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**FEEDBACK NASH EQUILIBRIA FOR DESCRIPTOR
DIFFERENTIAL GAMES USING MATRIX PROJECTORS**

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Feedback Nash Equilibria for Descriptor Differential games using Matrix Projectors

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Abstract

In this article we address the problem of finding feedback Nash equilibria for linear quadratic differential games defined on descriptor systems. First, we decouple the dynamic and algebraic parts of a descriptor system using canonical projectors. We discuss the effects of feedback on the behavior of the descriptor system. We derive necessary and sufficient conditions for the existence of the feedback Nash equilibria for index 1 descriptor systems and show that there exist many informationally non-unique equilibria corresponding to a single solution of the game. Further, for descriptor systems with index greater than 1, we give a regularization based approach and discuss the associated drawbacks.

Keywords linear-quadratic games, linear feedback Nash equilibrium, solvability conditions, matrix projectors
JEL-Codes C61, C72, C73.

1 Introduction

Dynamic game theory captures multi person decision making processes that occur in time. These problems arise in several disciplines such as monetary policy coordination, ecology and several others, refer [8] for a detailed overview. The dynamic environment where the players interact is often modeled by a set of ordinary differential equations and the related theory for such games is well established [3]. Complex systems, however, include modeling with both differential and algebraic equations, i.e., differential algebraic equations. Problems of this kind appear in studying systems which are constrained and which operate under different timescales, for example in environmental economics where global warming is assumed to be a system which has slow dynamics that is affected by various processes that have fast dynamics. Descriptor systems, linear differential algebraic equations, approximate such multiple time scale systems. Differential games for descriptor systems were e.g., already studied by Xu and Mizukami [26]. Further, these authors considered the leader follower information structure in [27]. Glizer [14] considers the asymptotic behavior of the zero-sum game solution from a cheap control perspective. More recently Engwerda et al. [12] give solvability conditions, with open loop information pattern, for the index 1 linear quadratic differential game and in [11] for higher order indices. The same authors, in [10], consider the game problem with feedback information pattern for the index 1 case. The main tool used to analyze the descriptor system in most of the works is Kronecker canonical form (KCF). The differential and algebraic parts of a descriptor system can be decoupled using KCF. It is known, however, that computing KCF for a descriptor system is still a challenge, see [16] for some recent developments.

In this report, we consider the linear quadratic differential game on a descriptor system with feedback information pattern. We consider a geometric approach towards decoupling the descriptor system instead of KCF. We lean heavily on the matrix projector techniques pioneered by März et al. [15, 20]. We review some important results on matrix projectors in appendix A. Further, we propose an algorithm to compute matrix projectors that completely

decouple a descriptor system into algebraic and differential parts. In section 2 we introduce the game problem. We analyze effect of feedback on a decoupled system and analyze the informational non uniqueness property of feedback strategies. In section 3 we give solvability conditions for the game problem for the index 1 case. Later, for descriptor systems with higher order indices, we give an approach to recast the problem as an index 1 case. We demonstrate the limitation of this method with an example. Finally section 4 concludes.

Notation: We use the following notation. X' represents the transpose of $X \in \mathbb{R}^{n \times m}$. $X^\dagger$ represents Moore-Penrose pseudo inverse of X . For $n > m$, $X^\dagger$ gives a left inverse of X , when X has full column rank. $\text{Im}X$ and $\text{ker}X$ represents the column space and null space of X respectively.

2 Preliminaries

In this section we assume that players $i = 1, 2$ like to minimize:

$$\lim_{t_f \rightarrow \infty} J_i(t_f, x_0, u_1, u_2),$$

where $J_i(t_f, x_0, u_1, u_2) = \int_0^{t_f} [x'(t) u'_1(t) u'_2(t)] M_i [x'(t) u'_1(t) u'_2(t)]' dt,$ (1)

$$M_i = \begin{bmatrix} D_i & V_i & W_i \\ V_i' & R_{1i} & N_i \\ W_i' & N_i' & R_{2i} \end{bmatrix}, D_i = D_i^T, R_{ii} > 0, i = 1, 2, \text{ and } x(t) \text{ satisfies}$$

$$E\dot{x}(t) = Ax(t) + B_1 u_1(t) + B_2 u_2(t), x(0) = x_0. \quad (2)$$

Where $x(t) \in \mathbb{R}^n$, $u_i(t) \in \mathbb{R}^{m_i}$, $i = 1, 2$, $A \in \mathbb{R}^{n \times n}$, $E \in \mathbb{R}^{n \times n}$, $B_i \in \mathbb{R}^{n \times m_i}$, $i = 1, 2$ and dimensions of other matrices are defined appropriately. Since the information pattern is feedback type we assume that players use feedback strategies of the type $u_i(t) = F_i x(t)$, $i = 1, 2$. $u^* := (u_1^*, u_2^*)$ is called a feedback Nash equilibrium (FBNE) if the usual inequalities apply, i.e., no player can improve his performance by a unilateral deviation from this set of equilibrium actions. We introduce the notation F_{-i}^* which corresponds to the strategies, $u_j(t) = F_j x(t)$, $j = 1, 2$, $j \neq i$, used by all the players the excluding player i . Now, formal definition of FBNE reads as follows:

Definition 2.1. $F^* := (F_1^*, F_2^*)$ is called a feedback Nash equilibrium if for $i = 1, 2$, $J_i(x_0, F^*) \leq J_i(x_0, (F_i, F_{-i}^*))$ for every input x_0 and F_i .

The following lemma follows directly from the definition of FBNE.

Lemma 2.1. Let $H \in \mathbb{R}^{n \times d}$. (F_1^*, F_2^*) is a FBNE for the game defined by the cost $J_i(F_1 H, F_2 H)$ and the system $\dot{x}(t) = (A + B_1 F_1 H + B_2 F_2 H)x(t)$, $x(0) = x_0$ if and only if (G_1^*, G_2^*) is a FBNE for the game defined by the cost $J_i(G_1, G_2)$ and the system $\dot{x}(t) = (A + B_1 G_1 + B_2 G_2)x(t)$, $x(0) = x_0$ and (F_1^*, F_2^*) solve the set of equations $F_1^* H = G_1^*$ and $F_2^* H = G_2^*$ ¹.

For $E = I$, we recall the following result from [9].

Theorem 2.1. Assume that matrix $G = \begin{bmatrix} R_{11} & N_1 \\ N_2' & R_{22} \end{bmatrix}$ is invertible. Then the differential game (1,2) with $E = I$, has a feedback Nash equilibrium (F_1, F_2) for every initial state if and only if

$$\begin{bmatrix} F_1 \\ F_2 \end{bmatrix} = -G^{-1} \begin{bmatrix} B_1' X_1 + V_1' \\ B_2' X_2 + W_2' \end{bmatrix}. \quad (3)$$

Here (X_1, X_2) are a symmetric stabilizing solution of the coupled algebraic Riccati equations

$$\begin{aligned} D_1 + W_1 F_2 + F_2' W_1' + F_2' R_{21} F_2 - F_1' R_{11} F_1 + X_1 (A + B_2 F_2) + (A + B_2 F_2)' X_1 &= 0 \\ D_2 + V_2 F_1 + F_1' V_2' + F_1' R_{12} F_1 - F_2' R_{22} F_2 + X_2 (A + B_1 F_1) + (A + B_1 F_1)' X_2 &= 0. \end{aligned}$$

¹ Solvable only if $\text{Ker}H \subset \text{Ker}G_i^*$, $i = 1, 2$.

If E is non-singular, the FBNE are obtained using theorem 2.1. The result given in theorem 2.1 is based on dynamic programming principle, see [3, 9]. However, optimal control of descriptor systems is still an active area of research, and except for some special cases, for example [18, 2], general results are not available. Generally, the approach followed is to first decouple algebraic and dynamic components of the descriptor system and later eliminate the algebraic components. We review some important details about the descriptor system.

The system (2) is solvable, under assumptions on smoothness of $u(t)$ and consistent initial states x_0 , if the pencil $\lambda E - A$ is regular i.e., $\det(\lambda E - A) \neq 0$ for atleast one λ . We call (E, A) a regular pair if the pencil $\lambda E - A$ is regular. The index, denoted by a whole number μ , of the descriptor system is the number differentiations required to solve the descriptor system as an ordinary differential equation, see section 2 of [5]. The solution of the descriptor system depends on the derivatives of the input up to an order of $\mu - 1$. There are several approaches used to analyze the descriptor system and most notable of such methods is the Kronecker canonical form. Though it serves the purpose of analyzing a descriptor system, computation of the KCF for a regular pencil is still a challenge, see [16] for details. We summarize the main problem addressed in this section as follows:

Problem: Consider the performance criterion (1) and system (2), where $\text{rank}(E) = r < n = \dim(x)$. Assume (E, A) is regular with index μ . Find conditions under which (1,2) has a feedback Nash solution $u_i = F_i x$.

The problem considered above is posed in a very general setting. We did not make assumptions with respect to stabilizing properties of the feedback strategies, which is generally the case when E is non-singular. Now, we consider a different approach, based on projector techniques, to decouple the descriptor system and offers a significant numerical advantage compared to KCF.

In his seminal paper Smale [23] puts forward the idea of interpreting differential algebraic equations as vector fields on manifolds. In the later works by Rheinboldt in [22] and Reich in [21] discuss the classes of DAEs which can be seen as vector fields on constraint manifolds (due to algebraic constraints). For linear DAEs, März et.al., [15] develop a matrix chain approach to decouple a descriptor system into differential and algebraic parts. If the pencil $\lambda E - A$ is regular, then it was shown in theorem 3.1 of [20] that the existence of so called *canonical projectors* is guaranteed. Using canonical projectors a descriptor system can be decoupled canonically as a vector field and a constraint manifold. Recently, Wong [24, 25] demonstrates the computational advantage of the matrix projector methods in engineering applications. In appendix A, we review important details of matrix projectors and discuss an algorithm to generate the canonical projectors for a regular matrix pencil. A regular descriptor system can be decoupled completely, using canonical projectors (Q_i, P_i) , $i = 0, 1, \dots, \mu - 1$, as follows:

$$\dot{m}(t) = P_0 \cdots P_{\mu-1} E_{\mu}^{-1} A m(t) + P_0 \cdots P_{\mu-1} E_{\mu}^{-1} B u(t), \quad m(0) = P_0 \cdots P_{\mu-1} x_0 \quad (4a)$$

$$n(t) = - \sum_{i=0}^{\mu-1} N_i E_{\mu}^{-1} B u^i(t), \quad n(0) = (I - P_0 \cdots P_{\mu-1}) x_0, \quad (4b)$$

$$x(t) = m(t) + n(t), \quad \dot{m}(t) = P_0 \cdots P_{\mu-1} \dot{x}(t), \quad n(t) = (I - P_0 \cdots P_{\mu-1}) \dot{x}(t) \quad (4c)$$

Where $m(t)$ and $n(t)$ are projections of the state $x(t)$. Here $P_i, Q_i = I - P_i, i = 0, 1, \dots, \mu - 1$ are canonical projectors generated by a regular pair (E, A) , refer to appendix A for details. This decomposition is unique for a regular pair (E, A) .

Remark 2.1. Since $m(0) \in \text{Im}(P_0 P_1 \cdots P_{\mu-1})$ and $\dot{m}(t) \in \text{Im}(P_0 P_1 \cdots P_{\mu-1})$, from lemma 3.2.1 of [4], we have $m(t) \in \text{Im}(P_0 P_1 \cdots P_{\mu-1}), \forall t \geq 0$. In other words, equation (4a) represents an invariant flow and $m(t)$ belongs to the invariant subspace $\text{Im}(P_0 P_1 \cdots P_{\mu-1}) \subset \mathbb{R}^n$ for all $t \geq 0$.

Using the above remark we have the following lemma:

Lemma 2.2. $m(t) \in \mathbb{R}^n$ is isomorphic to a vector $m_l(t) \in \mathbb{R}^d$, where $d = \dim(\text{Im}(P_0 P_1 \cdots P_{\mu-1})) < n$. Further, there exist matrices $Y_{d \times n}$ and $Z_{n \times d}$ such that $m_l(t) = Y m(t)$, $m(t) = Z m_l(t)$ and $ZY = P_0 P_1 \cdots P_{\mu-1}$.

Proof. First part of the lemma is obvious from remark 2.1. For the second part we know from [15] that $P_0 P_1 \cdots P_{\mu-1}$ is a projector. So, we have $m(t) = P_0 P_1 \cdots P_{\mu-1} m(t)$, taking SVD of $P_0 P_1 \cdots P_{\mu-1}$ as $U_{n \times d} \Sigma_{d \times d} V'_{d \times n}$ we have $m(t) = U \Sigma V' m(t)$. Identifying $Z = U \Sigma$ and $Y = V'$ we have that $m_l(t) = Y m(t)$ and $m(t) = Z m_l(t)$. $\square$

From (4a-4b) it is clear that $x(t)$ is continuous if $u(t)$ is at least $\mu - 1$ times differentiable. We denote $\mathcal{U}$ to be the set of admissible controls, i.e., $u(\cdot) \in \mathcal{U}$ is at least $\mu - 1$ times differentiable. Let $\mathcal{X}_0$ denote the consistent initial state manifold, then it is characterized in the following lemma:

Lemma 2.3. *For every $u(\cdot) \in \mathcal{U}$, the descriptor system (2) yields a unique continuous state trajectory if $x_0 \in \mathcal{X}_0$.*

Proof. Violation of the algebraic constraint (4b) at $t = 0$ results in a jump, i.e., $x(0^+) = \lim_{t \downarrow 0} x(t) \neq x(0^-) = x(0)$. Let us define

$$\mathcal{X}_0 := \left\{ x_0 \in \mathbb{R}^n \mid (I - P_0 P_1 \cdots P_{\mu-1}) x_0 = - \sum_{i=0}^{\mu-1} N_i E_{\mu}^{-1} B u^i(0) \right\}. \quad (5)$$

For an admissible $u(t)$, such jumps can be avoided if $x_0 \in \mathcal{X}_0$. $\square$

From lemma 2.2 the low dimensional representation of the inherent ODE (4a) is given by:

$$\dot{m}_l(t) = Y P_0 P_1 \cdots P_{\mu-1} E_{\mu}^{-1} (A Z m_l(t) + B_1 u_1(t) + B_2 u_2(t)), \quad m_l(0) = Y P_0 P_1 \cdots P_{\mu-1} x_0. \quad (6)$$

We define the following classes of feedback strategies:

- a) Class of partial state information feedbacks $\mathcal{P}$, where players use $u_i(t) = K_i m_l(t)$.
- b) Class of full state information feedbacks $\mathcal{F}$, where players use $u_i(t) = F_i x(t)$.

An application of a particular feedback strategy can alter the behavior of the descriptor system, for instance, change in the index and regularity may fail to hold. However, if players use strategies from class $\mathcal{P}$ we show, in the theorem 2.2 below, that both regularity and index are preserved. In the discussion that follows, taking $m = m_1 + m_2$, we denote $u(t) := [u'_1(t), u'_2(t)]' \in \mathbb{R}^m$, $B := [B_1 \ B_2] \in \mathbb{R}^{n \times m}$, $F := [F'_1 \ F'_2]' \in \mathbb{R}^{m \times n}$ and $K := [K'_1 \ K'_2]' \in \mathbb{R}^{m \times d}$. We denote by $K \in \mathcal{P}$ if players use strategies $u_i(t) = K_i m_l(t)$, $i = 1, 2$ and by $F \in \mathcal{F}$ if players use strategies $u_i(t) = F_i x(t)$, $i = 1, 2$. We need the following auxiliary result to prove theorem 2.2.

Lemma 2.4. *For the admissible projectors Q_i and P_i , $i = 0, 1, \dots, \mu - 1$ we have the following:*

- a) $P_0 P_1 \cdots P_{\mu-1} Q_i = 0$, $i \in [0, \mu - 1]$
- b) $P_0 P_1 \cdots P_{\mu-1} P_i = P_0 P_1 \cdots P_{\mu-1}$, $i \in [0, \mu - 1]$

Proof. a) For $i \in [0, \mu - 1]$, $P_0 P_1 \cdots P_{\mu-1} Q_i = P_0 P_1 \cdots P_{\mu-2} (I - Q_{\mu-1}) Q_i$, from admissibility of Q_i 's (see appendix A) we know $Q_j Q_i = 0$ for $j > i$. So, we have $P_0 P_1 \cdots P_{\mu-1} Q_i = P_0 P_1 \cdots P_{\mu-2} Q_i$, repeating this $\mu - i - 1$ times we have $P_0 P_1 \cdots P_{\mu-1} Q_i = P_0 P_1 \cdots P_i Q_i = 0$.

b) For $i \in [0, \mu - 1]$, $P_0 P_1 \cdots P_{\mu-1} P_i = P_0 P_1 \cdots P_{\mu-1} (I - Q_i)$, from part (a), derived above, we have $P_0 P_1 \cdots P_{\mu-1} P_i = P_0 P_1 \cdots P_{\mu-1}$. $\square$

Theorem 2.2. *The class $\mathcal{P}$ is index preserving, i.e., (E, A) and $(E, A + BKY P_0 P_1 \cdots P_{\mu-1})$ have same index for any $K \in \mathcal{P}$.*

Proof. Consider the regular index- μ descriptor system. From theorem A.1 we know E_{μ} is non singular. The descriptor system with partial information feedback is given by:

$$E \dot{x} = Ax + BK m_l = (A + BKY P_0 P_1 \cdots P_{\mu-1}) x.$$

Let $(\tilde{E}_i, \tilde{A}_i)$ be the matrix chain obtained by taking $\tilde{E}_0 = E$ and $\tilde{A}_0 = A + BKY P_0 P_1 \cdots P_{\mu-1}$. Then we need to show that the matrix chain stops after μ steps i.e., $\tilde{E}_{\mu}$ is nonsingular. We prove the result using an induction argument.

For $i = 0$, clearly $\tilde{E}_i = E_i = E$ and $\tilde{A}_i = A_i + BKYP_0P_1 \cdots P_{\mu-1}$. For $i = 1$ as $\ker \tilde{E}_0 = \ker E_0 = \ker E$, we take $\tilde{Q}_0 = Q_0$

$$\begin{aligned}\tilde{E}_1 &= E + (A + BKYP_0P_1 \cdots P_{\mu-1}) Q_0 \\ &= E + AQ_0 + BKYP_0P_1 \cdots P_{\mu-1}Q_0 \\ &= E_1 + BKYP_0P_1 \cdots P_{\mu-1}Q_0 = E_1 \quad (\text{follows from 2.4.a}),\end{aligned}$$

$$\begin{aligned}\tilde{A}_1 &= (A + BKYP_0P_1 \cdots P_{\mu-1}) P_0 \\ &= AP_0 + BKYP_0P_1 \cdots P_{\mu-1}P_0 \\ &= A_1 + BKYP_0P_1 \cdots P_{\mu-1} \quad (\text{follows from 2.4.b}).\end{aligned}$$

We assume $\tilde{E}_k = E_k$ and $\tilde{A}_k = A_k + BKYP_0P_1 \cdots P_{\mu-1}$ for $0 \leq k \leq i$ and we show that $\tilde{E}_k = E_k$ and $\tilde{A}_k = A_k + BKYP_0P_1 \cdots P_{\mu-1}$ for $0 \leq k \leq i+1$. For $i = 1$ the above assumption holds true since we already showed $\tilde{E}_1 = E_1$ and $\tilde{A}_1 = A_1 + BKYP_0P_1 \cdots P_{\mu-1}$. Now, since $\tilde{E}_k = E_k$ we have $\tilde{Q}_k = Q_k$ and $\tilde{P}_k = P_k$ for $0 \leq k \leq i$ as only the image of Q_k is fixed (as $\ker E_k$).

$$\begin{aligned}\tilde{E}_{i+1} &= \tilde{E}_i + \tilde{A}_i \tilde{Q}_i \\ &= E_i + (A_i + BKYP_0P_1 \cdots P_{\mu-1}) Q_i \\ &= E_i + A_i Q_i + BKYP_0P_1 \cdots P_{\mu-1} Q_i = E_{i+1} \quad (\text{follows from 2.4.a}), \\ \tilde{A}_{i+1} &= \tilde{A}_i \tilde{P}_i \\ &= (A_i + BKYP_0P_1 \cdots P_{\mu-1}) P_i \\ &= A_{i+1} + BKYP_0P_1 \cdots P_{\mu-1} P_i \quad (\text{follows from 2.4.b}).\end{aligned}$$

So, by induction we have $\tilde{E}_k = E_k$ and $\tilde{A}_k = A_k + BKYP_0P_1 \cdots P_{\mu-1}$ for $0 \leq k \leq \mu$. Further, E_μ is nonsingular, since the pencil (A, B) is regular with index $-\mu$ is nonsingular, we have $\tilde{E}_\mu$ is nonsingular as well. So, by theorem A.1 the pencil $(E, A + BKYP_0P_1 \cdots P_{\mu-1})$ is regular with index μ . $\square$

Remark 2.2. We cannot guarantee such index preserving property, in general, for any $F \in \mathcal{F}$. Let us denote the index preserving subclass by $\mathcal{F}_\mu$, of $\mathcal{F}$. We notice for any $K \in \mathcal{P}$ the players use $u(t) = Km_l(t) = KYm(t) = (KYP_0P_1 \cdots P_{\mu-1})x(t)$. Thus for every strategy $K \in \mathcal{P}$ there exists an index preserving strategy $(KYP_0P_1 \cdots P_{\mu-1}) \in \mathcal{F}_\mu \subset \mathcal{F}$.

2.1 Informational non uniqueness

When players use a strategy $F \in \mathcal{F}$, let the index of the resulting autonomous descriptor system (2) be $\tilde{\mu}$. Here, feedback alters the static and dynamic spectral properties of the pencil (A, B) and as a result $\tilde{\mu}$ is different from μ unless $\mathcal{F} = \mathcal{F}_\mu$. A canonical decomposition of the descriptor system, for $t > 0$, once the players use an $F \in \mathcal{F}$ is given by:

$$\begin{aligned}\dot{\tilde{m}}(t) &= \tilde{P}_0 \cdots \tilde{P}_{\tilde{\mu}-1} E_{\tilde{\mu}}^{-1} A \tilde{m}(t), \quad \tilde{m}(0) = \tilde{P}_0 \cdots \tilde{P}_{\tilde{\mu}-1} x_0 \\ \tilde{n}(t) &= 0.\end{aligned} \tag{7}$$

So, for $t > 0$ the players actually use $\tilde{m}(t)$, as $x(t) = \tilde{m}(t) + \tilde{n}(t) = \tilde{m}(t)$, and $\tilde{m}(t) \in \text{Im}(\tilde{P}_0 \cdots \tilde{P}_{\tilde{\mu}-1}) \subset \mathbb{R}^n$. From lemma 2.2 we have that $\tilde{m}(t)$ is isomorphic to $\tilde{m}_l(t)$. As a result, for $t > 0$, we see that $x(t) \in \mathbb{R}^n$ is isomorphic to $\tilde{m}_l(t) \in \mathbb{R}^d$ with $d < n$. The algebraic constraints in the descriptor system render some of the state variables in $x(t)$ redundant. This feature is captured in the canonical decomposition (7) by an invariant flow. However, we notice that a canonical decomposition (7) could be given only after applying the full state feedback $F \in \mathcal{F}$. Further, the canonical projectors and the players costs vary with F in a way that is not easily tractable for applications like optimal control. This limitation motivates an investigation for the existence of index preserving feedback

strategies. In theorem 2.2 we showed that the class $\mathcal{P}$ is index preserving. So, a question arises if players restrict their strategies $\mathcal{F}_\mu$, see remark 2.2, can we infer that it is sufficient for players to use a strategy in $\mathcal{P}$? We show in the theorem 2.3 below that this observation is true for $\mu = 1$. Consider an index 1 descriptor system (2), then with a canonical decomposition we have:

$$\dot{m}_l(t) = YP_0E_1^{-1}Am(t) + YP_0E_1^{-1}Bu(t), \quad m_0(0) = YP_0x_0 \quad (8a)$$

$$n(t) = -Q_0E_1^{-1}Bu(t), \quad n(0) = Q_0x_0. \quad (8b)$$

Lemma 2.5. Assume (E, A) regular and index 1. For any $F \in \mathcal{F}_1$ we have $(I + Q_0E_1^{-1}BF)$ is non-singular.

Proof. Since (E, A) is index 1, for any projector Q_0 such that $\text{Im}Q_0 = \text{Ker}E$ we have by theorem A.1, $E + AQ_0$ is non-singular. For any $F \in \mathcal{F}_1$ we have $(E, A + BF)$ is index 1. As a result, for any Q_0 with $\text{Im}Q_0 = \text{Ker}E$, we have $E + (A + BF)Q_0$ is non-singular.

$$E + (A + BF)Q_0 = E + AQ_0 + BFQ_0 = E_1 + BFQ_0 = E_1 (I + E_1^{-1}BFQ_0).$$

Clearly, $(I + E_1^{-1}BFQ_0)$ is non-singular. We know,

$$\det(I + E_1^{-1}BFQ_0) = \det\left(\begin{bmatrix} I & -E_1^{-1}BF \\ Q_0 & I \end{bmatrix}\right) = \det(I + Q_0E_1^{-1}BF),$$

so, $(I + Q_0E_1^{-1}BF)$ is also non-singular. $\square$

Theorem 2.3. Every full state information feedback $F \in \mathcal{F}_1$ can be realized as a partial state feedback $K = \Omega(F) = F(I + Q_0E_1^{-1}BF)^{-1}Z$.

Proof. Using an $F \in \mathcal{F}_1$, i.e., with $u(t) = Fx(t)$, in the system (8a-8b) we have:

$$\begin{aligned} \dot{m}_l(t) &= YP_0E_1^{-1}AZm_l(t) + YP_0E_1^{-1}BFx(t), \quad m_l(0) = YP_0x_0 \\ n(t) &= -Q_0E_1^{-1}BFx(t). \end{aligned}$$

From the second equation we have $n(t) = -Q_0E_1^{-1}BFZm_l(t) - Q_0E_1^{-1}BFn(t)$. Consequently, using lemma 2.5, we have $n(t) = -(I + Q_0E_1^{-1}BF)^{-1}Q_0E_1^{-1}BFZm_l(t)$. Thus, $x(t)$ is given as:

$$\begin{aligned} x(t) &= m(t) + n(t) \\ &= \left(I - (I + Q_0E_1^{-1}BF)^{-1}Q_0E_1^{-1}BF\right)Zm_l(t) \\ &= (I + Q_0E_1^{-1}BF)^{-1}Zm_l(t). \end{aligned} \quad (9)$$

Therefore the inherent ODE is given by:

$$\dot{m}_l(t) = YP_0E_1^{-1}\left(A + BF(I + Q_0E_1^{-1}BF)^{-1}\right)Zm_l(t).$$

Now, let us define a mapping $\Omega : \mathcal{F}_1 \rightarrow \mathcal{P}$, i.e., $K = \Omega(F)$, as

$$K = \Omega(F) = F(I + Q_0E_1^{-1}BF)^{-1}Z. \quad (10)$$

Using the identity $Q_0E_1^{-1}BF(I + Q_0E_1^{-1}BF)^{-1} = (I + Q_0E_1^{-1}BF)^{-1}Q_0E_1^{-1}BF$, we have:

$$\begin{aligned} \dot{m}_l(t) &= YP_0E_1^{-1}AZm_l(t) + YP_0E_1^{-1}BKm_l(t) \\ n(t) &= -(I + Q_0E_1^{-1}BF)^{-1}Q_0E_1^{-1}BFZm_l(t) \\ &= -Q_0E_1^{-1}BKm_l(t). \end{aligned}$$

$\square$

Remark 2.3. The autonomous regular descriptor system, after F is applied, can be seen canonically as a vector field on a manifold (a proper linear subspace $\text{Im}P_0$ here). What theorem 2.3 says is that applying an $F \in \mathcal{F}_1$ to (2) is same as applying $\Omega(F) \in \mathcal{P}$ to the vector field (4a). Since, $\Omega(\cdot)$ is a many to one mapping we expect to have more than one $F \in \mathcal{F}_1$ resulting in the same closed loop behavior. Further, for applications like optimal control or dynamic games it will suffice to regulate (4a) using a partial state feedback, say $K \in \mathcal{P}$, then reconstruct the full state feedback as $\Omega^{-1}(K) \in \mathcal{F}_1$.

We analyze some properties of $\Omega^{-1}(\cdot)$ in the following theorem.

Theorem 2.4. *For any $K \in \mathcal{P}$, the inverse map $\Omega^{-1}(\cdot)$ is given by*

$$\Omega^{-1}(K) = KS^\dagger + T(I - SS^\dagger), \quad (11)$$

where $S = Z - Q_0 E_1^{-1} B K$ and $T \in \mathbb{R}^{m \times n}$ is arbitrary. Further, $\Omega^{-1}(K)$ is non empty.

Proof. From (10) we have $Z - Q_0 E_1^{-1} B K = \left(I - Q_0 E_1^{-1} B F (I + Q_0 E_1^{-1} B F)^{-1} \right) Z = (I + Q_0 E_1^{-1} B F)^{-1} Z$. Since Z has full column rank and from the above we notice that $Z - Q_0 E_1^{-1} B K$ has full column rank. Now, taking $S = Z - Q_0 E_1^{-1} B K$, (10) is given by $K = FS$. Now, for given a $K \in \mathcal{P}$, all solutions $F \in \mathcal{F}_1$ are characterized, see pg. 295 [1], by

$$F = \Omega^{-1}(K) = KS^\dagger + T(I - SS^\dagger), \quad (12)$$

where $T \in \mathbb{R}^{m \times n}$ is arbitrary. For the non emptiness part, applying a feedback $K \in \mathcal{P}$ means $u(t) = Km_l(t) = KYm(t) = KYZm_l(t)$. So, K and KYZ are same strategies resulting in same closed-loop behavior. Replacing K with KYZ and choosing $T = KY = KYZY$ in (12) we have $F = KYZS^\dagger + KYZY(I - SS^\dagger) = KYZY + KYZS^\dagger - KYZYSS^\dagger = KY + KYZS^\dagger - KYZYSS^\dagger - KYP_0 Q_0 E_1^{-1} B KS^\dagger = KY$. So, for every $K \in \mathcal{P}$, there is a trivial solution $\Omega^{-1}(K) = F = KY \in \mathcal{F}_1$. Notice, that this observation coincides with remark 2.2. $\square$

Remark 2.4. The game (1,2) with strategy set $\mathcal{P}$ is informationally inferior (see section 6.3 [3]) to $\mathcal{F}_1$, since $\mathcal{P} \subseteq \mathcal{F}_1$. Moreover, we showed that for every $F \in \mathcal{F}_1$ there exists a $\Omega(F) \in \mathcal{P}$. We recall from proposition 6.3.2 [3], that players cannot realize a cost lesser than what is achieved with an equilibrium strategy from $\mathcal{P}$, by searching in $\mathcal{F}_1$. In fact, here $K = \Omega(F)$ encapsulates all the state information. In other words, being a deterministic optimization problem providing more information about the state does not improve the optimal solution. We show later that there exist many informationally non unique equilibrium strategies in $\mathcal{F}_1$ which correspond to a single solution of the game. However, the situation can be quite different in a stochastic setting, for example see section 6.7 [3].

Remark 2.5. Notice, that the analysis in this section is restricted to the index 1 case. For higher order indices the algebraic constraints cannot be eliminated easily as they include derivatives of inputs. However, we address this case with a different approach in section 3.2.

3 Feedback Nash equilibria

3.1 Index 1 case

In this section we collect all the results discussed in the previous section to derive FBNE for the differential game (1, 2). First we consider the index 1 case. The game (1, 2), when players use strategies $u_i = F_i x$, $i = 1, 2$, with $F = [F_1 \ F_2] \in \mathcal{F}_1$, can be written as:

$$\begin{aligned} J_i(x_0, F) &= \int_0^\infty x'(t) \begin{bmatrix} I \\ F_1 \\ F_2 \end{bmatrix}' \begin{bmatrix} D_i & V_i & W_i \\ V_i' & R_{i1} & N_i \\ W_i' & N_i' & R_{2i} \end{bmatrix} \begin{bmatrix} I \\ F_1 \\ F_2 \end{bmatrix} x(t) dt \\ E\dot{x}(t) &= (A + BF)x(t), \quad x(0) = x_0. \end{aligned}$$

Using theorem 2.3 the game (1, 2) is transformed as follows:

$$J_i(m_l(0), K) = \int_0^\infty m_l'(t) \begin{bmatrix} I \\ K_1 \\ K_2 \end{bmatrix}' \begin{bmatrix} \tilde{D}_i & \tilde{V}_i & \tilde{W}_i \\ \tilde{V}_i & \tilde{R}_{1i} & \tilde{N}_i \\ \tilde{W}_i & \tilde{N}_i & \tilde{R}_{2i} \end{bmatrix} \begin{bmatrix} I \\ K_1 \\ K_2 \end{bmatrix} m_l(t) dt \quad (14a)$$

$$\dot{m}_l(t) = (\tilde{A} + \tilde{B}K) m_l(t), m_l(0) = Y P_0 x_0, \quad (14b)$$

where $F := [F_1' F_2']'$, $B := [B_1 B_2]$, $K := [K_1' K_2']'$, $\tilde{A} = Y P_0 E_1^{-1} A X$, $\tilde{B} = Y P_0 E_1^{-1} B$ and

$$\begin{bmatrix} \tilde{D}_i & \tilde{V}_i & \tilde{W}_i \\ \tilde{V}_i & \tilde{R}_{1i} & \tilde{N}_i \\ \tilde{W}_i & \tilde{N}_i & \tilde{R}_{2i} \end{bmatrix} := \begin{bmatrix} Z & -Q_0 E_1^{-1} B_1 & -Q_0 E_1^{-1} B_2 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}' \begin{bmatrix} D_i & V_i & W_i \\ V_i' & R_{1i} & N_i \\ W_i' & N_i' & R_{2i} \end{bmatrix} \begin{bmatrix} Z & -Q_0 E_1^{-1} B_1 & -Q_0 E_1^{-1} B_2 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}.$$

Notice, that the game (1, 2) is transformed into a game where the players' objectives and system dynamics are given by (14a) and (14b) respectively. We are interested in, first finding FBNE of the resulting lower order game (14a, 14b) and later use theorem 2.4 to obtain FBNE of the game (1, 2). We seek for stabilizing strategies to keep the players' objectives bounded. To this end, we make an assumption that $K \in \mathcal{P}$ is stabilizing i.e., $\tilde{A} + \tilde{B}K$ is stable. Further, if $m_l(t) \rightarrow 0$ then from (9) we have that $x(t) \rightarrow 0$. Moreover, $K \in \mathcal{P}$ and $\Omega^{-1}(K) \in \mathcal{F}_1$ have the same closed-loop behavior and as a result the FBNE of the game (1, 2) are stabilizing too. Now, The main result is given as follows:

Theorem 3.1. Let $\mu = 1$ and assume that matrix $\tilde{G} = \begin{bmatrix} \tilde{R}_{11} & \tilde{N}_1 \\ \tilde{N}_2' & \tilde{R}_{22} \end{bmatrix}$ is invertible and the matrices $\tilde{R}_{ii} > 0$, $i = 1, 2$.

Then (F_1^*, F_2^*) is a stabilizing FBNE for (1, 2) for every consistent initial state if and only if $F^* \in \Omega^{-1}(K)$ where K is given by

$$K = \begin{bmatrix} K_1 \\ K_2 \end{bmatrix} = -\tilde{G}^{-1} \begin{bmatrix} \tilde{B}_1' X_1 + \tilde{V}_1' \\ \tilde{B}_2' X_2 + \tilde{W}_2' \end{bmatrix} \quad (15)$$

and (X_1, X_2) are a symmetric stabilizing solution of the coupled algebraic Riccati equations

$$\tilde{D}_1 + \tilde{W}_1 K_2 + K_2' \tilde{W}_1' + K_2' \tilde{R}_{21} K_2 - K_1' \tilde{R}_{11} K_1 + X_1 (\tilde{A} + \tilde{B}_2 K_2) + (\tilde{A} + \tilde{B}_2 K_2)' X_1 = 0 \quad (16a)$$

$$\tilde{D}_2 + \tilde{V}_2 K_1 + K_1' \tilde{V}_2' + K_1' \tilde{R}_{12} K_1 - K_2' \tilde{R}_{22} K_2 + X_2 (\tilde{A} + \tilde{B}_1 K_1) + (\tilde{A} + \tilde{B}_1 K_1)' X_2 = 0. \quad (16b)$$

Moreover, $J_i = x_0' P_0' Y' X_i Y P_0 x_0$.

Proof. Let us define $H = (I + Q_0 E_1^{-1} B F)^{-1} Z$ and we notice that H has full column rank. From the above reformulation it follows directly from lemma 2.1 that (F_1^*, F_2^*) is a FBNE for (1,2) for every consistent initial state if and only if (K_1, K_2) is a FBNE for the game (14a,14b) and (F_1^*, F_2^*) solve $F^* \in \Omega^{-1}(K)$. Further, by theorem 2.1 we have that (K_1, K_2) is a FBNE for every initial state for the game (14a,14b) if and only if (15) holds where (X_1, X_2) are a stabilizing solution of (16a,16b). $\square$

Remark 3.1. The algebraic constraint is given by $n(t) = -Q_0 E_1^{-1} B K m_l(t) = -Q_0 E_1^{-1} B K Y m(t)$. Then, the consistent initial state manifold is characterized as:

$$\mathcal{X}_0 = \left\{ x_0 \in \mathbb{R}^n \mid Q_0 (I + E_1^{-1} B K Y P_0) x_0 = 0 \right\},$$

where K is the lower order FBNE given by (15).

3.2 Index μ case

Lemma 2.5 and theorem 2.3 were obtained by restricting the class of feedbacks to $\mathcal{F}_1$ and as a result the projectors P_0 and Q_0 were retained even for the modified pencil $(E, A + BF)$. However, for $\mu > 1$ the projector chains change

with an exception for the class $\mathcal{P}$, as demonstrated in theorem 2.2. For the index 1 case, the map $\Omega(\cdot)$, as shown in (9), removes all the redundant state information. However, for $\mu > 1$, due to presence of derivatives in the algebraic constraints, even if the strategies are restricted to $\mathcal{F}_\mu$ it is not clear if a mapping $\Omega : \mathcal{F}_\mu \rightarrow \mathcal{P}$, similar to (10), exists. However, we have the following sufficient condition.

Proposition 3.1. *If $D_i(I - P_0 \cdots P_{\mu-1}) = 0$, $V_i'(I - P_0 \cdots P_{\mu-1}) = 0$ and $W_i'(I - P_0 \cdots P_{\mu-1}) = 0$ for $i = 1, 2$, then players' objectives do not include the algebraic constraints. Further, the game (1, 2) can be solved using a partial state feedback.*

Proof. Using (4c), the integrand in (1) is rewritten as:

$$\begin{aligned} & (m(t) + n(t))' D_i(m(t) + n(t)) + (m(t) + n(t))' V_i u_1(t) + (m(t) + n(t))' W_i u_2(t) \\ & + u_1'(t) V_i'(m(t) + n(t)) + u_1'(t) W_i'(m(t) + n(t)) + \cdots \end{aligned} \quad (17)$$

Again from (4c), we have $n(t) = (I - P_0 \cdots P_{\mu-1})x(t)$. Now, if the conditions given in the statement of the lemma hold true then (17) is given by

$$m'(t) D_i m(t) + m'(t) V_i u_1(t) + m'(t) W_i u_2(t) + u_1'(t) V_i' m(t) + u_1'(t) W_i' m(t) + \cdots$$

As $m(t) = Z m_I(t)$, the game (1, 2) is same as the game obtained by replacing (1) with

$$\int_0^{t_f} [m_I'(t) \quad u_1'(t) \quad u_2'(t)] \tilde{M}_i [m_I'(t) \quad u_1'(t) \quad u_2'(t)]' dt, \text{ where } \tilde{M}_i = \begin{bmatrix} Z' D_i Z & Z' V_i & Z' W_i \\ V_i' Z & R_{1i} & N_i \\ W_i' Z & N_i' & R_{2i} \end{bmatrix},$$

and (2) with (6) respectively. Clearly, we see that the algebraic part of the descriptor system does not influence player's objectives and the resulting game is solved using theorem 2.1 with partial state feedback strategies of the type $u_i(t) = K_i m_I(t)$, $i = 1, 2$. $\square$

The conditions given in the above proposition can be too restrictive and we suggest a different approach so as to recast the problem to the index 1 case. The response of a descriptor system is characterized by the eigen structure of the pencil (E, A) . In the classical pole placement problem for systems where E is non-singular a desired eigen structure can be achieved, under a controllability assumption of the system, by using a derivative or proportional feedback. For descriptor systems there are several notions of controllability, for example, see [7, 28]. A descriptor system (2) is *impulse controllable* if $\text{rank}([E \quad A \mathcal{N}_\infty \quad B]) = n$, where the columns of $\mathcal{N}_\infty$ span $\text{Ker} E$. If the descriptor system is impulse controllable then it was shown, in corollary 7 [6], that there exists a matrix $G \in \mathbb{R}^{m \times n}$ such that the pencil $(E, A + BG)$ is regular and has index at most 1. If the dynamic environment where the players interact is modeled by a descriptor system with index μ then the strategies of players should be sufficiently smooth, and as a result players cannot adapt their strategies quickly. We make an assumption that players are obliged first, with an incentive that they can adapt quickly later, so as to regularize the system using a proportional state feedback. The players use strategies of type $u_i = G_i x + v_i$ such that the resulting descriptor system, given by $E \dot{x} = (A + B_1 G_1 + B_2 G_2)x + B_1 v_1 + B_2 v_2 = (A + BG)x + Bv$, $x(0) = x_0$ has index 1. This involves finding matrices G such that the projector chain stops after one step, i.e., $E_1 = E + (A + BG)Q_0$ is non singular, where Q_0 is a projector such that $\text{Im} Q_0 = \text{Ker} E$. An SVD based algorithm is presented in [6] to construct such regularizing feedbacks.

Now, once the higher index descriptor system is regularized we can apply the theory developed in section 3.1 to compute the FBNE. This involves replacing the matrix A with $A + BG$ in the analysis of sections 2.1 and 3.1. Notice that there exists more than one regularizing matrix G and not all of them give stabilizing solutions to the Riccati equations. We demonstrate this drawback with an example in the next section.

3.3 Examples

Example 1: We consider [13] the issue of convergence of feedback Nash solution of the singularly perturbed system

$$\begin{aligned} \dot{x}_1(t) &= x_1(t) + 2x_2(t) + u_1(t) + u_2(t), \quad x_1(0) = 1 \\ \varepsilon \dot{x}_2(t) &= -x_1(t) - 2x_2(t) + 2u_1(t) + 2u_2(t), \quad x_2(0) = 2, \end{aligned}$$

with performance criteria

$$J_i = \int_0^\infty \{ [x'_1(t)x'_1(t)]Q_i[x'_1(t)x'_1(t)]' + u'_i(t)R_{ii}u_i(t) + u'_j(t)R_{ij}u_j(t) \} dt, \quad i \neq j, \quad i, j = 1, 2,$$

where $D_i = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$, $R_{ii} = 1$ and $R_{ij} = 1$. In this example the converged feedback Nash equilibrium strategies are $F_i^* = \lim_{\varepsilon \downarrow 0} F_i(\varepsilon) = [-1.50908, -0.7321]$, $i = 1, 2$. The reduced order system, obtained by taking $\varepsilon = 0$, is a descriptor system. With $Q_0 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ as the initial projector and $E_1 = E + AQ_0 = \begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix}$, the canonical projector is obtained as $Q_0 = \begin{bmatrix} 0 & 0 \\ 1/2 & 1 \end{bmatrix}$. Using the canonical projector we have $E_1 = \begin{bmatrix} 2 & 2 \\ -1 & -2 \end{bmatrix}$, $Z = \begin{bmatrix} -1 \\ 1/2 \end{bmatrix}$ and $Y = \begin{bmatrix} -1 & 0 \end{bmatrix}$. The transformed system parameters (cf. (14a, 14b)) are $\tilde{A} = 0$, $\tilde{B} = \begin{bmatrix} -3 & -3 \end{bmatrix}$, $\tilde{Q}_i = 3/2$, $\tilde{V}_i = 0$, $\tilde{W}_i = 0$, $\tilde{R}_{ii} = 3$, $\tilde{R}_{ij} = 4$, $j \neq i$ and $\tilde{N}_i = 2$, $i = 1, 2$. Straightforward calculations show that the set of FBNE for this game, defined by $F_i = [f_{i1} \ f_{i2}]$, $i=1,2$, satisfy:

$$f_{i2} = -\frac{2}{2\sqrt{2}+1}f_{i1} - \frac{\sqrt{2}}{2\sqrt{2}+1}, \quad i = 1, 2, \quad f_{i1} \text{ is arbitrary.}$$

As observed in [13], we see that the FBNE of the original game, i.e., $F_i^* = \lim_{\varepsilon \downarrow 0} F_i(\varepsilon)$, does not belong to the set of the FBNE obtained from the lower order game as shown in the figure 1. If $F_i^* = \lim_{\varepsilon \downarrow 0} F_i(\varepsilon) \in \Omega^{-1}(K)$, then the game is considered to be well posed and this property is desired as the lower order FBNE, i.e., $\Omega^{-1}(K)$, are robust against uncertainties of the parameter ε . See [13, 17] for a detailed discussion.

Figure 1: The FBNE of the singularly perturbed system, i.e., ($\varepsilon \neq 0$), is represented as the black dot and the set of FBNE of the lower order game is represented by the gray line.

Example 2: In this example we show that the suggested regularization approach, as given in section 3.2, for higher order index cases gives different solutions based on the choice of G used. The example, from [11], is a macro-economic stabilization problem. Assume that a monetary and fiscal authority like to stabilize some key macro-economic variables, i.e., the real interest rate, r , inflation, $\dot{p}$, and the output gap, y , after a shock has oc-

curred. The system is described by the following equations:

$$r(t) = i(t) - \dot{p}(t) \quad (18a)$$

$$\dot{y}(t) = -\alpha i(t) + \alpha \dot{p}(t) + \beta f(t) \quad (18b)$$

$$m(t) - p(t) = \gamma y(t) - \delta i(t) \quad (18c)$$

Here $p(t)$ is the price level, $i(t)$ denotes the nominal interest rate, $m(t)$ is the money supply and $f(t)$ the fiscal policy. The first two instruments, the nominal interest rate and money supply, are determined by the monetary authority of the country, whereas the level of the third instrument, the fiscal policy, is set by the government. Here, Equation (18a) models the real interest rate, (18b) is a simple growth equation of the output gap and (18c) models asset market equilibrium. Assume that an initial shock in the real interest rate, price level and output gap has occurred, all equal to one (in the respective units). Introducing as the state variable $x(t) := [r(t) \ p(t) \ y(t)]'$, $u_1(t) = [i(t) \ m(t)]'$ and $u_2(t) = f(t)$ the model can be rewritten as (2), where

$$E = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -\alpha & 1 \\ 0 & 0 & 0 \end{bmatrix}, A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & -\gamma \end{bmatrix}, B_1 = \begin{bmatrix} 1 & 0 \\ -\alpha & 0 \\ \delta & 0 \end{bmatrix}, B_2 = \begin{bmatrix} 0 \\ \beta \\ 0 \end{bmatrix} \text{ and } x_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

The performance criteria of the players are given as:

$$J_i = \int_0^\infty e^{-\theta t} [x'(t) \ u_1'(t) \ u_2'(t)] M_i [x'(t) \ u_1'(t) \ u_2'(t)]' dt, \ i = 1, 2.$$

Since the performance indices are discounted, we make the following change of variables $A \rightarrow A - \frac{\theta}{2}E$. It can be easily verified that the system is regular if $1 + \alpha\gamma \neq 0$. We observe that the pencil (E, A) has index 2. Taking

$\alpha = 1/2, \beta = 3/4, \gamma = 1, \delta = 1/2$ and $\theta = 0.15$, the initial choice of $Q_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and $G = (g_{ij})$ we see that

with $g_{11} + 2g_{21} \neq 0$, the system can be regularized. The matrices entering the players costs are $D_1 = \text{diag}\{2, 2, 1\}$, $D_2 = \text{diag}\{1, 1, 2\}$, $R_{11} = \text{diag}\{2, 2\}$, $R_{21} = 1$, $R_{12} = \text{diag}\{1, 1\}$, $R_{22} = 2$, $V_i = 0$, $W_i = 0$, $N_i = 0$, $i = 1, 2$. For

the following choices of regularizing matrices $G = \begin{bmatrix} 2 & -1 & -1 \\ 1/2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and $G = \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ the eigenvalues of

$\tilde{A} + \tilde{B}K$ are found to be $(-1.2292, -0.58473)$ and $(-1.7905, -0.57729)$ respectively. Further, if the choice of the

regularizing matrix is $G = \begin{bmatrix} 2 & 1 & 1 \\ 1/2 & 3 & 4 \\ 0 & 1 & 1 \end{bmatrix}$, we observe that the coupled Riccati equations do not have a solution.

So, we see that the choice of regularizing matrix G does affect the solution of the game. However, if players agree, before switching to non cooperative mode of play, upon a particular choice of the regularizing matrix then this method can be used to find FBNE for games defined with higher order descriptor systems.

4 Conclusions

In this article we considered the regular indefinite infinite planning horizon linear quadratic differential game for descriptor systems. Firstly, we developed an algorithm to generate canonical projectors for a regular matrix pair and using these projectors it is possible to canonically decouple a descriptor system into differential and algebraic parts. Later, we analyze the effect of feedback on a regular descriptor system. We show, for the index 1 case, that there exists a many to one mapping from full state index preserving feedbacks to the projected state feedbacks leading to informational non-uniqueness. Further, we discussed the properties of the inverse mapping and derived necessary and sufficient conditions for the existence of FBNE. These conditions were stated in terms of a projected system. A complete parametrization was derived for the set of FBNE. For the higher order index case, under an impulse controllability assumption, we suggested a regularization based approach to recast the problem to an index 1 case. However, it is unclear as to how the Riccati equations depend upon the class of regularizing matrices. We

demonstrated the drawback of the approach with an example. The obtained theoretical results can be generalized straightforwardly to the N-player case.

We observed that the closed-loop system evolves invariantly on a proper linear subspace of $\mathbb{R}^n$, the configuration space. So, the FBNE is an inverse projection of the FBNE obtained from the lower order (projected) system. Further, in case there exists a FBNE, usually, there exists an infinite number of feedback Nash equilibria which all give rise to the same closed-loop behavior of the system. For future work, it would be interesting, for instance, to investigate the question whether in a singularly perturbed game the full order FBNE belongs to the set of inverse projections of the lower order FBNE, this aspect was studied in [13, 17]. Further, it would be interesting to search for equilibria within this set that satisfy some additional properties, like e.g. robustness.

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A Appendix

For a regular index- μ pencil $\lambda E - A$, we consider the following sequences of matrices, subspaces and projectors

$$E_0 := E, A_0 := A, \tag{19a}$$

$$\text{for } i \geq 0 \quad E_{i+1} = E_i + A_i Q_i, A_{i+1} = A_i P_i, \tag{19b}$$

$$Q_i^2 = Q_i, \text{Im} Q_i = \text{Ker} E_i, P_i = I - Q_i. \tag{19c}$$

For the matrix chain (19a-19c) an important result, following [15], is given as:

Theorem A.1. *If (E, A) is a regular pencil with the index μ , then the matrices $E_0, E_1, \dots, E_{\mu-1}$ are singular, whereas E_μ is non-singular. Conversely, if E_μ is non-singular and $E_0, E_1, \dots, E_{\mu-1}$ are singular, then (E, A) is a regular pencil with index μ .*

The projectors Q_i , $i = 0, 1, \dots, \mu - 1$ are not unique since the range of Q_i is fixed as $\text{Ker}E_i$ and $\text{Ker}Q_i$ is arbitrary. Later, we see that this arbitrariness is helpful to construct an algorithm. We review some important properties of matrix projectors below. Refer [20, 24] for details.

1. $(\text{Ker}E_i \cap \text{Ker}A_i) = (\text{Ker}E_i \cap \text{Ker}E_{i+1}) \subseteq (\text{Ker}E_{i+1} \cap \text{Ker}E_{i+2})$
2. Following theorem 2.1 of [20], for a regular pencil with index μ , the projectors $Q_0, Q_1, \dots, Q_{\mu-1}$ can be constructed such that $Q_j Q_i = 0$ for $j > i$. Projectors satisfying this property are called *admissible* projectors.

The above properties hold true irrespective of the choice of projectors. We elaborate more on properties 1 and 2 here. For a regular pencil with index μ , E_μ is non-singular so $\text{Ker}E_\mu = \{0\}$. From property (1), given above, we have $(\text{Ker}E_0 \cap \text{Ker}E_1) \subseteq \dots \subseteq (\text{Ker}E_{\mu-2} \cap \text{Ker}E_{\mu-1}) \subseteq (\text{Ker}E_{\mu-1} \cap \text{Ker}E_\mu) = \{0\}$. This implies $(\text{Ker}E_0 \oplus \dots \oplus \text{Ker}E_{i-1}) \cap \text{Ker}E_i = \{0\}$ or $(\text{Im}Q_0 \oplus \text{Im}Q_1 \oplus \dots \oplus \text{Im}Q_{i-1}) \cap \text{Im}Q_i = \{0\}$, see lemma 2.6 [19] for details. We recall the following lemma:

Lemma A.1 (lemma 2.5, [20]). *For two subspaces of $\mathbb{R}^m$ $L = \text{span}\{l_1, l_2, \dots, l_s\}$, $N = \text{span}\{n_1, n_2, \dots, n_t\}$, $L \cap N = 0$, there is a projector U such that $\text{Im}U = L$, $\text{Ker}U \supseteq N$.*

Proof. Denote by R the $m \times (s+t)$ matrix consisting of the columns $l_1, \dots, l_s, n_1, \dots, n_t$. Since $N \cap L = 0$, $s+t \leq n$, and $l_1, \dots, l_s, n_1, \dots, n_t$ are linearly independent. Then, R is full column rank and the desired projector U is constructed as

$$U = R \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} (R'R)^{-1} R', \text{ where } R = [l_1, \dots, l_s \quad n_1 \dots, n_t]. \quad (20)$$

□

Using lemma A.1 we can choose Q_i such that $(\text{Im}Q_0 \oplus \text{Im}Q_1 \oplus \dots \oplus \text{Im}Q_{i-1}) \subseteq \text{Ker}Q_i$ and $\text{Im}Q_i = \text{Ker}E_i$. The constructed projectors, called as *admissible projectors*, satisfy $Q_j Q_i = 0$, $j > i$. Later we propose an algorithm to generate these *admissible* projectors given a regular pencil with index μ .

Illustration for index 1

In this section we demonstrate the application of matrix projectors towards decoupling a descriptor system. Consider the following regular index 1 descriptor system:

$$E\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0 \quad (21)$$

The matrix chain is given by $E_0 = E$, $A_0 = A$, $Q_0 = Q_0^2$, $\text{Im}Q_0 = \text{Ker}E_0$, $P_0 = I - Q_0$, $E_1 = E_0 + A_0 Q_0 = E + A Q_0$ and E_1 is non singular. We verify that $E_1 P_0 = E P_0 + A Q_0 P_0 = E P_0 \rightarrow P_0 = E_1^{-1} E P_0$ and $E_1 Q_0 = E Q_0 + A Q_0 Q_0 = A Q_0 \rightarrow Q_0 = E_1^{-1} A Q_0$. Further, $x(t)$ can be decomposed as $x(t) = Ix(t) = (P_0 + Q_0)x(t) = P_0 x(t) + Q_0 x(t)$ and we represent $m(t) = P_0 x(t)$ and $n(t) = Q_0 x(t)$. Now, pre-multiplying the above descriptor system with E_1^{-1} leads to

$$\begin{aligned} E_1^{-1} E \dot{x}(t) &= E_1^{-1} A x(t) + E_1^{-1} B u(t) \\ E_1^{-1} E (P_0 \dot{x}(t) + Q_0 \dot{x}(t)) &= E_1^{-1} A P_0 x(t) + E_1^{-1} A Q_0 x(t) + E_1^{-1} B u(t) \\ P_0 \dot{x}(t) &= E_1^{-1} A P_0 x(t) + Q_0 x(t) + E_1^{-1} B u(t). \end{aligned}$$

Multiplying the above equation with P_0 gives the inherent ODE

$$\dot{m}(t) = P_0 E_1^{-1} A m(t) + P_0 E_1^{-1} B u(t) \quad (22)$$

and with Q_0 gives the algebraic constraint

$$0 = Q_0 E_1^{-1} A m(t) + n(t) + Q_0 E_1^{-1} B u(t). \quad (23)$$

The above decoupling is not complete as $m(t)$ appears in (23). Let us define $\tilde{Q}_0 = Q_0 E_1^{-1} A$. To see $\tilde{Q}_0$ as a valid projector onto $\text{Ker} E_0$, we note that $\tilde{Q}_0 Q_0 = Q_0 E_1^{-1} A Q_0 = Q_0$ and $Q_0 \tilde{Q}_0 = Q_0 Q_0 E_1^{-1} A = \tilde{Q}_0$ which implies $\tilde{Q}_0$ is a projector onto the same range as Q_0 . Next, we discuss the canonicity² of $\tilde{Q}_0$. Let Q_{0_1} and Q_{0_2} be two projectors with range constrained to $\text{Ker} E$, then $Q_{0_1} Q_{0_2} = Q_{0_2}$ and $Q_{0_2} Q_{0_1} = Q_{0_1}$. Using this we can show $Q_{0_1} P_{0_2} = -Q_{0_2} P_{0_1}$. We write $E_{1_2} = E + A Q_{0_2} = E + A Q_{0_1} Q_{0_2} = E + A Q_{0_1} - A Q_{0_1} P_{0_2} = E + A Q_{0_1} + A Q_{0_2} P_{0_1} = (E + A Q_{0_1})(I + Q_{0_2} P_{0_1}) = E_{1_1}(I + Q_{0_2} P_{0_1})$. Thus we have $E_{1_2}^{-1} = (I - Q_{0_2} P_{0_1}) E_{1_1}^{-1}$. Now, $\tilde{Q}_{0_2} = Q_{0_2} E_{1_2}^{-1} A = Q_{0_2} (I - Q_{0_2} P_{0_1}) E_{1_1}^{-1} A = Q_{0_1} E_{1_1}^{-1} A = \tilde{Q}_{0_1}$. $\tilde{Q}_0$ is called a *canonical projector* and it is unique, so we have that $\tilde{Q}_0 = \tilde{Q}_0 \tilde{E}_1^{-1} A$, where $\tilde{E}_1 = E + A \tilde{Q}_0$. Repeating the above decoupling procedure by replacing Q_0 with $\tilde{Q}_0$ we arrive at the same decoupled equations (22, 23) as above, but the cross term disappears in (23), i.e., $n(t) = -Q_0 E_1^{-1} B u(t)$, leading to a complete decoupling.

Canonical Projectors

In the previous section we showed that an index 1 system can be decoupled completely with the existence of a unique canonical projector $\tilde{Q}_0$. For index $\mu > 1$, by theorem 3.1 [20], the existence of canonical projectors is guaranteed for a regular descriptor system. Furthermore, the canonical projectors are also admissible and satisfy:

$$Q_i = Q_i P_{i+1} \cdots P_{\mu-1} E_{\mu}^{-1} A_i, \quad i = 0, 1, \dots, \mu - 2 \quad (24)$$

$$Q_{\mu-1} = Q_{\mu-1} E_{\mu}^{-1} A_{\mu-1}, \quad (25)$$

where A_i, E_{μ} are defined in (19a-19c).

Algorithm

Collecting the ideas discussed above we present an algorithm to generate canonical projectors for an index μ regular matrix pair (E, A) as follows:

STEP 1 Start with $E_0 = E, A_0 = A, Q_0 = I - E_0^{\dagger} E_0, P_0 = I - Q_0$ and $V = \text{Ker} E_0$.

STEP 2 (**Admissible projectors**) for $i \in [1, \mu - 1]$

- (a) $E_i = E_{i-1} + A_{i-1} Q_{i-1}, A_i = A_{i-1} P_{i-1}, U = \text{Ker} E_i$
- (b) Define $R = \begin{bmatrix} U & V \end{bmatrix}$
- (c) $Q_i = R \begin{bmatrix} I & \\ & 0 \end{bmatrix} (R' R)^{-1} R', P_i = I - Q_i$
- (d) $V = \begin{bmatrix} V & U \end{bmatrix}$

STEP 3 (**Canonical projectors**)

- (a) Set for $i = 0, \dots, \mu - 1, Q_i^{(0)} = Q_i, E_i^{(0)} = E_i, A_i^{(0)} = A_i$
- (b) Make $Q_{\mu-1}^{(0)}$ canonical by $Q_{\mu-1}^{(0)} = Q_{\mu-1}^{(0)} \left(E_{\mu}^{(0)} \right)^{-1} A_{\mu-1}^{(0)}$
- (c) for $j = 0$ to $\mu - 1$
 - i. $Q_{\mu-1}^{(j)} = Q_{\mu-1}^{(j-1)} \left(E_{\mu}^{(j-1)} \right)^{-1} A$
 - ii. $Q_i^{(j)} = Q_i^{(j-1)} P_{i+1}^{(j-1)} \cdots P_{\mu-1}^{(j-1)} \left(E_{\mu}^{(j-1)} \right)^{-1} A, i = 0, 1, \dots, \mu - 2$
 - iii. $\left(E_{\mu}^{(j)} \right)^{-1} = \left(I - Q_0^j P_0^{j-1} - Q_1^j P_1^{j-1} - \cdots - Q_{\mu-1}^j P_{\mu-1}^{j-1} \right) \left(E_{\mu}^{(j-1)} \right)^{-1}$

After obtaining the set of admissible projectors from step (2), canonical projectors are obtained from step (3). For a discussion on the additional properties, such as admissibility of the intermediate projector chains $\left(Q_i^{(j)}, P_i^{(j)} \right), i = 0, 1, \dots, \mu - 1$ and canonicity of the resulting projectors $Q_i^{(\mu-1)}, i = 0, 1, \dots, \mu - 1$ refer to [24].

²This discussion follows from [25] and we repeat it for the sake of completeness.

Illustration for index $\mu > 1$

Similar to index 1 case, the canonical projectors completely decouple a regular descriptor system with index $\mu > 1$. For more details on the procedure refer [20]. We, however, give few important steps. Firstly, canonical projectors satisfy the following identity:

$$x = (P_0 + Q_0)x = (P_0P_1 + P_0Q_1 + Q_0)x = \dots = (P_0 \dots P_{\mu-1})x + \dots + P_0P_1Q_2x + P_0Q_1x + Q_0x. \quad (26)$$

We see $x(t)$ can be projected into $\mu + 1$ subspaces. The first projection constitutes a vector field that evolves invariantly in the subspace $\text{Im}(P_0 \dots P_{\mu-1})$ and the remaining μ projections constitute algebraic constraints. The descriptor system (21) is decoupled completely as follows:

$$\begin{aligned} (P_0 \dots P_{\mu-1})\dot{x} &= (P_0 \dots P_{\mu-1})E_\mu^{-1}A(P_0 \dots P_{\mu-1})x + (P_0 \dots P_{\mu-1})E_\mu^{-1}Bu \\ \begin{bmatrix} Q_0 & Q_0P_1 & Q_0P_1P_2 & \dots & (Q_0P_1 \dots P_{\mu-2}) \\ & Q_1 & Q_1P_2 & \dots & (Q_1P_2 \dots P_{\mu-2}) \\ & & \ddots & & \vdots \\ & & & Q_{\mu-3} & Q_{\mu-3}P_{\mu-2} \\ & & & & Q_{\mu-2} \end{bmatrix} \begin{bmatrix} Q_1 \\ Q_2 \\ \vdots \\ Q_{\mu-2} \\ Q_{\mu-1} \end{bmatrix} \dot{x} &= - \begin{bmatrix} Q_0 \\ Q_1 \\ \vdots \\ Q_{\mu-3} \\ Q_{\mu-2} \end{bmatrix} x - \begin{bmatrix} Q_0P_1 \dots P_{\mu-1} \\ Q_1P_2 \dots P_{\mu-1} \\ \vdots \\ Q_{\mu-3}P_{\mu-2}P_{\mu-1} \\ Q_{\mu-2}P_{\mu-1} \end{bmatrix} E_\mu^{-1}Bu \\ 0 &= Q_{\mu-1}x + Q_{\mu-1}E_\mu^{-1}Bu. \end{aligned}$$

The last two equations are written compactly as follows:

$$\begin{bmatrix} Q_0 \\ Q_1 \\ \vdots \\ Q_{\mu-3} \\ Q_{\mu-2} \\ Q_{\mu-1} \end{bmatrix} x = -\mathcal{M} \begin{bmatrix} Q_1 \\ Q_2 \\ \vdots \\ Q_{\mu-2} \\ Q_{\mu-1} \\ 0 \end{bmatrix} \dot{x} - \mathcal{M} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ I \end{bmatrix} E_\mu^{-1}Bu, \quad \mathcal{M} = \begin{bmatrix} Q_0 & Q_0P_1 & P_1P_2 & \dots & Q_0P_1 \dots P_{\mu-1} \\ & Q_1 & Q_1P_2 & \dots & Q_1P_2 \dots P_{\mu-1} \\ & & \ddots & & \vdots \\ & & & Q_{\mu-3}P_{\mu-2} & Q_{\mu-3}P_{\mu-2}P_{\mu-1} \\ & & & Q_{\mu-2} & Q_{\mu-2}P_{\mu-1} \\ & & & & Q_{\mu-1} \end{bmatrix}.$$

By careful elimination of derivatives, of terms Q_jx , $1 \leq j \leq \mu - 1$, on the right hand side, we have:

$$\begin{bmatrix} Q_0 \\ P_0Q_1 \\ \vdots \\ P_0 \dots P_{\mu-3}Q_{\mu-2} \\ P_0P_1 \dots P_{\mu-2}Q_{\mu-1} \end{bmatrix} x = - \begin{bmatrix} X & X & X & \dots & X & -Q_0P_1 \dots P_{\mu-1} \\ & X & X & \dots & X & -P_0Q_1P_2 \dots P_{\mu-1} \\ & & \ddots & & \vdots & \vdots \\ & & & X & -P_0 \dots P_{\mu-3}Q_{\mu-2}P_{\mu-1} \\ & & & & -P_0 \dots P_{\mu-2}Q_{\mu-1} \end{bmatrix} \begin{bmatrix} E_\mu^{-1}Bu^{\mu-1} \\ E_\mu^{-1}Bu^{\mu-2} \\ \vdots \\ E_\mu^{-1}Bu^1 \\ E_\mu^{-1}Bu \end{bmatrix}. \quad (27)$$

From property (26), representing $m(t) = (P_0 \dots P_{\mu-1})x(t)$ and $n(t) = (P_0 \dots P_{\mu-2}Q_{\mu-1} + \dots + P_0Q_1 + Q_0)x(t) = (I - P_0 \dots P_{\mu-1})x(t)$, the descriptor system can be decoupled as:

$$\dot{m}(t) = P_0 \dots P_{\mu-1}E_\mu^{-1}Am(t) + P_0 \dots P_{\mu-1}E_\mu^{-1}Bu(t), \quad m(0) = P_0 \dots P_{\mu-1}x_0 \quad (28a)$$

$$n(t) = - \sum_{i=0}^{\mu-1} N_i E_\mu^{-1}Bu^i(t). \quad (28b)$$

Where N_i in (28b) are obtained by manipulation of terms in (27).