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**SCRAP VALUE FUNCTIONS IN DYNAMIC DECISION
PROBLEMS**

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Scrap value functions in dynamic decision problems*

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Short title: Scrap value functions

Abstract: We introduce an accurate, easily implementable, and fast algorithm to compute optimal decisions in discrete-time long-horizon welfare-maximizing problems. The algorithm is useful when interest is only in the decisions up to period T , where T is small. It relies on a flexible parametrization of the relationship between state variables and optimal total time-discounted welfare through scrap value functions. We demonstrate that this relationship depends on the boundedness, half-boundedness, or unboundedness of the utility function, and on whether a state variable increases or decreases welfare. We propose functional forms for this relationship for large classes of utility functions and explain how to identify the parameters.

JEL Classification: C61, C63.

Keywords: Scrap value function, Dynamic optimization, Computation, Short horizon.

1 Introduction

Let us consider a Ramsey-type growth model, presented as a dynamic optimization problem. We denote by U a non-decreasing, concave utility function, and we define welfare W as

$$W = \sum_{t=0}^{\infty} \beta^t U(x_t), \quad (1)$$

where β is the time-discount factor and $x = \{x_t\}_{t=0}^{\infty}$ is a sequence of (univariate) control variables. A typical decision maker will maximize W subject to restrictions involving state and control variables. The latter include not only x but also other (possibly multivariate) control variables $y = \{y_t\}_{t=0}^{\infty}$. Optimal welfare subject to the restrictions is obtained at (x^*, y^*) .

In practice, with the exception of some simple specific cases, it is computationally not feasible to solve the infinite-horizon problem. One typically solves a finite-horizon problem, say over T periods. When T is large, for example 60 periods, then the long horizon may well approximate the infinite horizon, but when T is small, for example 2 periods, then the short horizon will not provide a good approximation.

Our interest is in the short-horizon problem, for two reasons. The long-horizon problem may also be unsolvable or may computationally not be feasible, especially in a stochastic context. But even if it were feasible, decision makers typically have a short horizon, if only because their term of office is short.

Supposing then that the decision maker has a finite T -period horizon, we write welfare (1) as

$$W = \sum_{t=0}^{T-1} \beta^t U(x_t) + \beta^T S_T, \quad S_T = \sum_{t=0}^{\infty} \beta^t U(x_{T+t}).$$

For every $T \geq 0$ we now define the *scrap value* as

$$S_T^* = \sum_{t=0}^{\infty} \beta^t U(x_{T+t}^*), \quad (2)$$

which depends on the optimal path $(x_T^*, x_{T+1}^*, \dots)$, but not on y^* or on $(x_0^*, \dots, x_{T-1}^*)$. If we would know the optimal path $\{x_t^*\}$ for $t \geq T$, then

maximizing W over all x and y can be achieved by maximizing

$$\overline{W} = \sum_{t=0}^{T-1} \beta^t U(x_t) + \beta^T S_T^* \quad (3)$$

over $x_0, \dots, x_{T-1}$ and $y_0, \dots, y_{T-1}$. This simple observation is our starting point. The scrap value S_T^* depends on the state variables at time T , and we shall call this functional relationship the *scrap value function*. In fact, this is precisely the *value function* in dynamic programming, and the recursion

$$S_0^* = \sum_{t=0}^{T-1} \beta^t U(x_t^*) + \beta^T S_T^*$$

is known as the Bellman equation.

Only in very special cases do we know the functional form of the scrap value S_T^* as a function of the state variables. Hence we need to approximate it. The purpose of this paper is to explore the relationship between the form of the utility function and the form of the scrap value function, to propose simple flexible forms for the scrap value functions, and to discuss how the parameters in these flexible forms can be estimated or calibrated. Throughout we shall distinguish between state variables that increase welfare (G for ‘good’), say capital, and state variables that decrease welfare (B for ‘bad’), say pollution. We shall see that the specification of the scrap value function depends on whether a state variable is good or bad in a nontrivial manner.

This paper can be viewed in two ways. One can see it as providing a solution to a *long-horizon* (stochastic) dynamic decision problem by considering a short horizon and treating the remainder (the scrap value function) appropriately. But one can also see it as providing a solution to a *short-horizon* problem where the decision maker has two objectives: to maximize welfare over a short horizon and to leave a ‘reasonable’ state for the next decision maker. The two views are conceptually different but mathematically equivalent.

There is a substantial literature on the theory of approximate dynamic programming. Without attempting to review this literature, we mention that projection methods, described in Judd (1992) and reviewed in Christiano and Fisher (2000), have become standard tools for solving dynamic models in economics. More recently, Lau et al. (2002) proposed a method to improve the approximation to the infinite-horizon problem through a complementarity formulation. Our paper avoids the sequential optimization that is required for their method (p. 586, footnote 5). Finally, Dorofeenko et al.

(2010) introduced an efficient algorithm for solving stochastic dynamic models. However, their algorithm assumes proximity to the steady state, which is not reasonable for small T .

The plan of the remainder of this paper is as follows. In Section 2 we explore the relationship between utility function and scrap value function in more detail. In Sections 3 and 4 we discuss our solution method, given the choice of scrap value function, first in a deterministic framework and then in a stochastic framework. Different utility functions require different specifications of the scrap value functions. In Sections 5–8, we discuss unbounded, partially bounded, and bounded utility functions in detail. Section 9 concludes.

2 Scrap value and utility functions

Since we do not know the scrap value S_T^* in (3), we need to approximate it. Some authors take T to be large but finite, and add a terminal condition to the model. For example, the 60-period (600 years) DICE model in Nordhaus (2008) includes a terminal condition, namely that at least 2% of the capital stock at the beginning of period T should be invested annually during period T ; see also Doroodian and Boyd (2003, Section 3.6) and Leach (2009, Section 3). In fact, this terminal condition is equivalent to a linear scrap value function for capital (Eyckmans and Tulkens, 2003).

When T is large then $\beta^T S_T^* \approx 0$, and a poor approximation to the scrap value may not have large effects on the values of the optimal controls. But when T is small, we are forced to look for good approximations, thus emphasizing the fact that the decision maker has the double objective of maximizing time-discounted welfare over a finite number of periods T , while also leaving a reasonable economy for the next decision maker, based on the remaining stocks of something good ($G > 0$), say capital, and something bad ($B > 0$), say pollution.

With these two state variables we write the scrap value S_T^* as

$$S_T^* = S^*(G_T, B_T).$$

The simplest specification for the scrap value function S^* is the linear function

$$S_T^* = \nu_0 + \nu_1 G_T - \nu_2 B_T \quad (\nu_1 > 0, \nu_2 > 0), \quad (4)$$

where the parameters ν_1 and ν_2 denote the scrap prices of capital and pollution at the beginning of period T . This scrap value function satisfies the minimum requirement that the decision maker will be happier if there is more

capital and less pollution at the end of the period. But the function has two problems. First, it is more common and more realistic to assume that the scrap value function is concave rather than linear. Second, the linear scrap value function is unbounded, both towards $+\infty$ and $-\infty$, although the utility function on which welfare is based may be bounded or half-bounded, and this mathematical property should carry over to the scrap value function.

The main purpose of this paper is to propose scrap value functions that are appropriate for different types of utility functions. Throughout we shall retain separability, so that we can write

$$S_T^* = \nu_0 + \nu_1 S_g^*(G_T) - \nu_2 S_b^*(B_T) \quad (\nu_1 > 0, \nu_2 > 0). \quad (5)$$

Our task then is to specify the functions S_g^* and S_b^* . Without loss of generality we normalize the scrap value functions such that

$$\left. \frac{\partial S_g^*(G)}{\partial G} \right|_{G=G_0} = \left. \frac{\partial S_b^*(B)}{\partial B} \right|_{B=B_0} = 1, \quad (6)$$

which guarantees that if we linearize $S^*(G_T, B_T)$ around (G_0, B_0) , we find

$$S^*(G_T, B_T) \approx \text{constant} + \nu_1 G_T - \nu_2 B_T,$$

so that ν_1 and ν_2 can be interpreted as scrap prices, just as in the linear case.

Supported by findings in Wirl (1991), Eyckmans and Tulkens (2003), and Krawczyk (2005), we shall choose S_g^* to be increasing and concave, so that the more capital is left, the better, but at a decreasing rate.

More difficult is the choice of S_b^* . When utility is unbounded from below (as in Sections 5 and 6) we shall choose S_b^* to be increasing and convex, so that the more pollution is left, the worse, and at an increasing rate. This is supported by Montgomery (1972) who argued formally in favor of a monotonic and convex abatement cost function; see also Hoel and Karp (2002), Feng and Zhao (2006), and de Zeeuw (2008). But when utility is bounded from below (as in Sections 7 and 8), then S_b^* must be bounded from above, and this causes a problem because there are no increasing convex functions that are bounded from above. In that case we shall assume that S_b^* is convex-concave, that is, convex for low values of B_T and concave for higher values. A rigorous analysis of the optimal management of a convex-concave resource was provided by Skiba (1978). Several of the papers in Dasgupta and Mäler (2003) contain economic analyses of ecosystems whose natural regeneration functions are convex-concave. The introduction of a convex-concave scrap value function is closely related to the idea of a threshold; see for example Ranjan and Shortle (2007) and Leandri (2009).

The specification of the scrap value functions S_g^* and S_b^* depends on the utility function. A utility function can be:

- unbounded (for example, $U(x) = \log x$);
- bounded from above but not from below ($U(x) = 1 - 1/x$);
- bounded from below but not from above ($U(x) = \sqrt{x}$, $U(x) = \log(x + 1)$); or
- bounded ($U(x) = 1 - e^{-x}$, $U(x) = x/(x + 1)$).

Two of these four types are unbounded from above, in which case infinite welfare can occur. This can happen, in principle, when $x \rightarrow \infty$ or $G \rightarrow \infty$, but also when $B \rightarrow 0$. Although we can understand that infinite pollution gives infinite misery, it is less credible that no pollution gives infinite welfare. Hence we make the following assumption throughout.

ASSUMPTION 1: *The scrap value function S_b^* is bounded from below.*

One of the features of this paper is that we distinguish between two types of scrap value function, associated with a good and a bad stock, while most of the literature only allows one type (good). We shall see that the modeling of the scrap value function associated with pollution (the ‘bad’) is more complex than for capital (the ‘good’).

3 Deterministic framework

Although the introduction and treatment of scrap value functions is particularly important in a stochastic framework, it will be useful to consider the deterministic framework first. A well-known example of such a framework is Nordhaus’ (2008) economic model of climate change (the DICE model). This model has three state variables: capital, CO2 concentration, and temperature; and two controls: per capita consumption (x) and the abatement fraction for CO2 (y). More capital is good, more CO2 is bad, and a higher temperature is also bad (at a global level). Nordhaus considers 60 periods (600 years), and he imposes a terminal condition in order to obtain the optimal paths for the two control variables.

If we are only interested in short-horizon decisions (say 2 periods, 20 years), we have two options. We can still calculate, from (1), the optimal paths over 60 periods, and then only consider the optimal paths over the first 2 periods. Alternatively, we can set $T = 2$ and introduce scrap value functions, one for capital (good) and one for CO2 concentration (bad). (In this model it appears to be unnecessary to have a scrap value for temperature.) The

assumed utility function is $U(x) = 1 - 1/x$, and the functional forms of the two scrap value functions $S_g^*(G_T)$ and $S_b^*(B_T)$ depend on the functional form of the utility function, as will be made more precise in Section 6. Given the functional form of $S_g^*(G_T)$ and $S_b^*(B_T)$, the total scrap value function is then given by (5):

$$S^*(G_2, B_2) = \nu_0 + \nu_1 S_g^*(G_2) - \nu_2 S_b^*(B_2) \quad (\nu_1 > 0, \nu_2 > 0).$$

We need to estimate (calibrate) ν_1 , ν_2 , and the parameters in S_g^* and S_b^* . This is done by solving the model for different values of (G_0, B_0) . Each set of initial state values will generate solutions $(G_2, B_2, S^*(G_2, B_2))$, and from these generated data we estimate the scrap value parameters by some standard line fitting method such as nonlinear least squares. Once we have estimated the parameters in the scrap value function S^* , we obtain the optimal solutions for the first two periods from (3) instead of (1).

In a deterministic framework there is no computational advantage in using scrap value functions instead of optimizing over all (in this case 60) periods. However, the formulation using scrap values highlights the essence of the short-horizon decision problem by introducing the policy period of the current decision maker and the desired state at the end of this policy period. In an analysis of current versus future decisions, this allows us to investigate the sensitivity of current optimal controls with respect to small changes in the scrap value parameters.

4 Stochastic framework

The main advantage of using scrap value functions becomes apparent in a stochastic framework, for example a stochastic version of the DICE model, as proposed by Ikefuji et al. (2010a) in the context of catastrophic risk in economy-climate models. Suppose again that $T = 2$ and that there is only one random shock, ϵ , with some known cumulative distribution function F . The decision maker has two decisions to make, one at the beginning of period 0 and one at the beginning of period 1. The shock is observed at the end of period 0, and its observed value is taken into account for the second decision. But for the first decision the decision maker does not know ϵ , and will therefore take into account its distribution but not its realization. Welfare is a function of the chosen sequence of controls, $x = \{x_t\}_{t=0}^\infty$, the initial values of the state variables (G_0, B_0) , and the shock ϵ . Thus, conditional on (G_0, B_0) , $W = W(x, \epsilon)$. At time 0 all that is known about ϵ is its distribution, and

hence the decision maker maximizes expected welfare,

$$E(W) = \int W(x, \epsilon) dF(\epsilon).$$

subject to the restrictions involving state and control variables.

Without introducing scrap value functions, the long-horizon problem may not be solvable or it may be computationally too burdensome. But with scrap value functions we can maximize expected welfare in four steps as follows. First, determine the scrap value function $S^*(G_2, B_2)$ (both its structural form and the values of its parameters) by ignoring stochasticity, using the perturbation method described in the previous section. Then, write welfare as

$$\overline{W}(x_0, x_1, \epsilon) = U(x_0, \epsilon) + \beta U(x_1, \epsilon) + \beta^2 S^*(G_2, B_2),$$

and maximize $\overline{W}$ with respect to x_1 conditional on (x_0, ϵ) . This gives x_1^* and concentrated welfare

$$\overline{W}_c(x_0, \epsilon) = \overline{W}(x_0, x_1^*, \epsilon).$$

Next, compute the expectation $E(\overline{W}_c(x_0, \epsilon))$, if it exists. Finally, maximize this expectation with respect to x_0 using a standard grid-based approach.

The big computational advantage lies in the fact that for every iteration of the maximization and for each grid cell, we now only need to solve a short-horizon problem. Since the number of perturbations required to estimate the scrap value function is very much smaller than the number of grid cells required for the maximization, the computational burden of our problem is much reduced, and this reduction increases as T decreases.

5 Unbounded utility

The infinite-horizon problem with a given utility function implies the functional form of the scrap value function in the finite-horizon problem. But, in general, we can not derive this functional form. There are, however, certain properties of the utility function that carry over to the scrap value function, namely whether the function is bounded, half-bounded, or unbounded. By employing this simple fact we obtain much-improved approximations to the scrap value functions. In this and the next three sections we propose flexible functional forms for scrap value functions (both for ‘good’ and for ‘bad’ state variables) for each of four types of utility functions.

In the class of unbounded utility functions, the function $U(x) = \log(x)$ is probably the best-known example. It is unbounded from below and from

above, and the corresponding scrap value functions should therefore also be unbounded. Writing the scrap value function as

$$S_T^* = \nu_0 + \nu_1 S_g^*(G_T) - \nu_2 S_b^*(B_T),$$

the simplest option is the linear scrap value function (4) where $S_g^*(G_T) = G_T$ and $S_b^*(B_T) = B_T$. Although the linear scrap value function seems to be unbounded, in fact it is not, because $G_T > 0$ and $B_T > 0$, so that both S_g^* and S_b^* are only half-bounded.

A suitable alternative would be a scrap value function in the power class, namely

$$S_T^* = \nu_0 + \zeta_1 \log(G_T) - \zeta_2 B_T^q \quad (\zeta_1 > 0, \zeta_2 > 0, q \geq 1).$$

Choosing ζ_1 and ζ_2 according to the normalization (6), we obtain

$$S_g^*(G_T) = G_0 \log(G_T), \quad S_b^*(B_T) = \frac{B_0}{q} \left(\frac{B_T}{B_0} \right)^q, \quad (7)$$

where $q \geq 1$. Notice that S_g^* is increasing and strictly concave on $(0, \infty)$, and that S_b^* is increasing and convex on $(0, \infty)$ in accordance to the arguments in Section 2 and Assumption 1.

Figure 1: Calibrated scrap value functions: unbounded utility.

As an example, we graph the scrap value functions (7) in Figure 1 for a typical case. We see that S_g^* is concave and unbounded (left panel), and that S_b^* is convex, bounded from below, and unbounded from above (right panel). For S_g^* we need to estimate only ν_1 ; for S_b^* we need to estimate both ν_2 and q . Estimating these parameters as outlined in Sections 3 and 4, we obtain the two curves in Figure 1, where the circles represent the data generated from the infinite-horizon model.

6 Utility bounded from above but not from below

Consider the power utility function

$$U(x) = \frac{x^{1-\alpha} - 1}{1-\alpha} \quad (\alpha > 0). \quad (8)$$

In the previous section we considered the case $\alpha = 1$, that is, $U(x) = \log x$. Let us now consider the class $\alpha > 1$. Such utility functions are often used, in particular the case $\alpha = 2$ where $U(x) = 1 - 1/x$. Welfare is then bounded from above but not from below. The linear scrap value function does not share this feature. Hence we consider the nonlinear scrap value function (5), and choose the scrap value functions within the family of power functions. This gives

$$S_g^*(G_T) = -\zeta_1 G_T^{-p}, \quad S_b^*(B_T) = \zeta_2 B_T^q$$

with $p > 0$, $q \geq 1$, $\zeta_1 > 0$, and $\zeta_2 > 0$. Then, S_g^* is increasing and strictly concave on $(0, \infty)$, and S_b^* is increasing and convex on $(0, \infty)$. Inserting these two functions in (5) we obtain

$$S_g^*(G_T) = -\frac{G_0}{p} \left(\frac{G_T}{G_0} \right)^{-p}, \quad S_b^*(B_T) = \frac{B_0}{q} \left(\frac{B_T}{B_0} \right)^q, \quad (9)$$

where $p > 0$ and $q \geq 1$, and we have normalized ν_1 and ν_2 as before, so that they can be interpreted as scrap prices.

Let us also consider an alternative specification for S_b^* , namely

$$S_b^*(B_T) = \frac{\gamma_0 B_T^{q+r}}{1 + B_T^q} \quad (q \geq 1, 0 < r < 1), \quad (10)$$

where

$$\gamma_0 = \frac{(1 + B_0^q)^2}{B_0^{q+r-1}(rB_0^q + q + r)}.$$

For B_T close to zero the function S_b^* behaves like B^{q+r} (convex) and for B_T close to ∞ it behaves like B^r (concave), and there is precisely one point where S_b^* turns from convex to concave. Hence, S_b^* is now increasing and convex-concave on $(0, \infty)$.

The scrap value functions (9) are graphed in Figure 2. We see that S_g^* is concave, bounded from above, but unbounded from below (left panel), and that S_b^* is convex, bounded from below, but unbounded from above (right panel). We need to estimate the parameters (ν_1, p) for S_g^* , and (ν_2, q) for S_b^* . In this case it is clear from the data that the convex specification (9) for S_b^* fits the data better than the alternative convex-concave specification (10), but this depends of course on the case under investigation.

Figure 2: Calibrated scrap value functions: utility bounded from above.

7 Utility bounded from below but not from above

This class of utility functions, which includes $U(x) = x^r$ for $0 < r < 1$ and $U(x) = \log(x + \lambda)$ for $\lambda > 0$, is not used as often as the other three classes. It is not difficult to find a suitable scrap value function for G_T , namely

$$S_g^*(G_T) = \frac{G_0^{1-r} G_T^r}{r} \quad (0 < r < 1). \quad (11)$$

The function S_g^* is increasing and strictly concave on $(0, \infty)$.

It is more difficult to find a suitable scrap value function for B_T . As discussed in Section 2, when utility is bounded from below, as is the case in this and the next section, then S_b^* must be bounded from above, and this causes a problem because there are no increasing convex functions that are bounded from above. By Assumption 1, S_b^* is also bounded from below. Hence we are looking for a function S_b^* that is increasing, convex-concave, and bounded, in other words, we are looking for a general class of distribution functions. The Burr cumulative distribution function (Burr, 1942; Burr and Cislak, 1968), defined for $z > 0$ as

$$F(z) = 1 - (1 + (z/\lambda)^c)^{-p} \quad (\lambda > 0, p > 0, c > 0),$$

is a three-parameter family of distribution functions with the property that many of the known distribution functions are special or limiting cases. It is therefore an appropriate function to approximate an unknown distribution

function. The function F is increasing between $F(0) = 0$ and $F(\infty) = 1$, and

$$F''(z) < 0 \iff \left(\frac{z}{\lambda}\right)^c > \frac{c-1}{cp+1}.$$

Hence, F is concave when $0 < c \leq 1$ and convex-concave when $c > 1$. We therefore specify

$$S_b^*(B_T) = \gamma_1 \left[1 - (1 + (B_T/\lambda)^c)^{-p}\right] \quad (\lambda > 0, p > 0, c > 1), \quad (12)$$

where

$$\gamma_1 = \frac{\lambda}{cp} \cdot \frac{(1 + (B_0/\lambda)^c)^{p+1}}{(B_0/\lambda)^{c-1}}.$$

The function S_b^* is then increasing and convex-concave on $(0, \infty)$.

Figure 3: Calibrated scrap value functions: utility bounded from below.

The scrap value functions (11) and (12) are graphed in Figure 3. We see that S_g^* is concave, bounded from below, but not from above (left panel), and that S_b^* is convex-concave and bounded (right panel). For S_g^* we need to estimate (ν_1, r) ; for S_b^* we need to estimate (ν_2, λ, p, c) .

8 Bounded utility

If we assume an exponential utility function

$$U(x) = 1 - e^{-\beta x} \quad (\beta > 0) \quad (13)$$

or the ‘Burr’ utility function (Ikefuji et al., 2010b)

$$U(x) = 1 - \left(\frac{\lambda}{x + \lambda}\right)^k \quad (\lambda > 0, k > 0), \quad (14)$$

then welfare is bounded from above and from below. Hence we require S_g^* to be increasing, concave, and bounded; and S_b^* to be increasing, convex-concave, and bounded. Using again the flexibility of the Burr cumulative distribution function, we obtain

$$S_g^*(G_T) = \gamma_2 [1 - (1 + G_T/\lambda_1)^{-p}] \quad (\lambda_1 > 0, p > 0) \quad (15)$$

and

$$S_b^*(B_T) = \gamma_3 [1 - (1 + (B_T/\lambda_2)^c)^{-q}] \quad (\lambda_2 > 0, q > 0, c > 1), \quad (16)$$

where

$$\gamma_2 = \frac{\lambda_1}{p} (1 + G_0/\lambda_1)^{p+1}, \quad \gamma_3 = \frac{\lambda_2}{cq} \cdot \frac{(1 + (B_0/\lambda_2)^c)^{q+1}}{(B_0/\lambda_2)^{c-1}}.$$

We see that S_g^* is increasing and concave on $(0, \infty)$, and that S_b^* is increasing and convex-concave on $(0, \infty)$.

Figure 4: Calibrated scrap value functions: bounded utility.

The scrap value functions (15) and (16) are graphed in Figure 4. We see that S_g^* is concave and bounded (left panel), and that S_b^* is convex-concave and bounded (right panel). For S_g^* we need to estimate (ν_1, λ_1, p) ; for S_b^* we need to estimate (ν_2, λ_2, q, c) .

9 Conclusion

In infinite-horizon dynamic stochastic models, such as discrete long-horizon welfare-maximizing problems, it is typically not possible to determine the

optimal policy analytically, and numerical solutions may be computationally expensive. In this paper, we present a method which greatly reduces this computational effort.

Our method employs the idea of a scrap value function, which we estimate based on the deterministic version of the model. The form of the scrap value function depends on the form of the utility function, and this is explicitly taken into account by considering four types of utility function. A finer distinction does not appear to be useful. We also distinguish between state variables that are ‘good’ (like capital) and ‘bad’ (like pollution). In our analysis we have assumed a single random shock, but our method is easily generalized to multiple random shocks, as long as they appear before period T .

In our experience the estimated functions typically fit the data very well over the entire support, especially when the utility function belongs to the HARA class (as all our examples do), that is the class of utility functions with linear absolute risk tolerance $T(x) = -U'x/U''(x)$.

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