^q(S) \leq \sum_{q=1}^{l_k} B_k^q(z_k^q(S)) \text{ and the second inequality follows}$$

from the fact that $\hat{y}$ is feasible for the dual problem which determines $v(S)$ because the constraints in the dual problems are independent of S . It follows that $z \in C(v)$.

□

That the model with committee control is indeed a generalization of the model described by Owen can be seen as follows. For every resource k the number of portions of k is taken to be equal to the number of players, that is, $l_k = n$. For every $k \in \{1, \dots, m\}$ and $i \in \{1, \dots, n\}$ the simple game w_k^i which describes the control of B_k^i is defined by $w_k^i(S) = 1$ iff $i \in S$, so w_k^i is the simple game with player i

as dictator. The linear production game which arises from this situation is the same as the game in the model of Owen with every player owning a vector of resources where the resource vector of player i is $(B_1^i, B_2^i, \dots, B_m^i)$.

If all the simple games which describe the control don't have veto players they all have an empty core and the following theorem states that the core of the linear production game is also empty then.

Theorem 2. (Curiel, Derks, Tijs [3]).

Let $\langle N, v \rangle$ be a linear production game with committee control described by the simple games w_k^q . If there is not any w_k^q with a veto player and $v(N) > 0$ then the core of v is empty.

Proof: Because w_k^q has no veto player we have that $w_k^q(N-i) = w_k^q(N)$ for every $i \in N$, $k \in \{1, \dots, m\}$, $q \in \{1, \dots, l_k\}$ and hence $B(N-i) = B(N)$ for every $i \in N$. From this follows that $v(N-i) = v(N)$ for every $i \in N$. For z to be an element of $C(v)$ it is required that $z(N-i) = z(N)$ for every $i \in N$, but then $z_i = 0$ for every $i \in N$ and $z(N) = 0 < v(N)$ and hence z cannot be an element of $C(v)$. So $C(v) = \emptyset$.

□

In the following section we will look at allocation methods in these cases where the core is empty.

3. Reward allocations in non-stable situations

In situations where the core of a game is empty there are various escapes. In this section we will look at an extension of the core introduced by Tijs and Driessen [11]. They consider a multiplicative way of imposing taxes on the formation of a coalition $S \subseteq N$. They take

these taxes to be proportional to the relative reward that the formation of S yields. Let us make this formal. Let $\langle N, v \rangle$ be a game in characteristic function form such that $v(N) \geq \sum_{i \in N} v(\{i\})$. If this inequality doesn't hold then it is clear that the players will be better off if the grand coalition isn't formed and they all work separately. The relative reward that the formation of a coalition S yields is $v(S) - \sum_{i \in S} v(\{i\})$, that is, the difference between the worth of S and the sum of the worths of the members of S when they work alone. Let $0 \leq \epsilon \leq 1$, a tax of $\epsilon[v(S) - \sum_{i \in S} v(\{i\})]$ is imposed on the formation of every $S \subsetneq N$. In this way we get a new game, the *multiplicative ϵ -tax game* $\langle N, v^\epsilon \rangle$ defined by

$$v^\epsilon(S) = \begin{cases} v(S) & \text{if } S = N, \phi \\ v(S) - \epsilon[v(S) - \sum_{i \in S} v(\{i\})] & \text{if } S \neq N, \phi \end{cases}$$

Now we consider the smallest ϵ for which the game $\langle N, v^\epsilon \rangle$ has a non-empty core. If $\langle N, v \rangle$ has a non-empty core then this ϵ is equal to 0. Further, for $\epsilon = 1$ we have $v^1(S) = \sum_{i \in S} v^1(\{i\})$ for $S \neq N, \phi$ and it is easily seen that $C(v^1) \neq \phi$. For every game $\langle N, v \rangle$ with $\sum_{i \in N} v(\{i\}) \leq v(N)$ we define

$$\epsilon(v) := \min \{ \epsilon : 1 \geq \epsilon \geq 0, C(v^\epsilon) \neq \phi \}.$$

Then $\epsilon(v)$ is well defined since the class of all n -person cooperative games with a non-empty core is a convex polyhedral cone and we already saw that $C(v^1) \neq \phi$. Now an extension of the core, the *least-tax-core* $LTC(v)$ of the game $\langle N, v \rangle$ is defined as the core of the game $v^{\epsilon(v)}$. If $C(v) \neq \phi$ then $\epsilon(v) = 0$ and $v^{\epsilon(v)} = v$ and $LTC(v) = C(v)$. Let $\langle N, v \rangle$ be a simple game with an empty core and $v(N) \geq \sum_{i \in N} v(\{i\})$. We distinguish

two cases.

(1) There is an $i^* \in N$ such that $v(\{i^*\}) = 1$.

(2) For all $i \in N$: $v(\{i\}) = 0$.

In case (1) $\epsilon(v) = 1$. This can be seen as follows. Let $\epsilon < 1$ then $v^\epsilon(\{i^*\}) = 1$. Because i^* is no veto player there is an S with $i^* \notin S$ and $v(S) = 1$. Then $v^\epsilon(S) = 1 - \epsilon > 0$. So if $x \in C(v^\epsilon)$ then $x_{i^*} = 1$, $x(S) \geq 1 - \epsilon > 0$ and $x(N) = 1$, $x_i \geq 0$ for all $i \in N$ and this is not possible. Hence $C(v^\epsilon) = \emptyset$ and $\epsilon(v) = 1$. It follows that in case (1)

$$v^{\epsilon(v)}(S) = \begin{cases} 1 & \text{if } i^* \in S \\ 0 & \text{if } i^* \notin S \end{cases}$$

In case (2)

$$v^{\epsilon(v)}(S) = \begin{cases} 0 & \text{if } v(S) = 0 \\ 1 - \epsilon(v) & \text{if } v(S) = 1, S \neq N \end{cases}$$

We see that in case (1) $v^{\epsilon(v)}$ is a simple game with i^* as unique veto player. In case (2) $v^{\epsilon(v)}$ is no longer a simple game and we have to find a new interpretation if we see such a game as modelling a situation of committee control. An interpretation is as follows. Suppose w is a simple game which describes the control of a portion B of a certain resource. Then we can view the game $w^{\epsilon(w)}$ as describing the situation where a coalition S with $w(S) = 1$ can only control part of the portion B , namely the part given by $(1-\epsilon)B$. Let $\langle N, v \rangle$ be a linear production game with committee control described by the simple games w_k^q . Then we define the linear production game with least tax committee control $\langle N, v^\epsilon \rangle$ by

$$v^E(S) = \max x \cdot p \text{ subject to}$$

$$x \in \mathbb{R}^r, x \geq 0, xA \leq B^E(S).$$

Here $B^E(S)$ is the vector in $\mathbb{R}^m$ with coordinates

$$B_K^E(S) := \sum_{q=1}^l B_K^q w^{E_{Kq}}(S) \text{ where } \varepsilon_{Kq} := \varepsilon(w_K^q) \text{ and } \underline{\varepsilon} := (\varepsilon_{11}, \varepsilon_{12}, \dots, \varepsilon_{ml_m}).$$

The following theorem states that the core of this game is not empty.

Theorem 3.

The core of a linear programming game with least tax committee control is not empty.

Proof: Let $\langle N, v^E \rangle$ be a linear programming game with least tax committee control described by $w^{E_{Kq}}$. Then by the definition of ε_K every $w^{E_{Kq}}$ has a non-empty core. Let $z^{E_{Kq}}$ be a core element of $w^{E_{Kq}}$. In the same way as in the proof of theorem 1 we can construct now with the $z^{E_{Kq}}$'s a core element of $\langle N, v^E \rangle$.

□

A way of handling a non-stable situation is hence to impose suitable multiplicative ε -taxes on the formation of the committees which control the portions of the resources. In this way we get a game with a non-empty core by theorem 3. Now any core element of this game can be used as a payoff vector for the original game. As $v^E(i) = v(i)$ this payoff vector will still be individual rational.

Another way of handling a non-stable situation is with the aid of power indices. A *power index* is a function ϕ which assigns to every n -person simple game a vector in $\mathbb{R}^n$. For every player $i \in N$ $\phi_i(w)$ is a measure of his power in the simple game $\langle N, w \rangle$. Some power indices

are the Shapley-Shubik index [10], the Banzhaf index [1], the Deegan-Packel index [4] and a power index recently introduced by Curiel [2].

Now a power index ϕ can be used to compute the power of the players in the controlling committees. If $\phi_i(w_k^q)$ is a measure of the power of player i in the game w_k^q then player i can regard this as describing the fact that he can control $\phi_i(w_k^q) B_k^q$ of the portion B_k^q . Then in total he can control $\sum_{q=1}^{l_k} \phi_i(w_k^q) B_k^q$ of the k -th resource. Let $\hat{y}$ be a solution of the linear programming problem

$$B(N) \cdot y \text{ subject to}$$

$$y \in \mathbb{R}^m, y \geq 0, Ay \geq p$$

The solution $\hat{y}$ can be regarded as a price vector for the resources. So instead of producing anything player i can decide to sell his resources against the prices $\hat{y}_1, \dots, \hat{y}_m$ and receive $z_i := \sum_{k=1}^m \hat{y}_k \sum_{q=1}^{l_k} \phi_i(w_k^q) B_k^q$. Now

if the power index ϕ is *efficient*, that is, if $\sum_{i \in N} \phi_i(w) = 1$ for every

simple game w , we have $\sum_{i=1}^n \phi_i(w_k^q) B_k^q = B_k^q$ for every $k \in \{1, \dots, m\}$,

$q \in \{1, \dots, l_k\}$ and hence $\sum_{i=1}^n z_i = B(N) \cdot \hat{y} = v(N)$ and $z = (z_1, \dots, z_n)$

is a payoff vector.

An advantage of the two methods described above is that just as in the case when the core is not empty it is not necessary to compute all the $v(S)$, which would mean solving as many linear programs, but that it is sufficient to find a solution of the minimization problem given above and to analyse the simple games involved. Applying other game-theoretic solution concepts, such as the Shapley-value [9] or the nucleolus [8]

directly to the linear programming game $\langle N, v \rangle$, would involve computing all the $v(S)$.

4. Conclusion

In this paper we looked at linear production games with committee control. When all the committees involved have at least one veto player the core of the linear production game is not empty.

This case is related to the generalized linear production model introduced by Granot [6]. It can be shown that the two models are equivalent.

In their paper on totally balanced games arising from controlled programming problems Dubey and Shapley [5] also look at committee control and obtain a result on the non-emptiness of the core of the game.

Their proof, however, is not constructive. In this paper a constructive proof is given for the linear case. Further two methods to deal with the non-stable situation are treated.

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