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**ON THE SCALAR FEEDBACK NASH
EQUILIBRIA IN THE INFINITE HORIZON LQ-
GAME**

By Jacob C. Engwerda

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On the scalar feedback Nash equilibria in the infinite horizon LQ-game

by

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Abstract

In this paper we consider linear stationary feedback Nash equilibria of the scalar linear-quadratic differential game. The planning horizon considered is assumed to be infinite. We present both necessary and sufficient conditions on the system parameters for existence of a unique solution of the associated algebraic Riccati equations (ARE) that stabilizes the closed-loop system. Moreover, we show that in case there exists more than one solution of the (ARE) that stabilizes the closed-loop system, the spectra of the corresponding closed-loop systems almost always differ. A numerical algorithm is given to calculate the solution which yields the most stable closed-loop system.

Keywords: Linear quadratic games, feedback Nash equilibrium, solvability conditions, Riccati equations

I. Introduction

During the last decade there has been an increasing interest to study several problems in economics using a dynamic game theoretical setting. In particular in the area of environmental economics and macro-economic policy coordination this is a very natural framework to model problems (see e.g. de Zeeuw et al. (1991), Mäler (1992), Kaitala et al. (1992) and Dockner et al. (1985), Tabellini (1986), Fershtman et al. (1987), Petit (1989), Levine et al. (1994), van Aarle et al. (1995), Neck et al. (1995), Douven et al (1996)). In, e.g., policy coordination problems usually two basic questions arise i.e., first, are policies coordinated and, second, which information do the participating parties have. Usually both these points are rather unclear and, therefore, strategies for different possible scenarios are calculated and compared with each other. One of these scenarios is the so-called feedback Nash scenario. The scenario we will study here assumes that the involved parties act noncooperative, that is, each party is only interested in the minimization of his own cost. This bare fact, however, does not completely specify the game. The cost each party incurs depends on the decisions made by the other parties. So, there is still some freedom in specifying which decisions will be made by the parties. We will assume that the ultimate policies that will be played are such that no party can improve his outcome by altering his decision unilaterally. In literature this is known as the Nash equilibrium concept. Furthermore, it is assumed that the strategy used by the parties is either based on the complete history of the state of the game, or just the initial state and the current state of the game, or just the current state of the game. This requirement on the information structure the players have to determine their strategy leaves however, in general, still much freedom for the party in choosing his policy. Therefore, additional requirements have been formulated to restrict the class of possible policies. A requirement we make here is that the chosen policy must be strongly time consistent. That is, the parties have no reason at any future stage of the game, to deviate from the adopted policy even if there have been deviations in the past from the actions which are dictated by the original policy.

It turns out that it is possible to define a refinement of the Nash equilibrium concept which has the above stated requirements, the so-called feedback Nash equilibrium concept (see e.g. Başar and Olsder (1995) for a precise definition and survey of relevant literature). In fact, it can be shown that the class of strongly time consistent Nash equilibria coincides with the class of feedback Nash equilibria (see e.g. Weeren (1995, theorem 3.24)).

Note that, since according to this scenario the participating parties can react to each other's policies, its economic relevance is mostly larger than that of the open-loop Nash scenario. In particular the feedback Nash scenario is very popular in studying problems where the underlying model can be described by a (set of) linear differential equation(s) and the individual objectives, the parties are striving for, can be approximated by functions which quadratically penalize deviations from some (equilibrium) targets. Under the assumption that the parties only have a finite-planning horizon, this problem was first analyzed by Starr and Ho in (1969) (see also Lukes (1971) for a result on uniqueness

within the class of affine memoryless strategies).

In this paper we study the infinite-planning horizon case. The reason for studying an infinite-planning horizon can be motivated from at least two reasons. First, in economic growth theory it is usually difficult to justify the assumption that a firm (or government) has a finite-planning horizon t_f ; for, why should it ignore profits earned after t_f (or utility of generations alive beyond t_f). Second, we will see that from a computational point of view the equilibrium strategies are much easier to implement and analyze than those for a finite-planning horizon.

We will concentrate here on solving the with this problem associated algebraic Riccati equations. In Weeren et al. (1997) it was shown that in the scalar case these equations have either one or three solutions which solve the optimization problem. However, a detailed study under which conditions on the system parameters these different situations occur is lacking. In this paper we study this subject in detail, and consider the question under which parameter conditions it is possible to discriminate between the three solutions if we impose some additional stability requirements. Furthermore, we consider numerical algorithms to calculate the solutions.

The outline of the paper is as follows. In section two we start by stating the problem analysed in this paper. Section three analyzes the solutions of the algebraic Riccati equations. Then, in section 4, we analyze the stabilization properties of the solutions and present in the subsequent section a numerical algorithm to calculate the most stabilizing solution. The paper ends with some concluding remarks.

II. Problem statement

In this paper we consider the problem where two parties (henceforth called players) try to minimize their individual quadratic performance criterion. Each player controls a different set of inputs to a single system. The system is described by the following differential equation

$$\dot{x} = ax + b_1 u_1 + b_2 u_2, \quad x(0) = x_0. \quad (1)$$

Here x is the state of the system, u_i is a (control) variable player i can manipulate, x_0 is the arbitrarily chosen initial state of the system, a , b_1 , and b_2 are constant system parameters, and $\dot{x}$ denotes the time derivative of x .

The performance criterion player $i = 1, 2$ aims to minimize is:

$$J_i(u_1, u_2) := \frac{1}{2} \int_0^\infty \{x(t)^T q_i x(t) + u_i(t)^T r_{ii} u_i(t)\} dt.$$

We assume that both q_i and r_{ii} are positive and b_i differs from zero.

In this paper we consider in detail the existence of a feedback Nash equilibrium of this differential game (see Başar and Olsder (1995) for a precise definition of this equilibrium concept).

Next, consider the following set of coupled algebraic Riccati equations (ARE):

$$-ak_1 - k_1 a - q_1 + k_1 s_1 k_1 + k_1 s_2 k_2 + k_2 s_2 k_1 = 0; \quad (2)$$

$$-ak_2 - k_2a - q_2 + k_2s_2k_2 + k_2s_1k_1 + k_1s_1k_2 = 0; \quad (3)$$

where $s_i := b_i r_{ii}^{-1} b_i$.

From Başar and Olsder (1995, proposition 6.8), we have:

Theorem 1:

Let $k_i \geq 0$ solve the set of Riccati equations.

Then the pair of strategies given by

$$u_i = -r_{ii}^{-1} b_i \bar{k}_i x \quad (4)$$

$i = 1, 2$, provide a feedback Nash equilibrium, leading to the cost $J_i(u_1, u_2) := x_0 \bar{k}_i x_0$, for player i .

Moreover, the resulting system dynamics described by $\dot{x} = a_{cl}x$; $x(0) = x_0$, with $a_{cl} := a - s_1 \bar{k}_1 - s_2 \bar{k}_2$, is asymptotically stable. $\square$

In fact, one can immediately deduce from Weeren (1995, p.96) that when the players are restricted at the outset to memoryless strategies (cf. Lukes (1971)) then existence of a positive solution to the above scalar Riccati equations is a both necessary and sufficient condition for existence of a feedback Nash equilibrium.

A natural question which arises is how many solutions the above set of algebraic Riccati equations (ARE) have. To analyze this question we introduce (for notational convenience) the variables:

$$\sigma_i := s_i q_i \text{ and } \kappa_i := s_i k_i, \quad i = 1, 2.$$

Using this notation (2,3) can be rewritten as

$$-2a\kappa_1 - \sigma_1 + \kappa_1^2 + 2\kappa_1\kappa_2 = 0; \quad (5)$$

$$-2a\kappa_2 - \sigma_2 + \kappa_2^2 + 2\kappa_2\kappa_1 = 0. \quad (6)$$

The above question can therefore be reformulated as under which conditions the two hyperbola

$$\kappa_2 = a - \frac{1}{2}\kappa_1 + \frac{\sigma_1}{2} \frac{1}{\kappa_1}; \quad (7)$$

and

$$\kappa_1 = a - \frac{1}{2}\kappa_2 + \frac{\sigma_2}{2} \frac{1}{\kappa_2}; \quad (8)$$

intersect in the first quadrant of the (κ_1, κ_2) -plane. In the next section we will study this problem in detail.

III. The intersection points

To visualize the situation, we sketched in figure 1 the hyperbola (7) and (8), respectively.

Figure 1: The hyperbola in case $a = 2, \sigma_1 = 1, \sigma_2 = \frac{1}{2}$

From (5) we have that $\kappa_2 = \frac{2a\kappa_1 + \sigma_1 - \kappa_1^2}{2\kappa_1}$. Substitution of this into (6) shows that $\kappa_1 > 0$ must satisfy the following equation

$$-a \frac{2a\kappa_1 + \sigma_1 - \kappa_1^2}{\kappa_1} - \sigma_2 + \frac{(2a\kappa_1 + \sigma_1 - \kappa_1^2)^2}{4\kappa_1^2} + (2a\kappa_1 + \sigma_1 - \kappa_1^2) = 0.$$

Elementary calculation shows that this equation can be rewritten as

$$3\kappa_1^4 - 8a\kappa_1^3 + (4a^2 - 2\sigma_1 + 4\sigma_2)\kappa_1^2 - \sigma_1^2 = 0. \quad (9)$$

From this equation it is immediately clear that the set of (ARE) equations has at most four solutions. To analyze the number of positive solutions in some more detail we consider the Hurwitz matrix H (see e.g. Lancaster et al. (1985, pp.480)) corresponding with this polynomial (9). This matrix equals

$$H = \begin{pmatrix} -8a & 0 & 0 & 0 \\ 3 & 4a^2 - 2\sigma_1 + 4\sigma_2 & -\sigma_1^2 & 0 \\ 0 & -8a & 0 & 0 \\ 0 & 3 & 4a^2 - 2\sigma_1 + 4\sigma_2 & -\sigma_1^2 \end{pmatrix}.$$

Let Δ_i denote the principal minors of this matrix and a_4 be the leading coefficient of the polynomial (9). Then, according the Routh-Hurwitz stability result (see e.g. Lancaster

et al. (1985, theorem 13.9.1)), provided Δ_i differs from zero the number of positive roots (counted with algebraic multiplicities) of the polynomial (9) is given by the number of alterations of sign in the sequence $a_4, \Delta_1, \frac{\Delta_2}{\Delta_1}, \frac{\Delta_3}{\Delta_2}, \frac{\Delta_4}{\Delta_3}$ and has the polynomial no roots on the imaginary axis. It is easily verified that the principal minors of matrix H are

$$\Delta_1 = -8a; \Delta_2 = -8a(4a^2 - 2\sigma_1 + 4\sigma_2); \Delta_3 = 64a^2\sigma_1^2 \text{ and } \Delta_4 = -(8a\sigma_1^2)^2.$$

So, under the assumption that both a and $(4a^2 - 2\sigma_1 + 4\sigma_2)$ differ from zero, the number of positive roots equals the number of alterations of sign, π , in the sequence

$$3, -8a, 4a^2 - 2\sigma_1 + 4\sigma_2, \frac{-8a\sigma_1^2}{4a^2 - 2\sigma_1 + 4\sigma_2}, -\sigma_1^2, \quad (10)$$

and the number of negative roots is $4 - \pi$. Using this, the next result is easily obtained

Theorem 2:

If $a \leq 0$, (ARE) has exact 1 solution satisfying $k_1 > 0, k_2 > 0$.

If $a > 0$, (ARE) has exact 1 solution satisfying $k_1 < 0, k_2 < 0$ and either 1 or 3 solutions satisfying $k_1 > 0, k_2 > 0$.

Proof:

First consider the case $a < 0$. From (10) it is then easily verified that in case $(4a^2 - 2\sigma_1 + 4\sigma_2)$ differs from zero, $\pi = 1$. So, in this case the polynomial has one positive real zero. Next assume that the corresponding $\kappa_2 < 0$. From (6) we then conclude that $\kappa_2 = a - \kappa_1 - \sqrt{(a - \kappa_1)^2 + \sigma_2}$, whereas from (7) we have that $\kappa_2 = a - \frac{1}{2}\kappa_1 + \frac{\sigma_1}{2\kappa_1}$. This yields a contradiction (see also Weeren et al. Lemma 2.6). We therefore conclude that (ARE) has exact one solution in the first quadrant. Similarly it follows for the case $a > 0$ and $(4a^2 - 2\sigma_1 + 4\sigma_2)$ different from zero that $\pi = 3$. Analogously it follows then that (ARE) has exact 1 solution in the third quadrant and at least 1 solution in the first quadrant. Since in the right half (κ_1, κ_2) -plane the hyperbole (7) has the asymptote $\kappa_2 = a - \frac{1}{2}\kappa_1$ and the κ_2 -axis is a vertical asymptote of the hyperbole (8) it is easily verified that both curves intersect either once or thrice.

Finally, we have to consider the degenerated cases $a = 0$ and $4a^2 - 2\sigma_1 + 4\sigma_2 = 0$, respectively. An elementary detailed study of the corresponding polynomials obtained from (9) yields then the advertised result. $\square$

Theorem 2 does not completely specify how many solutions the (ARE) equations have in the first quadrant in case $a > 0$. Therefore, we analyze the function

$$f(x) := 3x^4 - 8ax^3 + (4a^2 - 2\sigma_1 + 4\sigma_2)x^2 - \sigma_1^2 \quad (11)$$

in more detail now.

Differentiation of f yields

$$f'(x) := 12x^3 - 24ax^2 + 2(4a^2 - 2\sigma_1 + 4\sigma_2)x.$$

So, the stationary points of f are

- 1) if $a^2 - 2\sigma_2 + \sigma_1 < 0$: $x_1 = 0$
- 2) if $a^2 - 2\sigma_2 + \sigma_1 \geq 0$: $x_1 = 0$; $x_2 = a + \frac{1}{3}\sqrt{3}\sqrt{a^2 - 2\sigma_2 + \sigma_1}$ and $x_3 = a - \frac{1}{3}\sqrt{3}\sqrt{a^2 - 2\sigma_2 + \sigma_1}$.

In case 1) it is clear that f can only have one positive zero.

In case 2) things are somewhat more complicated. To analyze this case, first consider the situation that $x_3 \leq 0$ or, equivalently, $2a^2 + 2\sigma_2 - \sigma_1 \leq 0$. Elementary analysis shows then that f has one positive zero again. Finally consider the case $x_3 > 0$. It is easily verified that in this situation f has exact one positive zero if and only if the product $f(x_2)f(x_3) > 0$. Elementary calculations show that $f(x_2)f(x_3)$ equals

$$\frac{16}{27} \{-a^8 + 6a^4(\sigma_2^2 - \sigma_1\sigma_2 + \sigma_1^2) + 3(\sigma_2^2 - \sigma_1\sigma_2 + \sigma_1^2)^2 + 4a^2(\sigma_1 - 2\sigma_2)(\sigma_1 + \sigma_2)(2\sigma_1 - \sigma_2)\}.$$

We like to mention here that this polynomial also showed up in the analysis of Weeren et al. in (1997). Combining these results with those of theorem 2 yields then

Theorem 3A:

(ARE) has exact 1 solution in the first quadrant if and only if the system parameters satisfy either one of the following conditions:

- 1) $a \leq 0$;
- 2) $a^2 - 2\sigma_2 + \sigma_1 \leq 0$;
- 3) $2a^2 + 2\sigma_2 - \sigma_1 \leq 0$;
- 4) $-a^8 + 6a^4(\sigma_2^2 - \sigma_1\sigma_2 + \sigma_1^2) + 3(\sigma_2^2 - \sigma_1\sigma_2 + \sigma_1^2)^2 + 4a^2(\sigma_1 - 2\sigma_2)(\sigma_1 + \sigma_2)(2\sigma_1 - \sigma_2) > 0$.

In all other cases, (ARE) has three solutions in the first quadrant. $\square$

A more detailed analysis shows that the above conditions 2) and 3) can be skipped. In fact the theorem can be reformulated as follows:

Theorem 3B:

(ARE) has exact 1 solution in the first quadrant if and only if the system parameters satisfy either one of the following two conditions:

- 1) $a \leq 0$;
- 2) $-a^8 + 6a^4(\sigma_2^2 - \sigma_1\sigma_2 + \sigma_1^2) + 3(\sigma_2^2 - \sigma_1\sigma_2 + \sigma_1^2)^2 + 4a^2(\sigma_1 - 2\sigma_2)(\sigma_1 + \sigma_2)(2\sigma_1 - \sigma_2) > 0$.

In all other cases, (ARE) has three solutions in the first quadrant. $\square$

Proof:

Assume that condition 2) of theorem 3A is satisfied. We show that then condition 4) holds. To that end we rewrite σ_2 as $\sigma_2 = \frac{\sigma_1 + a^2}{2} + \mu$ for some positive μ . Substitution of this expression into $-a^8 + 6a^4(\sigma_2^2 - \sigma_1\sigma_2 + \sigma_1^2) + 3(\sigma_2^2 - \sigma_1\sigma_2 + \sigma_1^2)^2 + 4a^2(\sigma_1 - 2\sigma_2)(\sigma_1 + \sigma_2)(2\sigma_1 - \sigma_2)$ (i), yields $27 * \{\frac{1}{4}(a^4 - \sigma_1^2) + a^2 * \mu\}^2 + \frac{9}{2}\mu^2(\sigma_1^2 - a^4) + 14a^2\mu^3 + 3\mu^4$. Now, if $a^4 - \sigma_1^2 \geq 0$, then $27 * \{\frac{1}{4}(a^4 - \sigma_1^2) + a^2 * \mu\}^2 - \frac{9}{2}\mu^2 a^4 \geq 27 * a^4 \mu^2 - \frac{9}{2}\mu^2 a^4 \geq 0$. So, (i) is positive. On the other hand, if $a^4 - \sigma_1^2 < 0$, it is obvious that (i) is positive too, which proves the claim.

In a similar way one can show that 4) holds in case 3) is satisfied by rewriting σ_1 as $\sigma_1 = 2a^2 + 2\sigma_2 + \mu$, for some positive μ . Substitution into (i) yields $135a^8 + (216 + 324\sigma_2)a^6 + (126\mu^2 + 378\mu\sigma_2 + 270\sigma_2^2)a^4 + (108\sigma_2^3 + 32\mu^3 + 216\mu\sigma_2^2 + 144\mu^2\sigma_2)a^2 + 18\mu^3\sigma_2 + 27\sigma_2^4 + 3\mu^4 + 45\mu^2\sigma_2^2 + 54\mu\sigma_2^3$, which obviously is positive. $\square$

To get an impression how the parameter surface satisfying condition 2) of theorem 3B (for $a > 0$) looks like, we first note that this condition can be rewritten as:

$$3((a^2 + \sigma_1 + \sigma_2)^2 - 3\sigma_1\sigma_2)^2 - 4a^2(a^2 + \sigma_1 + \sigma_2)^3 > 0.$$

By using the reparametrization $\sigma_i = a^2\tau_i$, it is easily verified that for a fixed a we can visualize this set in a two-dimensional plot from the area $3((1 + \tau_1 + \tau_2)^2 - 3\tau_1\tau_2)^2 - 4(1 + \tau_1 + \tau_2)^3 > 0$. This set is visualized in figure 2.

Figure 2: Graph of $3((1 + \tau_1 + \tau_2)^2 - 3\tau_1\tau_2)^2 - 4(1 + \tau_1 + \tau_2)^3 = 0$.

We conclude this section by stressing the fact

Corollary 4:

If the system parameters do not satisfy any of the conditions mentioned in theorem 3, the differential game has at least three feedback Nash solutions. $\square$

IV. Stability of solutions

A natural question that arises in case none of the conditions of theorem 3 is satisfied is whether it is possible to formulate some (natural) additional requirement on the feedback Nash solution that is satisfied by only one of the solutions. In Weeren et al. (1997) this issue was also raised, and it is shown that the additional requirement of dynamic stability of the solutions is not a sufficient criterion to eliminate this nonuniqueness. If there are three equilibria, see e.g. Figure 3, the smallest and largest equilibrium (in terms of κ_1) are stable equilibria whereas the remaining equilibrium is a saddle-point. An interesting point, that is not elaborated in the above mentioned paper but which is easily verified, is that in the symmetric case, $\sigma_1 = \sigma_2$, the origin $(0, 0)$ is located on a saddle-path. This suggests that, on the one hand, one can expect computational difficulties in finding the appropriate equilibrium in a finite planning horizon setting with no penalties on the final state. On the other hand, one can conclude that the equilibrium outcomes of the game will depend in that case crucially on the assumptions how to penalize outcomes of the final state of the game.

Next, we consider the closed-loop stability of the solutions as a criterion to eliminate nonuniqueness. Our first result is

Theorem 5:

Assume that $\sigma_1 \neq \sigma_2$.

Then the closed-loop "matrix" a_{cl} differs for all solutions.

Moreover, in case (ARE) has three different solutions (k_1, k_2) , (l_1, l_2) and (m_1, m_2) , with (without loss of generality) $k_1 < l_1 < m_1$, then the smallest closed-loop matrix is attained by either (k_1, k_2) or (m_1, m_2) .

Proof:

From theorem 1 we have that if (k_1, k_2) solve (ARE), the corresponding closed-loop matrix a_{cl} equals $a - s_1 k_1 - s_2 k_2$ or, equivalently, $a - \kappa_1 - \kappa_2$.

Using this we note that (5,6) can be rewritten as

$$2a_{cl}\kappa_1 + \sigma_1 + \kappa_1^2 = 0; \quad (12)$$

$$2a_{cl}\kappa_2 + \sigma_2 + \kappa_2^2 = 0. \quad (13)$$

Now assume, both the two different solutions (k_1, k_2) and (l_1, l_2) satisfy the above equation yielding the same closed-loop matrix a_{cl} . From (12) we then get:

$$2a_{cl}k_1 + \sigma_1 + k_1^2 = 0 \text{ and } 2a_{cl}l_1 + \sigma_1 + l_1^2 = 0.$$

Subtracting these two equations yields: $2a_{cl}(k_1 - l_1) + (k_1^2 - l_1^2) = 0$. Since $k_1 \neq l_1$ we conclude that $2a_{cl} = -(k_1 + l_1)$. Substituting this into (12) gives $-(k_1 + l_1)k_1 = -\sigma_1 - k_1^2$ or, equivalently, $k_1 l_1 = \sigma_1$. Similarly, one can show that under these assumptions also

the equality $k_2 l_2 = \sigma_2$ must hold.

By assumption $a_{cl} = a - k_1 - k_2 = a - l_1 - l_2$. So, $k_1 + k_2 = l_1 + l_2$. Substitution of the above derived equalities $k_i l_i = \sigma_i$ into this equation yields then

$$\frac{\sigma_1}{k_1} + \frac{\sigma_2}{k_2} = k_1 + k_2. \quad (14)$$

On the other hand we have from (12,13) that

$$-2a_{cl} = \frac{\sigma_1}{k_1} + k_1 = \frac{\sigma_2}{k_2} + k_2. \quad (15)$$

Addition of (14) and (15) yields $k_1 k_2 = \sigma_2$ (i), whereas subtraction of them gives rise to the equality $k_1 k_2 = \sigma_1$ (ii). Comparing (i) and (ii) we conclude that the equality $\sigma_1 = \sigma_2$ must hold, which violates our assumption. So, our assumption that both solutions yield the same closed-loop system must be wrong, which completes the first part of the proof. To prove the second part of the theorem, we first note that all three solutions lie on the hyperbole (7). Note that on the interval $[k_1, m_1]$ this hyperbole is a convex linesegment. It is easily seen that the maximum of the function $g(\kappa_1, \kappa_2)$ defined by $g(\kappa_1, \kappa_2) := \kappa_1 + \kappa_2$ on this convex linesegment is attained at either one of the endpoints of the segment. Since both closed-loop matrices do not coincide (as we proved above), this shows that the minimum value of $a_{cl} = a - \kappa_1 - \kappa_2$ is attained at either one of the points (k_1, k_2) or (m_1, m_2) . $\square$

Next, consider the symmetric case, i.e. $\sigma_1 = \sigma_2$. Obviously, when this condition holds then, whenever (k_1, k_2) satisfies (ARE) also (k_2, k_1) solves these equations (see e.g. (12,13)). In particular the symmetric solution (k, k) is obtained for $k = \frac{a + \sqrt{a^2 + 3\sigma}}{3}$.

With respect to the corresponding closed-loop systems we have the following

Theorem 6:

Let $\sigma_1 = \sigma_2$. Assume that (ARE) has three different solutions.

Then the closed-loop systems of both non-symmetric solutions coincide. Moreover, the closed-loop matrix a_{cl} of the non-symmetric solution is smaller than that of the symmetric solution.

Proof:

First we consider the closed-loop system of the non-symmetric solutions.

Subtracting (13) from (12) yields the equality $2a_{cl}(\kappa_1 - \kappa_2) = -(\kappa_1^2 - \kappa_2^2)$. Since we assume that $\kappa_1 \neq \kappa_2$, we obtain from this equality that $2a_{cl} = -(\kappa_1 + \kappa_2)$. Using the definition of $a_{cl} = a - \kappa_1 - \kappa_2$, we conclude that $2a = \kappa_1 + \kappa_2$. Substitution of this into a_{cl} shows that $a_{cl} = -a$ (note that $a > 0$ since we assumed that (ARE) has three different solutions!). Note that this result does not depend on the specific structure of the asymmetric solution. So, we conclude that the closed-loop matrix for both asymmetric solutions equals $-a$.

To prove the second statement of the theorem, we calculate the closed-loop matrix of the

system resulting from the symmetric solution (k, k) , which we will denote by $a_{cl}(k, k)$. Simple calculation shows that $a_{cl}(k, k) = \frac{1}{3}(a - 2\sqrt{a^2 + 3\sigma})$. Since by assumption (ARE) has three different solutions, it follows immediately from theorem 3 that $a^2 > \sigma$. Using this, elementary calculations show then that $a_{cl}(k_1, k_2) < a_{cl}(k, k)$. Which completes the proof. $\square$

From theorems 5 and 6 we conclude that closed-loop stability is also not a sufficient criterion to eliminate nonuniqueness completely. In the symmetric case ($\sigma_1 = \sigma_2$) things go wrong. On the other hand, we observe that always either at the smallest or largest equilibrium (in terms of κ_1) the most stable closed-loop system is achieved. Finally, we consider minimum of total cost incurred by both players in the equilibrium as a criterion to eliminate non-uniqueness. To that end we first note that since the cost incurred by a player is given by $J_i = \kappa_i x_0^2$, $i = 1, 2$. Since these equilibrium points lie on one hyperbola, it is clear that never one of these equilibria will be Pareto efficient (that is both players have lower cost in an equilibrium compared to those in another equilibrium). We first prove the next lemma

Lemma 7:

Both $(\bar{k}_1, \bar{k}_2)$ and $(\bar{k}_2, \bar{k}_1)$ are equilibria of the game if and only if $\sigma_1 = \sigma_2$.

Proof:

Assume that both points are equilibria of the game. Then from (5) we have $-2a\bar{k}_1 - \sigma_1 + \bar{k}_1^2 + 2\bar{k}_1\bar{k}_2 = 0$ and from (6) we have $-2a\bar{k}_1 - \sigma_2 + \bar{k}_1^2 + 2\bar{k}_1\bar{k}_2 = 0$. Subtracting both equations yields then $\sigma_1 = \sigma_2$. The other implication was already noted before. $\square$

Now, assume $\sigma_1 \neq \sigma_2$. Subtracting (5) from (6) yields

$$-2a(\kappa_1 - \kappa_2) - \sigma_1 + \sigma_2 + (\kappa_1 - \kappa_2)(\kappa_1 + \kappa_2) = 0.$$

or, stated differently,

$$\kappa_1 + \kappa_2 = \frac{\sigma_1 - \sigma_2}{\kappa_1 - \kappa_2} + 2a. \quad (16)$$

If both (k_1, k_2) and $(\bar{k}_1, \bar{k}_2)$ are two different equilibria yielding the same total cost, i.e. $(k_1 + k_2)x_0^2 = (\bar{k}_1 + \bar{k}_2)x_0^2$ (i), it follows then straightforwardly from (16) that also $k_1 - k_2 = \bar{k}_1 - \bar{k}_2$ (ii) should hold. Obviously (i) and (ii) together imply that $k_1 = \bar{k}_1$ and $k_2 = \bar{k}_2$, which contradicts the assumption that both equilibria differ. In case $\sigma_1 = \sigma_2$, it is clear from the proof of theorem 6 that total cost in the smallest and largest equilibrium point are the same, i.e. $2ax_0^2$. Furthermore elementary calculation shows that in the symmetric equilibrium (k, k) total cost are $\frac{2}{3}(a + \sqrt{a^2 + 3\sigma})x_0^2$. Since $\sigma < a^2$ (see theorem 3), total cost are in this equilibrium point always smaller than $2ax_0^2$. This yields

Theorem 8:

If $\sigma_1 \neq \sigma_2$ then total cost differ in all equilibria.

If $\sigma_1 = \sigma_2$ then total cost are minimized in the symmetric equilibrium point. $\square$

So we conclude that the additional requirement, that amongst all equilibria we look for an equilibrium which minimizes total cost, always gives rise to a unique equilibrium.

IV. Numerical algorithms

In the previous section we saw that the solution that stabilizes the system most, is either the smallest or largest solution (in terms of κ_1) in the first quadrant of the (κ_1, κ_2) -plane. We will now briefly discuss the question how both these solutions can be calculated numerically.

One way is to calculate the roots of the fourth degree polynomial (9) directly. Note that there exist analytic formula to do this. So, this method seems to suit best in this case. Another way, which has the advantage that it can be straightforwardly generalized to the multidimensional case, is presented in the following algorithm

Algorithm 9A:

- Step 1 Let $i := 0$. Take $\kappa_1^{(0)} = 0$, and let $\kappa_2^{(0)}$ be the unique positive definite solution of the corresponding algebraic Riccati equation (6): $2a\kappa_2^{(0)} + \sigma_2 - \kappa_2^{(0)^2} = 0$.
- Step 2 Substitute $\kappa_2^{(i)}$ into (5). Denote the unique positive definite solution of the corresponding Riccati equation by $\kappa_1^{(i+1)}$: $2(a - \kappa_2^{(i)})\kappa_1^{(i+1)} + \sigma_1 - \kappa_1^{(i+1)^2} = 0$.
- Step 3 Substitute $\kappa_1^{(i+1)}$ into (6). Denote the unique positive definite solution of the corresponding Riccati equation by $\kappa_2^{(i+1)}$: $2(a - \kappa_1^{(i+1)})\kappa_2^{(i+1)} + \sigma_2 - \kappa_2^{(i+1)^2} = 0$.
- Step 4 Update i and return to step 2 of the algorithm until the process has converged.

$\square$

In figure 3, we sketched the figure obtained by enlarging the first quadrant of figure 1.

To illustrate how algorithm 9A works, we copied in figure 4 the left part of this figure and indicated how the successive iterations in the algorithm are geometrically obtained. Figure 5 illustrates for the same example algorithm 9B for finding the largest solution of the (ARE) equations. This figure is obtained from figure 3 by copying the right part of it.

Figure 3: The first quadrant of fig.1

Figure 4: Algorithm for finding the smallest solution cf. fig.1,3

Figure 5: Algorithm for finding the largest solution cf. fig.1,3

Algorithm 9B:

- Step 1 Let $i := 0$. Take $\kappa_2^{(0)} = 0$, and let $\kappa_1^{(0)}$ be the unique positive definite solution of the corresponding algebraic Riccati equation (5): $2a\kappa_1^{(0)} + \sigma_1 - \kappa_1^{(0)2} = 0$.
- Step 2 Substitute $\kappa_1^{(i)}$ into (6). Denote the unique positive definite solution of the corresponding Riccati equation by $\kappa_2^{(i+1)}$: $2(a - \kappa_1^{(i)})\kappa_2^{(i+1)} + \sigma_2 - \kappa_2^{(i+1)2} = 0$.
- Step 3 Substitute $\kappa_2^{(i+1)}$ into (5). Denote the unique positive definite solution of the corresponding Riccati equation by $\kappa_1^{(i+1)}$: $2(a - \kappa_2^{(i+1)})\kappa_1^{(i+1)} + \sigma_1 - \kappa_1^{(i+1)2} = 0$.
- Step 4 Update i and return to step 2 of the algorithm until the process has converged.

□

It is now not difficult to show the following theorem (see also figure 4 for an illustration of the basic idea of the algorithm)

Theorem 10:

In algorithm 9A, $\kappa_1^{(i)}$ is monotonically increasing and $\kappa_2^{(i)}$ is monotonically decreasing to the smallest solution of (ARE).

In algorithm 9B, $\kappa_1^{(i)}$ is monotonically decreasing and $\kappa_2^{(i)}$ is monotonically increasing to the largest solution of (ARE).

Proof:

The proof can be given by induction. We will only give an outline for the monotonicity of the $\kappa_1^{(i)}$ sequence.

Obviously, $\kappa_1^{(1)} - \kappa_1^{(0)} > 0$. So, assume the inequality holds for $i = N$. Then,

$$\begin{aligned}\kappa_1^{(N+1)} - \kappa_1^{(N)} &= a - \kappa_2^{(N)} - \sqrt{(\kappa_2^{(N)} - a)^2 + \sigma_1} - (a - \kappa_2^{(N-1)} - \sqrt{(\kappa_2^{(N-1)} - a)^2 + \sigma_1}) \\ &= \kappa_2^{(N-1)} - \kappa_2^{(N)} + \sqrt{(\kappa_2^{(N-1)} - a)^2 + \sigma_1} - \sqrt{(\kappa_2^{(N)} - a)^2 + \sigma_1} \\ &> 0,\end{aligned}$$

since by induction $\kappa_2^{(N-1)} - \kappa_2^{(N)} > 0$ and $\kappa_2^{(i)} > a$, $\forall i$. That $\kappa_1^{(i)}$ converges to the smallest solution of the (ARE) is easily verified too. $\square$

Remark 11:

The generalization of algorithm 9 to the multivariable case is straightforward. In fact, the corresponding multivariable algorithm, without specification of the initial values, was also proposed by Freiling et al. in (1996) to find a solution of the (ARE) equations. However, in general, the monotonic convergence properties mentioned in theorem 10 do not hold. $\square$

V. Concluding remarks

In this paper we studied feedback Nash equilibria in the two-player linear quadratic scalar differential game. We showed that the corresponding set of algebraic Riccati equations has either one or three different solutions in the first quadrant and we gave necessary and sufficient conditions on the system parameters for existence of a unique solution of these equations.

In order to be able to discriminate between different equilibria we considered three different additional requirements: dynamic stability, stability of the closed-loop system and minimal total cost for the players.

We noted that always the smallest and largest solution (measured w.r.t. κ_1) are dynamically stable and the other equilibrium is a saddle point. Furthermore we pointed out that if $\sigma_1 = \sigma_2$ the origin is situated on a saddle path. With respect to closed-loop stabilization we saw that either the largest or smallest solution will stabilize the system most. A numerical algorithm was devised to calculate both these solutions. In case $\sigma_1 = \sigma_2$ closed-stability at the largest and smallest equilibrium point is the same. So in that case, using this criterion, we still have non-uniqueness. If one uses the additional requirement to choose that equilibrium for which the sum of the cost of both players is minimal we showed that always a unique equilibrium results. In particular we saw that if $\sigma_1 = \sigma_2$ the minimal cost are attained in the symmetric equilibrium. However, as we already noted, this equilibrium point is dynamically unstable and its closed-loop stability is worse than in the other equilibria. So, definite answers on which equilibrium

one should choose seem to be difficult in that situation.

We hope that the obtained results may be helpful in analyzing the more general multi-player and multi-dimensional case.

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