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Heuts, R.M.J.; de Klein, J.

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**AN (s, q) INVENTORY MODEL
WITH STOCHASTIC AND
INTERRELATED LEAD TIMES**

by Ruud Heuts and
Jan de Klein

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An (s, q) Inventory Model with Stochastic and Interrelated Lead Times

Ruud Heuts Jan de Klein

August 17, 1993

Abstract

In this paper we introduce a lead time mechanism that allows simultaneous order arrivals. This mechanism ensures that orders never cross each other. For the continuous review (s, q) inventory system with constant demand, we show that the allowance of batch arrivals gives rise to a discontinuous cost function. By exploiting the special structure of this cost function, a search algorithm is derived that yields the optimal order strategy in a reasonable amount of time. The search is restricted to integer order strategies only. We also consider two approximate methods of solution. The first approximation is based on a related model with a continuous cost function. The results obtained by this approximation are, in general, very satisfactorily. The second approximation is based on the economic order quantity adapted to allow backlogging. This approximation, in general, leads to very poor results, especially if the lead time variation is large. The main drawback of the EOQ approximation is its inflexibility in dealing with inventory situations in which the lead times are unreliable.

1 Introduction

We consider the continuous review (s, q) inventory system with constant demand and stochastic lead time. To exclude the possibility of crossing of orders, we introduce a lead time mechanism in which batch arrivals are allowed. That is, if a stackup of orders occurs at the supplier, all these orders will be delivered in the same batch. For a detailed description of the model see Section 2.

The motivation for introducing the present model is as follows. Consider an (s, q) inventory system in which the lead times are unreliable, and suppose that a request for a replenishment is made during the lead time of an order placed previously. If the successive lead times of delivery are assumed to be independent, then the new order may arrive before the preceding one. In a real business situation, however, crossing of orders rarely occurs. The first approach to deal with this difficulty is to ignore the possibility that orders can cross. It is evident that such ignorance may lead to an underestimation of the stockout cost, and hence to a considerable financial loss, especially if the lead time variability is large. The second approach to get around the difficulty of crossing of orders is to require that the order quantity is sufficiently large, so that no additional replenishments are needed during an arbitrary lead time. This approach has for instance been taken by Kulcsár [4]. (We also refer to Spiccas and Nasri [6], and Friedman [2], who derive for the model with constant demand and stochastic lead time a stochastic extension of the classic EOQ formula. However, the conditions they impose on the system parameters are very restrictive. For instance, they require, in analogy with Kulcsár, that the order quantity is sufficiently large to exclude crossing of orders.) Such a large order

quantity, however, may result in high average inventory and backlogging levels, and hence also in high inventory and backlogging costs. This is especially the case, if the lead time variability is large. In consequence, the second approach may also lead to a high financial loss, and is, therefore, not very useful for inventory systems in practice. The third approach to get around the problem of crossing of orders is to drop the assumption of independence of the successive lead times, and to construct a lead time process in which orders cannot cross each other. The lead time process involving simultaneous order arrivals, introduced in the present study, has the important property that crossing of orders is impossible. Although the lead time process being considered, in essence, is very simple, it adequately describes many inventory systems that occur in practice. A frequently occurring situation that we observed in practice is namely as follows. Assume that a retailer is ordering by a supplier that produces on order. A major source of uncertainty from the retailer's point of view, is the time that elapses until the supplier starts a production run to fulfil the retailer's order. When this time is long, follow-up orders are likely to occur. Moreover, the concept can be generalized in several ways, of which some will be discussed in the concluding section.

Until now most attention in inventory theory has been paid to incorporating demand variability. Analytic research of inventory systems with unreliable lead times is sparse. The main reason for this sparseness of attention is that it is difficult to incorporate lead time variability into a tractable model. In many studies dealing with single item inventory systems it is therefore assumed that the involved lead times are constant. If this is not the case in a real application, then ignoring lead time variability may lead to a very high cost penalty and a poor system performance. For two case studies concerning the effects of ignoring lead time variability, we refer the reader to Vinson [7], and Kumar and Arora [5]. Gross and Soriano [3] conclude that, in general, lead time variability has a much greater impact on system performance than demand variability (see also Vinson [7]). For this reason we restrict ourselves to the model in which only the lead times are unreliable. The incorporation of stochastic demand into the model with simultaneous order arrivals will be left for future research.

The main difficulty of incorporating unreliable lead times into a tractable model is that in a practical situation successive lead times are usually dependent of each other. Otherwise, it may be possible that an order arrives before the preceding one, which rarely occurs in practice. For a more extensive discussion of this aspect, we refer the reader again to Vinson [7] (see also the discussion above). If successive lead times are not mutually independent, the construction of the relevant cost function may be very difficult. For the model with simultaneous order arrivals we show that the cost function can be obtained in a straightforward manner.

Consider the single item (s, q) inventory system with stochastic demand and fixed lead time. Most studies dealing with this model are based on the fundamental property that the total demand during an arbitrary lead time is independent of the inventory position at the beginning of this lead time. If this property holds, the optimal order strategy can be determined once the (stationary) distributions of the inventory position and the lead time demand are known. (Commonly, the inventory position at an arbitrary epoch is uniformly distributed on the integers $s + 1, s + 2, \dots, s + q$.) Federgruen and Zheng [1] derive an efficient algorithm for computing the optimal order strategy. If the lead times are unreliable, the situation is much more complicated. The main problem is, because of the necessary no-crossing assumption, that successive lead times are not mutually independent. Zipkin [10] introduces a mechanism for the arrival of orders, which ensures that orders never cross, while in general lead times will be dependent. In essence, this mechanism guarantees that the only orders outstanding at an arbitrary epoch are the orders placed during a random 'lead time' interval immediately preceding this

epoch. If the lead times are generated according to this mechanism, the optimal order strategy can be determined by means of the algorithm described in Federgruen and Zheng [1]. The lead time process involving simultaneous order arrivals cannot be characterized by means of the mechanism introduced by Zipkin. In the situation considered in this study, it is possible that orders placed during a random 'lead time' interval immediately preceding an arbitrary epoch arrive *before* this epoch. This behaviour violates one of the assumptions made by Zipkin concerning the lead time mechanism, and hence, complicates an analysis based on the lead time demand considerably. For this reason we choose another and more direct approach to deal with the model involving simultaneous order arrivals (see Section 3).

Due to the simultaneous order arrival mechanism the cost function for the present model turns out to be discontinuous in the order quantity q . The determination of the optimal order strategy requires the search of a possibly large number of regions of continuity. By exploiting the special structure of the problem we show in Section 4 that the optimum can be obtained in a reasonable amount of time. Because of the discontinuous behaviour of the cost function we also consider two approximate solutions. The first approximation is based on a related model with a continuous cost function. It turns out that this continuous cost function, in general, is nonconvex. Essentially, this nonconvexity is a consequence of the possibility that orders arrive simultaneously. We see a similar effect on the cost function when the standard (s, q) model is generalized to the (s, nq) model to allow for random demand (see Federgruen and Zheng [1], and Zheng and Chen [9]). In fact, the nonconvexity of the cost function for the (s, nq) model is a consequence of the possibility of ordering an integer multiple of a base lot size. The second approximation is based on the economic order quantity adapted to allow backlogging. It will turn out that this approximation, in general, leads to very poor results. The reason for this pooriness of the EOQ approximation is the involved *fixed* order quantity, which depends only on the inventory, backlogging and ordering costs, but not on the lead time process. In contrast, the EOQ approximation for the standard (s, q) model with stochastic demand and fixed lead time yields a reasonably accurate result. Zheng [8] shows that the relative error in cost of the EOQ approximation for this model is at most $\frac{1}{8}$. This contrast confirms the above-mentioned conclusion of Gross and Soriano [3] that lead time variability is more important than demand variability.

The organization of this study is as follows. In Section 2 we describe the inventory model with simultaneous order arrivals in detail. An expression for the cost function associated with this model is derived in Section 3. Due to the discontinuous behaviour of the involved cost function, the determination of the optimal order strategy requires special attention. This is discussed in Section 4. In Section 5 we derive a continuous approximation of the involved cost function and show that the optimum of this approximation can be determined very efficiently. The applicability of the model with simultaneous order arrivals is studied in Section 6 by means of several numerical examples. In particular, a numerical comparison between the exact method and the approximate method is made. Finally, in Section 7 we discuss some generalizations.

2 Model Description

A single item (s, q) inventory system is analyzed. The demand is deterministic and occurs at a constant rate d . Each time the inventory position drops to level s a new order of q units is placed with the supplier (s and q are the decision variables). It is assumed that it takes at least a fixed and nonnegative time l_0 for each order to arrive, say a fixed handling and transportation time. The residual lead time is assumed to be a stochastic variable which in general depends on preceding lead times. To

avoid crossing of orders, we introduce the following dependence with respect to the successive lead times. If at an order epoch t all orders placed previously have arrived or will arrive before time $t + l_0$ (note that l_0 represents the minimal lead time), then the lead time of the order placed at time t is a discrete stochastic variable $\mathbf{L}$ defined at the points $l_0, l_1, \dots, l_{M-1}$ with distribution

$$p_j = P\{\mathbf{L} = l_j\},$$

where the possible lead times are numbered in ascending order, that is, $l_0 < l_1 < \dots < l_{M-1}$. In particular, the lead time of the order placed at time t is independent of preceding lead times. Let us call an order for which the lead time has distribution p_j a *start-up* order. The reason for this convention is made clear below. If at an order epoch t all orders placed previously but not yet arrived will not arrive before time $t + l_0$, then the order placed at time t will be delivered simultaneously with the first order not yet arrived (necessarily a start-up order). An order incorporated with preceding orders is called a *follow-up* order. It indicates the situation that an additional order has to be placed, because the current inventory position is insufficient to satisfy present and future demand. Consequently, if at time t a start-up order with lead time l_j is placed, then all orders placed during the time period $(t, t + l_j - l_0]$ will arrive simultaneously with this start-up order. For the lead time process introduced above, it is easily verified that crossing of orders is excluded.

In order to gain insight in the actual inventory process, we consider the model described above for a given order strategy s and q . It is supposed that at time 0 a new order is placed and all orders placed previously have arrived. Hence, the order placed at time 0 represents a start-up order. Suppose further that the lead time of this start-up order is given by l_j with $j > 0$. From the model description above it is easily seen that all orders placed before or at time $l_j - l_0$ will also arrive at time l_j . A typical realization of the inventory process is depicted in Figure 1.

Figure 1: A realization of the inventory process

For the realization depicted in Figure 1 it is seen that 4 orders arrive simultaneously at time l_j (1 start-up order and 3 follow-up orders). Since the fifth order is placed at a moment beyond $l_j - l_0$, this order cannot be included in the batch arriving at time l_j , and hence will initiate a new order sequence. From Figure 1 it is seen that there are no follow-up orders corresponding to this start-up order, and hence the fifth order arrives singly.

The cost structure is as follows. The inventory and backlogging costs per unit per unit of time are given by c_1 and c_2 , respectively. Each time a new order is placed an ordering cost of c_3 is charged if this order represents a start-up order, and of c_4 if this order represents a follow-up order. Without restricting the generality it is assumed that $c_3 \geq c_4$. This assumption reflects the fact that in practical cases (make-to-order situations), it is usually less expensive to place a follow-up order than a start-up order, when transportation costs are also included in the ordering cost factors. In that case the follow-up orders have to bear a smaller part of the incorporated transportation costs, when the orders are sent together.

The model described above reflects the inventory system in many real business situations. Frequently, if a stackup of orders occurs at a factory, all these orders will be delivered in the same batch. So successive lead times are rarely independent of each other. Ignoring this dependence may lead to an unreliable estimation of the expected total cost, and hence to a significant financial loss. In the present study we deal explicitly with a certain kind of dependency between successive lead times.

3 Derivation of the Cost Function

In this section we derive an expression for the long-run expected cost per unit of time associated with an arbitrary order strategy (s, q) . Suppose at time t a new start-up order is placed. Then the inventory position at time t is equal to s and all orders placed previously but not yet arrived are bound to arrive before time $t + l_0$ (see the model description in the preceding section). Since the demand rate is given by d , it follows that the net inventory at time $t + l_0$ is equal to $s - dl_0$. With regard to the order placed at time t , it is assumed that the net inventory is affected by this order only after its arrival. (In other words, the net inventory process is assumed to be left continuous.) In particular, if the order placed at time t has a lead time of l_0 , the net inventory will be increased by the q units belonging to this order immediately after time epoch $t + l_0$. Let us suppose that the lead time of the start-up order placed at time t is given by l_j . The probability that such an event occurs is equal to p_j . By definition of the lead time process (see preceding section), all additional orders placed during the time interval $(t, t + l_j - l_0]$ will arrive simultaneously with the start-up order placed at time t . Since new orders are placed at time epochs $t + q/d, t + 2q/d, \dots$ the number of orders n_j arriving simultaneously at time $t + l_j$ satisfies for $j = 0, \dots, M-1$ the relation:

$$(n_j - 1) \frac{q}{d} \leq l_j - l_0 < n_j \frac{q}{d}. \quad (1)$$

So the batch arriving at time $t + l_j$ consists of $n_j q$ units. The first order which cannot be incorporated in this batch is generated at time $t + n_j q/d$. This order represents a new start-up order and will not arrive before time $t + n_j q/d + l_0$. By virtue of relation (1) we have

$$t + n_j \frac{q}{d} + l_0 > t + l_j, \quad (2)$$

so that the net inventory at time $t + n_j q/d + l_0$ is given by

$$s - d(n_j \frac{q}{d} + l_0) + n_j q = s - dl_0. \quad (3)$$

Again, it is assumed that the net inventory level can be affected by the start-up order placed at time $t + n_j q/d$ only after its arrival, which does not occur before time $t + n_j q/d + l_0$. Note that relation (2) also implies that the start-up order placed at time $t + n_j q/d$ will arrive after the preceding batch has arrived. Consequently, the net inventories at time epochs $t + l_0$ and $t + n_j q/d + l_0$ are equal. It is easily seen that the inventory positions at these time epochs are also equal. It follows that the inventory process from time $t + n_j q/d + l_0$ onward is a probabilistic replica of the inventory process from time $t + l_0$ onward. Since the start-up order epoch t is chosen arbitrarily, it is seen that the inventory process is regenerative with respect to the successive times at which the net inventory is equal to $s - dl_0$. Let $\mathbf{T}$ denote an arbitrary time between two successive occurrences of state $s - dl_0$, and $\mathbf{C}$ the total cost during this interoccurrence time. By applying the theory of regenerative processes it now follows that the long-run expected cost per unit of time $K(s, q)$, associated with an order strategy (s, q) , is given by

$$K(s, q) = \frac{E\{\mathbf{C}\}}{E\{\mathbf{T}\}}. \quad (4)$$

By using that the interoccurrence time $\mathbf{T}$ is equal to $n_j q/d$, if the lead time of the corresponding start-up order is equal to l_j , we obtain for the expected interoccurrence time $E\{\mathbf{T}\}$:

$$E\{\mathbf{T}\} = \frac{q}{d} \sum_{j=0}^{M-1} p_j n_j, \quad (5)$$

where the numbers n_j are uniquely determined by relation (1).

We now compute the expected cost $E\{\mathbf{C}\}$ during an interoccurrence time. Suppose a start-up order is placed at time 0. Thus, the inventory position at time 0 is equal to s . With probability p_j , the corresponding lead time is given by l_j . The number of orders n_j arriving simultaneously at time l_j is uniquely determined by relation (1). Because the net inventory at time l_0 is equal to $s - dl_0$, this epoch represents a regeneration epoch. The next regeneration occurs at time $n_j q/d + l_0$ (compare relation (3)), so that the time interval $[l_0, n_j q/d + l_0]$ constitutes an interoccurrence time. For the computation of the expected cost $E\{\mathbf{C}\}$, we may restrict ourselves to this time interval. We have to distinguish four different cases, which are depicted in Figure 2. By using this figure the conditional expectations $E\{\mathbf{C} \mid \mathbf{L} = l_j\}$ are readily computed for the four different cases. For the case $dl_0 \leq s < dl_j$, we obtain

$$E\{\mathbf{C} \mid \mathbf{L} = l_j\} = \frac{c_1}{2d}(s - dl_j + n_j q)^2 + \frac{c_2}{2d}(s - dl_j)^2 + c_3 + (n_j - 1)c_4.$$

Note that for this case the total length of the periods with positive inventory is given by $s/d - l_j + n_j q/d$ and the mean inventory during these periods by $(s - dl_j + n_j q)/2$, whereas the length of the period with negative inventory is given by $l_j - s/d$ and the mean inventory during this period by $(s - dl_j)/2$. Further, recall that for each start-up order a fixed cost of c_3 is charged and for each follow-up order a fixed cost of c_4 . Analogously, we obtain: for the case $s \geq dl_j$,

$$E\{\mathbf{C} \mid \mathbf{L} = l_j\} = \frac{c_1}{2d}(s - dl_j + n_j q)^2 - \frac{c_1}{2d}(s - dl_j)^2 + c_3 + (n_j - 1)c_4;$$

for the case $dl_j - n_j q \leq s < dl_0$,

$$E\{\mathbf{C} \mid \mathbf{L} = l_j\} = \frac{c_1}{2d}(s - dl_j + n_j q)^2 + \frac{c_2}{2d}(s - dl_j)^2 + c_3 + (n_j - 1)c_4;$$

Case i: $dl_0 \leq s < dl_j$ Case ii: $s \geq dl_j$ Case iii: $dl_j - n_j q \leq s < dl_0$ Case iv: $s < dl_j - n_j q$

Figure 2: Net inventory process during an interoccurrence time

and for the case $s < dl_j - n_j q$,

$$E\{C \mid L = l_j\} = -\frac{c_2}{2d}(s - dl_j + n_j q)^2 + \frac{c_2}{2d}(s - dl_j)^2 + c_3 + (n_j - 1)c_4.$$

Note that in the last case demands are never satisfied from on-hand inventory. By conditioning on the length of the lead time it now follows that the expected cost during the interoccurrence time $[l_0, n_j q/d + l_0]$ is given by

$$\begin{aligned} E\{C\} &= \sum_{j=0}^{M-1} p_j E\{C \mid L = l_j\} \\ &= \frac{c_1}{2d} \sum_{\substack{j=0 \\ s \geq dl_j - n_j q}}^{M-1} p_j (s - dl_j + n_j q)^2 - \frac{c_1}{2d} \sum_{\substack{j=0 \\ s \geq dl_j}}^{M-1} p_j (s - dl_j)^2 \\ &\quad + \frac{c_2}{2d} \sum_{\substack{j=0 \\ s < dl_j}}^{M-1} p_j (s - dl_j)^2 - \frac{c_2}{2d} \sum_{\substack{j=0 \\ s < dl_j - n_j q}}^{M-1} p_j (s - dl_j + n_j q)^2 \\ &\quad + c_3 + \left(\sum_{j=0}^{M-1} p_j n_j - 1 \right) c_4. \end{aligned} \quad (6)$$

By the regenerative character of the inventory process, expression (6) also represents the expected cost during an arbitrary interoccurrence time. Substituting the expressions (5) and (6) into expression (4), we obtain for the long-run expected cost per unit of time:

$$\begin{aligned} K(s, q) &= c_4 \frac{d}{q} + (c_3 - c_4) \frac{d}{q \sum_{j=0}^{M-1} p_j n_j} \\ &\quad + \frac{c_1 + c_2}{2q \sum_{j=0}^{M-1} p_j n_j} \left\{ \sum_{j=0}^{M-1} p_j (s - dl_j + n_j q)^2 \left[I_{s \geq dl_j - n_j q} - \frac{c_2}{c_1 + c_2} \right] \right. \\ &\quad \left. + \sum_{j=0}^{M-1} p_j (s - dl_j)^2 \left[I_{s < dl_j} - \frac{c_1}{c_1 + c_2} \right] \right\}, \end{aligned} \quad (7)$$

where I_A stands for the indicator function of the event A .

Next we discuss two important aspects of the cost function $K(s, q)$. From relation (1) it is seen that the number of orders n_j corresponding to lead time l_j depends on q discontinuously. So the cost function $K(s, q)$ is also discontinuous in q . This discontinuous behaviour of the cost function really stems from the way it is decided whether a new order can be included in the preceding batch, or whether it initiates a new order sequence. For example, consider the realization of the inventory process depicted in Figure 1. If the order quantity q is decreased until the fifth order is placed just before time $l_j - l_0$, then the expected cost during a cycle time increases with an amount $p_j c_4$, and the expected cycle time with an amount $p_j q/d$. (Here it is assumed that the numbers n_i with $i \neq j$ remain unchanged.) By using relation (4) it is readily verified that, in general, these increases lead to a nonzero cost difference at the value of q for which the fifth order is placed exactly at time $l_j - l_0$. (Actually, this value is equal to $d(l_j - l_0)/4$.) An example of the behaviour of a typical cost function is given in Figure 3, where a mesh plot in the neighbourhood of the optimum is depicted. This picture clearly illustrates the existence of break lines at which certain jumps occur in the cost function. The discontinuous behaviour of

Figure 3: Mesh plot of a typical cost function near the optimum

the cost function slightly complicates the determination of the order strategy with minimal total cost. In the next section we study this problem in some detail.

Let $\tilde{K}(s, q)$ denote the cost function corresponding to the same model but with the lead times l_j replaced by $\tilde{l}_j = l_j - l_0$. Then it is easily verified by using expression (7) that

$$\tilde{K}(s - dl_0, q) = K(s, q),$$

so that the optimal order strategy for the general case can be obtained by solving the related model in which the minimum lead time is equal to 0.

4 Marginal Analysis

Because the cost function $K(s, q)$, derived in (7), is not continuous in q , the problem of finding the optimal order strategy requires special attention. Let the lines Γ_{jn} for $j = 1, \dots, M-1$ and $n = 1, 2, \dots$ be defined by

$$\Gamma_{jn} = \{ (s, q) : nq = d(l_j - l_0) \}. \quad (8)$$

It is easily verified by using relation (1) that the lines Γ_{jn} represent lines of discontinuity of the cost function $K(s, q)$. Note that some of the lines Γ_{jn} may coincide. These lines of discontinuity can be clearly seen in the example plotted in Figure 3. The various boundary lines divide the s - q plane into horizontal strips. On each strip, the cost function $K(s, q)$ is continuous. Moreover, it can be shown that the cost function $K(s, q)$ is convex on each strip. This implicates, among other things, that a local optimum is also global. The convexity property of the cost function $K(s, q)$ considerably simplifies the problem of finding the order strategy with minimal total cost.

The optimal solution of the overall problem will be obtained by solving the problem for each strip separately. It is noted that we may restrict our attention to

the finite number of strips lying above a certain boundary line, say $q = q_{\min}$ with $q_{\min} > 0$, since the total ordering cost in the remaining strips is too high for finding the optimum is these strips (compare the term $c_4 d/q$ occurring in expression (7)). Because the number of strips to be searched may be quite large, the application of a general procedure to find the optimum in each strip may require a large amount of computation time, and hence, restricts the usefulness of the model considerably. However, by using the special structure of the problem, the optimum in each strip can be determined very efficiently. This is shown below.

Consider an arbitrary strip $\{(s, q) : q_1 \leq q \leq q_2\}$ with $q_1 < q_2$ in which the cost function $K(s, q)$ is continuous. The numbers n_j are constant in this strip and uniquely determined by relation (1). Differentiating (7) with respect to s yields

$$\frac{\partial}{\partial s} K(s, q) = \frac{c_1 + c_2}{q \sum_{j=0}^{M-1} p_j n_j} \left\{ \sum_{j=0}^{M-1} p_j (s - dl_j) \right. \\ \left. + q \sum_{j=0}^{M-1} p_j n_j \left[I_{s \geq dl_j - n_j q} - \frac{c_2}{c_1 + c_2} \right] \right\}. \quad (9)$$

It is not difficult to verify that the function $(\partial/\partial s)K(s, q)$ with q fixed is a piecewise linear, continuous and increasing function in s . Since

$$\left[\frac{\partial}{\partial s} K(s, q) \right]_{s=-q+dl_0} = -c_2, \\ \left[\frac{\partial}{\partial s} K(s, q) \right]_{s=dl_{M-1}} = c_1,$$

it is seen that, for fixed q , the function $(\partial/\partial s)K(s, q)$ possesses a unique zero $s(q)$ in the interval $(-q + dl_0, dl_{M-1})$. This zero can be easily determined by reducing the piecewise linear equation $(\partial/\partial s)K(s, q) = 0$ to a (simple) linear equation by means of bisection with respect to s . The zero $s(q)$, in turn, is a continuous and piecewise linear function of q , which can be shown by using again the equation $(\partial/\partial s)K(s, q) = 0$. Obviously, the zero $s(q)$ represents the optimal order level corresponding to a fixed order quantity q . It is noted that for the order strategy $(s(q), q)$ the service level (fraction of demand to be satisfied from on-hand inventory) is equal to $c_2/(c_1 + c_2)$. This can be shown as follows. By using Figure 2 the average time of positive inventory during a regeneration cycle is easily computed. The long-run expected service level then follows by dividing the resulting expression by the expected cycle time. For $s = s(q)$, the resulting expression for the service level reduces to $c_2/(c_1 + c_2)$. (For a similar derivation see Section 3.)

For the determination of the optimum of the cost function $K(s, q)$ in the region $\{(s, q) : q_1 \leq q \leq q_2\}$, it obviously suffices to take $s = s(q)$. Substituting $s = s(q)$ into expression (7) and using that $s(q)$ is a zero of $(\partial/\partial s)K(s, q)$, we obtain after a lengthy but straightforward computation:

$$K(q) := K(s(q), q) = a_0 + a_1 q + a_2 \frac{1}{q}, \quad (10)$$

with

$$a_0 = \frac{c_1 + c_2}{\sum_{j=0}^{M-1} p_j n_j} \left\{ \sum_{j=0}^{M-1} p_j n_j \left[I_{s(q) \geq dl_j - n_j q} - \frac{c_2}{c_1 + c_2} \right] \sum_{j=0}^{M-1} p_j dl_j \right. \\ \left. - \sum_{j=0}^{M-1} p_j n_j dl_j \left[I_{s(q) \geq dl_j - n_j q} - \frac{c_2}{c_1 + c_2} \right] \right\},$$

$$a_1 = \frac{c_1 + c_2}{2 \sum_{j=0}^{M-1} p_j n_j} \left\{ \sum_{j=0}^{M-1} p_j n_j^2 \left[I_{s(q) \geq dl_j - n_j q} - \frac{c_2}{c_1 + c_2} \right] - \left(\sum_{j=0}^{M-1} p_j n_j \left[I_{s(q) \geq dl_j - n_j q} - \frac{c_2}{c_1 + c_2} \right] \right)^2 / \sum_{j=0}^{M-1} p_j \right\},$$

$$a_2 = dc_4 + \frac{d(c_3 - c_4)}{\sum_{j=0}^{M-1} p_j n_j} + \frac{c_1 + c_2}{2 \sum_{j=0}^{M-1} p_j n_j} \left\{ \sum_{j=0}^{M-1} p_j (dl_j)^2 - \left(\sum_{j=0}^{M-1} p_j dl_j \right)^2 / \sum_{j=0}^{M-1} p_j \right\},$$

where $\sum'$ indicates summation over all j satisfying the condition $dl_j - n_j q \leq s(q) < dl_j$. By using that the zero $s(q)$ is a continuous and piecewise linear function of q (see above), it is not difficult to verify that the functions a_0 , a_1 and a_2 are piecewise constant, so that the function $K(q)$ is a piecewise rational function of q . Note that $a_2 > 0$ in each piece, which follows by applying the Cauchy-Schwarz inequality. It can be shown that the function $K(q)$ is continuously differentiable on the interval $[q_1, q_2]$. This implies, since $a_2 > 0$, that the function $K(q)$ is convex on the interval $[q_1, q_2]$. The last conclusion also follows from the convexity of the cost function $K(s, q)$ in each region of continuity.

Because of the convexity of the function $K(q)$, the optimum of the cost function $K(s, q)$ in the region $\{(s, q) : q_1 \leq q \leq q_2\}$ can now be determined as follows. We have to distinguish three cases:

1. If $K'(q_1) \geq 0$, the optimum lies on the lower boundary of the region and is given by $(s(q_1), q_1)$.
2. If $K'(q_2) \leq 0$, the optimum lies on the upper boundary of the region and is given by $(s(q_2), q_2)$.
3. If $K'(q_1) < 0$ and $K'(q_2) > 0$, the optimum lies in the interior of the region. In this case, the optimal order quantity can be found by reducing the piecewise rational equation $K'(q) = 0$ to a (simple) rational equation by means of bisection with respect to q , and then solving the resulting rational equation. The corresponding optimal order level now immediately follows from the equation $(\partial/\partial s)K(s, q) = 0$, because the indicator functions $I_{s < dl_j}$ and $I_{s \geq dl_j - n_j q}$ are both known at the optimum (compare expression (9)).

In summary, the overall optimum can be found by dividing the s - q plane into horizontal strips of continuity, and then solving the problem for each strip separately by using the method described above. If the number of strips to be searched is not too large, this solution procedure yields the optimal order strategy within a reasonable amount of computation time. For most strips it is sufficient to determine the optimal order level corresponding to the lower or upper boundary of the strip, and to check condition $K'(q_1) \geq 0$ or $K'(q_2) \leq 0$. A significant reduction in the number of strips to be searched is achieved by restricting the search to integer order quantities only, so that strips containing no integer order quantity can be skipped. Clearly this is allowed, since in practical situations only integer order quantities are of interest. It is noted that in Section 6 we encounter some numerical examples for which the number of strips to be searched is very large, so that, although the optimum in each strip can be determined very efficiently, the total time required to find the overall optimum may be considerable. For this reason we derive in the next section a continuous approximation of the cost function $K(s, q)$.

5 A Continuous Approximation

The inventory model studied in the preceding sections gives rise to a discontinuous cost function $K(s, q)$. For this model, the order strategy with minimal total cost can be determined by dividing the s - q plane into horizontal strips, such that the involved cost function is continuous in each strip, and then solving the problem for each strip separately. It is noted that for several cases the total variation of the involved cost function with respect to q is small in a large region containing the overall optimum, and as a consequence, the total variation of the optimal values corresponding to the strips in this region is small also. For such cases, the solution method described in the preceding section may require a considerable amount of computation time, if the region of small variability contains a large number of strips. However, the relative robustness of the optimal solution with respect to the order quantity q , immediately suggests that an approximate solution may yield satisfactory results. In the present section we discuss an approximation based on a related model with a continuous cost function.

We first describe the inventory model underlying the approximation. The demand is assumed to be deterministic with a constant rate d . The order strategy is as follows. Suppose at time t the inventory position drops to level s and all orders placed previously have arrived or will arrive before time $t + l_0$. Again it is assumed that it takes at least a fixed time l_0 for an order to arrive. Then at time t a new order of variable size is placed with the supplier. The size of this order depends on the length of the corresponding lead time. We assume that the lead time of the order placed at time t is a stochastic variable L with discrete distribution $p_j = P\{L = l_j\}$, where the possible lead times are numbered in ascending order, that is, $l_0 < l_1 < \dots < l_{M-1}$. If the lead time of the order placed at time t is equal to l_j , then the number of units to be delivered at time $t + l_j$ is equal to $q + d(l_j - l_0)$, where q denotes a positive decision variable. Note that the order quantity is chosen such that the net inventory immediately after the time of delivery is equal to $s + q - dl_0$. So by using a variable order quantity, the net inventory at the conclusion of a delivery is always the same. The next order will be placed at the first epoch after time $t + l_j - l_0$ at which the inventory position drops to level s , so that orders never cross each other.

In accordance with the main model described in Section 2, we impose the following cost structure upon the present model. The inventory and backlogging costs per unit per unit of time are fixed and given by c_1 and c_2 , respectively. The replenishment cost per order depends on the length of the corresponding lead time. Note that for an order with lead time l_j , the total order quantity is given by $q + d(l_j - l_0)$. It is assumed that for the first q units involved in an order a fixed cost of c_3 is charged, and for each additional number of q units (in general noninteger) a fixed cost of c_4 . Hence, the total replenishment cost for an order with lead time l_j is given by

$$c_3 + \frac{d(l_j - l_0)}{q} c_4. \quad (11)$$

Note that for the model studied in the preceding sections, the total replenishment cost incurred per batch is equal to $c_3 + (n_j - 1)c_4$, if this batch consists of n_j orders. From relation (1) it is now seen that, if the first order in a batch has lead time l_j , expression (11) represents a continuous approximation of the total replenishment cost associated with this batch.

The cost function for the present model can be derived in an analogous way as for the main model. Denote by $K_1(s, q)$ the long-run expected cost per unit of time associated with an order strategy (s, q) . Note that an order strategy consists of an order level s and a minimum order quantity q . Since the net inventory immediately

after a delivery is always equal to $s + q - dl_0$ (see above), the net inventory process is regenerative with respect to the successive times of order deliveries. Hence, for the derivation of the cost function $K_1(s, q)$, we may restrict ourselves to a regeneration cycle. Analogously as in Section 3, by conditioning on the length of the lead time, an expression for the expected cost during a regeneration cycle can be derived. It is easily verified that the expected cycle length is given by $E\{\mathbf{L}\} - l_0 + q/d$. By dividing the resulting expression for the expected cost during a cycle by the expected cycle length, we obtain for the cost function $K_1(s, q)$:

$$K_1(s, q) = \frac{c_1}{2}(2s + q - dE\{\mathbf{L}\} - dl_0) + \frac{dc_4}{q} + \frac{c_1 + c_2}{2(dE\{\mathbf{L}\} - dl_0 + q)} \left\{ \frac{2d(c_3 - c_4) - c_1 d^2 \text{var}(\mathbf{L})}{c_1 + c_2} + \sum_{\substack{j=0 \\ s < dl_j}}^{M-1} p_j (s - dl_j)^2 \right\}, \quad (12)$$

where $\text{var}(\mathbf{L})$ stands for the variance of the lead time. It is easily seen that the cost function $K_1(s, q)$ is continuous in both variables. Note that, if the lead time variability is large, the cost function $K_1(s, q)$ is not convex. The nonconvexity of the cost function $K_1(s, q)$ slightly complicates the determination of the order strategy with minimal total cost. Below we deal with this problem in some detail. The analysis proceeds along the same lines as the marginal analysis for the cost function $K(s, q)$, discussed in the preceding section.

We first reduce the cost minimization problem with respect to the variables s and q to one in which the variable s is eliminated. Differentiating (12) with respect to s we obtain

$$\frac{\partial}{\partial s} K_1(s, q) = c_1 + \frac{c_1 + c_2}{dE\{\mathbf{L}\} - dl_0 + q} \sum_{\substack{j=0 \\ s < dl_j}}^{M-1} p_j (s - dl_j). \quad (13)$$

It is not difficult to see that the derivative $(\partial/\partial s)K_1(s, q)$ with q fixed is a piecewise linear, continuous and increasing function in s . Since

$$\frac{\partial}{\partial s} K_1(s, q) = c_1 + \frac{c_1 + c_2}{dE\{\mathbf{L}\} - dl_0 + q} (s - dE\{\mathbf{L}\}) \leq -c_2,$$

for $s \leq -q + dl_0$, and

$$\frac{\partial}{\partial s} K_1(s, q) = c_1,$$

for $s \geq dl_{M-1}$, it follows that, for fixed q , the function $(\partial/\partial s)K_1(s, q)$ possesses a unique zero $s_1(q)$ in the interval $(-q + dl_0, dl_{M-1})$. This zero can be easily determined by reducing the piecewise linear equation $(\partial/\partial s)K_1(s, q) = 0$ to a (simple) linear equation by means of bisection with respect to s . The zero $s_1(q)$, in turn, is a continuous and piecewise linear function of q , which can be shown by considering a q -interval on which the indicator function $(s_1(q) < dl_j)$ is constant, and then solving the equation $(\partial/\partial s)K_1(s, q) = 0$ for s on this interval. Note that the zero $s_1(q)$ represents the optimal order level corresponding to a fixed order quantity q .

For the determination of the optimum of the cost function $K_1(s, q)$, it obviously suffices to take $s = s_1(q)$. Substituting $s = s_1(q)$ into expression (12) and using that $s_1(q)$ is a zero of $(\partial/\partial s)K_1(s, q)$, we obtain after a lengthy but straightforward computation:

$$K_1(q) := K_1(s_1(q), q) = b_0 + b_1 q + b_2 \frac{1}{q} + b_3 \frac{1}{q + dE\{\mathbf{L}\} - dl_0}, \quad (14)$$

with

$$\begin{aligned}
 b_0 &= \left(c_1 \sum_{j=0}^{M-1} p_j dl_j - \frac{dc_1^2(E\{\mathbf{L}\} - l_0)}{2(c_1 + c_2)} \right) / \sum_{j=0}^{M-1} p_j - \frac{c_1}{2} d(E\{\mathbf{L}\} + l_0), \\
 b_1 &= \frac{c_1}{2} - \frac{c_1^2}{2(c_1 + c_2)} / \sum_{j=0}^{M-1} p_j, \\
 b_2 &= dc_4, \\
 b_3 &= d(c_3 - c_4) - \frac{c_1}{2} d^2 \text{var}(\mathbf{L}) + \frac{c_1 + c_2}{2} \left\{ \sum_{j=0}^{M-1} p_j (dl_j)^2 - \left(\sum_{j=0}^{M-1} p_j dl_j \right)^2 / \sum_{j=0}^{M-1} p_j \right\},
 \end{aligned} \tag{15}$$

where $\sum'$ indicates summation over all j satisfying the condition $s_1(q) < dl_j$. By using that the zero $s_1(q)$ is a continuous and piecewise linear function of q (see above), it is not difficult to verify that the functions b_0 , b_1 , b_2 and b_3 are piecewise constant, so that the function $K_1(q)$ is a piecewise rational function of q . Moreover, it can be shown that the function $K_1(q)$ is continuously differentiable. Because the coefficient b_3 may be negative, the cost function $K_1(q)$ is, in general, not convex. As a consequence, this function may have several optima. Due to the special form of the cost function $K_1(q)$, the intervals on which the function $K_1(q)$ is convex can be easily identified. Then by computing the optimum in each interval of convexity, the overall optimum can be obtained. Note that, since the function $K_1(q)$ is continuously differentiable, the intervals on which the function $K_1(q)$ is concave need not to be considered. In the following we will describe the optimization procedure in some detail.

Consider an interval $[q_1, q_2]$ on which the coefficients b_0 , b_1 , b_2 and b_3 are constant. Differentiating (14) twice with respect to q we obtain

$$\begin{aligned}
 K_1'(q) &= b_1 - b_2 \frac{1}{q^2} - b_3 \frac{1}{(q + dE\{\mathbf{L}\} - dl_0)^2}, \\
 K_1''(q) &= 2 \left\{ b_2 \frac{1}{q^3} + b_3 \frac{1}{(q + dE\{\mathbf{L}\} - dl_0)^3} \right\}.
 \end{aligned}$$

To determine the subinterval on which the function $K_1(q)$ is convex, we distinguish two cases:

1. $b_2 + b_3 \geq 0$: In this case, $K_1''(q) > 0$ for $q > 0$, so that the function $K_1(q)$ is convex on the entire interval $[q_1, q_2]$.
2. $b_2 + b_3 < 0$: In this case, the second derivative $K_1''(q)$ has a unique zero ζ in $(0, \infty)$, which is given by

$$\zeta = \frac{dE\{\mathbf{L}\}}{\sqrt[3]{-b_3/b_2 - 1}}.$$

Obviously, $K_1''(q) > 0$ for $q < \zeta$ and $K_1''(q) < 0$ for $q > \zeta$. It follows that, if $\zeta \geq q_2$, the function $K_1(q)$ is convex on the entire interval $[q_1, q_2]$; if $q_1 < \zeta < q_2$, the function $K_1(q)$ is convex on the subinterval $[q_1, \zeta]$; and if $\zeta \leq q_1$, there does not exist a subinterval of $[q_1, q_2]$ on which the function $K_1(q)$ is convex.

Consequently, the subinterval of $[q_1, q_2]$ on which the function $K_1(q)$ is convex, if any, can be easily identified. For the case such a subinterval exists, the optimum

in this subinterval can be obtained as follows. If the first derivative $K'_1(q)$ does not change sign on this subinterval, then the optimum is given by one of its endpoints. If the first derivative $K'_1(q)$ does change sign on this subinterval, then the optimum lies in the interior of the interval and can be found by solving the rational equation $K'_1(q) = 0$ numerically. This equation will be solved by applying a combination of the method of bisection and Newton's method, which yields the optimum very fast.

In conclusion, the overall optimum of the cost function $K_1(s, q)$ can be obtained by considering the piecewise rational function $K_1(q)$. The overall optimum of the function $K_1(q)$ can be determined by distinguishing the intervals on which the coefficients b_0, b_1, b_2 and b_3 are constant, in particular the subintervals, if any, on which the function $K_1(q)$ is convex, and then solving the problem for each interval separately. The way to compute each local optimum has been described above. Because there are at most M subintervals on which the function $K_1(q)$ is convex (compare expressions (14) and (15)), the solution procedure to find the optimum of the cost function $K_1(q)$ is very efficient. Once the optimal order quantity has been obtained, the corresponding optimal order level then immediately follows from the equation $(\partial/\partial s)K_1(s, q) = 0$, because the indicator function $I_{s < d_l}$ is known at the optimum (compare expression (13)). Hence, the optimal order strategy for one model with a continuous cost function can be determined very efficiently.

6 Numerical Results

In order to gain insight in the applicability of the inventory model in which orders may arrive simultaneously, several numerical cases are considered. In particular, we investigate the effectiveness of the method, described in Section 4, to compute the optimal solution. This investigation includes a numerical comparison between the exact method and the approximate method, discussed in the preceding section. For some cases we also make a comparison with an approximation based on the economic order quantity with backlogging.

The model with simultaneous order arrivals has been mainly introduced to deal with inventory situations in which the lead times are very uncertain (see Section 1). To illustrate this motivation, we first study the effect of the order quantity on the various cost components. In the sequel we denote by c the row vector consisting of the four cost coefficients c_1, c_2, c_3 and c_4 ; by l the row vector consisting of the lead times $l_0, l_1, \dots, l_{M-1}$; and by p the row vector consisting of the lead time probabilities $p_0, p_1, \dots, p_{M-1}$. The expectation and squared coefficient of variation of the lead time $\mathbf{L}$ are denoted by $E\{\mathbf{L}\}$ and c_L^2 , respectively. Note that $\mathbf{L}$ represents the lead time of a start-up order and not that of an arbitrary order, which depends on the choice of the decision variables. Further we introduce

$$B := \max_{0 \leq j \leq M-1} d(l_j - l_0).$$

From relation (1) it is easily verified that B denotes the lower bound on the order quantity to ensure that no additional orders are placed during an arbitrary lead time. The first numerical example concerns the following model:

$$\begin{aligned} d &= 100, \\ c &= [0.05, 0.95, 100, 50], \\ l &= [1.2, 56], \\ p &= [0.4, 0.4, 0.2], \end{aligned} \quad (16)$$

with $E\{\mathbf{L}\} = 12.4$ and $c_L^2 = 3.09$. This example reflects an inventory situation in which most orders can be delivered within two days from a nearby depot, but

Figure 4: Effect of order quantity on cost components

when this depot is out of stock, the required units have to come from a remote depot, resulting in a long lead time of delivery. For model (16), the optimal order levels $s(q)$ are computed for various values of q . Note that $s(q)$ denotes the root of a piecewise linear equation, which can be easily solved by bisection (see Section 4). The inventory, backlogging, (total) ordering and total costs associated with the order strategy $(s(q), q)$ are then determined by using the expression (7) for the cost function $K(s, q)$. In Figure 4 we display the obtained results. From this figure it is clearly seen that the requirement $q \geq B$ leads to high inventory and backlogging costs, and hence also to a high total cost. In this case, the cost increase due to the requirement $q \geq B$ is approximately 84.7%. So it is very profitable to allow also order quantities $q < B$. It is noted that, in general, the relative difference in cost between the optimum in $q \geq B$ and the overall optimum increases as the lead time variability increases. In conclusion, the order restriction $q \geq B$ may lead to a high cost penalty, and is, therefore, an impractical way to ensure that orders cannot cross each other.

The inventory model, introduced in the present study, gives rise to a discontinuous cost function $K(s, q)$. The optimal solution is obtained by dividing the s - q plane into horizontal strips of continuity, and then solving the problem for each strip separately. The method to compute the optimum in each strip has been described in Section 4. We recall that the search for the optimum is restricted to integer order quantities only. The strips are sequentially searched in a downward manner starting with the strip $\{q : q > B\}$. The procedure is stopped, whenever the upper bound of the strip being considered is less than some prespecified value. Unless stated otherwise, the minimum allowable order quantity is chosen equal to the demand d per unit of time. For example, applying this stop criterion to the case considered in Figure 4, results in a total number of 56 strips to be searched and a total computation time of 3.24 seconds. All numerical computations have been per-

formed on an IBM computer with a 386 processor (16-MHz) and a 387 coprocessor. The reasons for introducing a minimum order quantity are the following: First, in real business situations, because of transportation or production considerations, the supplier usually specifies a minimum allowable order quantity. (The supplier may also require that replenishment quantities are integer multiples of a certain packing unit. The cost consequences of this restriction is studied at the end of this section.) Second, because the ordering cost dominates only for very small q (see Figure 4), and the number of discontinuities of the cost function $K(s, q)$ increases without bound as the order quantity q tends to 0 (compare expression (8)), a stop criterion based on the behaviour of the cost function may result in a very large number of strips to be searched. For example, if the procedure is continued until the ordering cost coefficient c_4d/q (compare expression (7)) exceeds the optimal cost obtained so far, the solution of the case considered in Figure 4 requires the search of 120 strips with a total computation time of 7.80 seconds.

We now investigate the performance of the solution method. This aspect is of interest, because the number of strips to be searched may be quite large. In particular, we compare the exact method with two approximate methods. The first approximate method is based on using the optimal order quantity according to the model with continuous cost function, discussed in the preceding section, while the second is based on using the economic order quantity adapted to allow backlogging, which is given by

$$q = \sqrt{\frac{2dc_3(c_1 + c_2)}{c_1c_2}}.$$

The optimal order levels corresponding to both order quantities are computed by solving the piecewise linear equation $(\partial/\partial s)K(s, q) = 0$ (see Section 4). Note that for the first method we have not used the equation $(\partial/\partial s)K_1(s, q) = 0$, because this would lead to a slightly less accurate approximation. The order strategies obtained in these ways are then substituted into the expression (7) for the main cost function to find the associated costs. The second method reflects the approach generally used in practice to deal with uncertainty in the lead times. In order to reduce the model complexity, however, it is usually assumed that orders cannot cross each other. If this assumption holds, then for a fixed order quantity (mostly EOQ), the corresponding order level that achieves the required level of customer service can be obtained by solving a service equation. Because this approach is comparable with the second approximate method, the numerical results to be presented shortly, also give some insight in the circumstances under which the service equation approach yields satisfactory results.

We first compare the results obtained by the three methods for lead time distributions with the same expectation but with different coefficients of variation. In the sequel, the order strategies according to the continuous and EOQ approximations are denoted by order strategy 1 and order strategy 2, respectively. The comparison is carried out for the following model family:

$$\begin{aligned} d &= 100, \\ c &= [0.05, 0.95, 100, 50], \\ l &= [0, 7, 14j], \\ p &= \left[0.05j, 1 - 0.1 \frac{j^2}{2j-1}, 0.05 \frac{j}{2j-1} \right], \end{aligned} \quad (17)$$

where the parameter j is varied from 1 up to 13. It is easily verified that $E\{L\} = 7$ and $c_L^2 = 0.1j^2$. The numerical results for model family (17) are presented in

Figure 5: Cost comparison main model with approximate model for $j = 5$ Figure 6: Cost comparison main model with approximate model for $j = 10$

Table 1: Comparison of optimal order strategy and order strategy 1 for model family (17)

c_L^2	optimal order strategy				order strategy 1				relative difference $(K_1 - K)/K$
	s	q	K	t	s_1	q_1	K_1	t_1	
0.1	687	895	46.7	1.0	675	589	51.3	0.4	9.9×10^{-2}
0.4	677	1764	89.3	1.8	910	1181	94.5	0.4	5.8×10^{-2}
0.9	660	2701	135.8	2.5	695	2067	139.1	0.4	2.4×10^{-2}
1.6	640	3657	183.2	3.2	681	3017	186.6	0.4	1.8×10^{-2}
2.5	617	4622	230.9	3.9	660	3985	233.4	0.4	1.1×10^{-2}
3.6	590	5593	278.8	4.7	636	4963	280.7	0.4	7.0×10^{-3}
4.9	558	6570	326.7	5.2	607	5948	328.3	0.4	4.8×10^{-3}
6.4	521	7552	374.6	5.8	573	6939	375.9	0.4	3.5×10^{-3}
8.1	476	8542	422.6	6.5	532	7938	423.7	0.4	2.6×10^{-3}
10.0	421	9540	470.4	7.5	482	8946	471.4	0.4	2.0×10^{-3}
12.1	353	10549	518.3	8.1	419	9965	519.1	0.4	1.5×10^{-3}
14.4	265	11575	566.0	8.6	339	11001	566.7	0.5	1.2×10^{-3}
16.9	150	12624	613.5	9.7	232	12060	614.1	0.4	9.7×10^{-4}

Figure 7: Relative errors in cost of order strategies 1 and 2 for model family (17)

Figures 5, 6 and 7, and in Table 1. In Figures 5 and 6 we display for the cases corresponding to $j = 5$ and $j = 10$ the cost functions $K(q)$ and $K_1(q)$, which are introduced in (10) and (14), respectively. In Table 1 the numerical results obtained by the exact method and by the first approximate method are compared. Note that K and K_1 represent the costs associated with the optimal and the approximate order strategy, that is, $K = K(s(q), q)$ and $K_1 = K(s(q_1), q_1)$. Further, the times required to compute the optimal and the approximate solution are denoted by t and t_1 . Figure 7 shows the relative errors in cost of both approximate methods against the coefficient of lead time variation, on a logarithmic scale, for model family (17).

We start with a discussion of the cost functions $K(q)$ and $K_1(q)$, plotted in Figures 5 and 6. It is concluded again that it may be attractive to allow also order quantities $q < B$. However, for the cases considered in these figures, the cost increase owing to the order restriction $q \geq B$ is less significant than for the case considered in Figure 4. (In fact, the cost increases are about 9.2% and 8.0%, respectively.) Concerning the cost functions, displayed in Figures 5 and 6, it is concluded that the continuous approximation $K_1(q)$ is quite accurate. Notice that the cost function $K_1(q)$ is nonconvex for both cases. In essence, this behaviour is a consequence of the discontinuity of the cost function $K(q)$. For most inventory situations, the (continuous) cost function $K_1(q)$ represents a fairly accurate approximation of the (discontinuous) cost function $K(q)$. However, there exist situations for which the function $K_1(q)$ yields a poor approximation. In particular, the deviation from the cost function $K(q)$ may be considerable, if the order quantity q is large. (For instance, this is the case for the model corresponding to $j = 1$.) The question whether the approximation procedure, discussed in the preceding section, can be refined to obtain a fit with a higher degree of accurateness, is beyond the scope of the present study.

For the cases considered in Figures 5 and 6, the optimum of the cost function $K_1(q)$ yields a quite good approximation of the optimum of the cost function $K(q)$. Because of the above, this is of course to be expected. In contrast, the EOQ approximation is very poor for both cases resulting in a high cost penalty. This is a consequence of the fact that the EOQ approximation takes no account of lead time variability. For the two examples it turns out that, because of the high lead time variabilities ($c_L^2 = 2.5$ and $c_L^2 = 10.0$, respectively), the order quantity which minimizes expected total cost is much larger than the (adapted) economic order quantity. However, it is also possible that the EOQ approximation overestimates the optimal order quantity. For example, the optimal order quantity and adapted economic order quantity corresponding to the model considered in Figure 4 are given by 262 and 649, respectively.

We now discuss the efficiency of the two approximate methods. From Table 1 it is seen that the effort required to compute the optimal solution increases as the lead time variation increases. (Note that all computation times are given in seconds.) In view of the discontinuous behaviour of the cost function $K(s, q)$, resulting in a large number of strips to be searched when the lead time variability is large, this effect on the total effort is of course to be expected. In contrast, the effort required to compute the approximation based on a continuous cost function remains nearly constant and is comparatively small. For this approximate method only a few convex regions, in general, need to be searched. It is noted that the effort required to compute the EOQ approximation, which is not considered in Table 1, is negligible. For this method only the optimal order level corresponding to the (adapted) economic order quantity need to be computed. In conclusion, both approximate methods require considerably less computation time than the exact method.

For the cases considered in Table 1, the relative error of the approximation based on a continuous cost function decreases as the lead time variation increases. Further, the approximate results are quite accurate except when the lead time variation

is too small. This exception on the applicability of the approximation, however, is no real drawback, because the time required to compute the exact solution is reasonably small for the case with moderately varying lead times. Moreover, the model involving simultaneous order arrivals has been developed mainly to deal with situations in which the lead times are very uncertain. From Figure 7 we see that, for all cases considered, the second approximation, based on the (adapted) economic order quantity, yields a poor result and is less accurate than the first approximation. Because an error of approximately 10% may lead to a high cost penalty, the EOQ approximation is, in general, inappropriate for situations with uncertain lead times. This inappropriateness of the EOQ approximation is explained by the involved *fixed* order quantity, which depends only on the inventory, backloging and ordering costs but not on the lead time process (see also the above discussion concerning Figures 5 and 6).

It is apparent from Table 1 that the cost penalty due to lead time variability may be very high. Note that the total cost for the case the lead times are constant is given by 30.8. Hence, the profits to be gained by reducing the lead time variability may be enormous. (For instance, by setting up a close relationship with the supplier, on a mutual beneficial basis.) In the present study we deal with the inventory situation where (high) lead time variation cannot be avoided.

We next compare the numerical results obtained by the three methods for varying expected lead time. The comparison concerns the following model family:

$$\begin{aligned} d &= 100, \\ c &= [0.01, 0.19, 50, 25], \\ l &= [1, 2, 3, 4, 5, 14j - 7, 14j, 14j + 7], \\ p &= \left[\frac{10}{35}, \frac{8}{35}, \frac{6}{35}, \frac{4}{35}, \frac{2}{35}, \frac{1}{28}, \frac{2}{28}, \frac{1}{28} \right], \end{aligned} \quad (18)$$

where the parameter j is varied from 1 up to 13. It is easily verified that $E\{L\} = 2 + 2j$ and $c_L^2 = (6j^2 - 2j + 11/8) / (j + 1)^2$. The numerical results for model family (18) are presented in Table 2 and Figure 8. In Table 2 we compare the exact method and the first approximate method. Figure 8 shows the relative cost errors of both approximate methods against the expected lead time, on a logarithmic scale, for model family (18). The following conclusions are made.

From Table 2 it is seen that the time required to compute the exact solution increases as the expected lead time increases, whereas the time required to compute the first approximation remains nearly constant and is comparatively small. A similar conclusion holds with regard to the cases considered in Table 1 (see above). Notice that for model family (18) the variation of the lead time increases with increasing expectation, which is a simple consequence of the fact that only the long lead times are varied. The (slight) increase of the computation times in comparison with those in Table 1 is mainly due to the increase in the number of different lead times.

It is evident from Figure 8 that, for all cases considered, the results obtained by the first approximate method are very good and more accurate than those obtained by method 2. For most cases, the EOQ approximation has an error of about 10% resulting in a high cost penalty. So we conclude again that, in general, the EOQ approximation is inappropriate to deal with inventory situations in which the lead times are very uncertain. In contrast, the results obtained by the approximate method based on a continuous cost function are mostly very satisfactory. Consequently, in inventory situations with lead time uncertainty where a small cost penalty is acceptable, the first approximate method is generally preferable, because this method requires considerably less computation time than the exact method.

Table 2: Comparison of optimal order strategy and order strategy 1 for model family (18)

		optimal order strategy				order strategy 1				relative difference
$E\{L\}$	c_L^2	s	q	K	t	s_1	q_1	K_1	t_1	$(K_1 - K)/K$
4.0	1.3	962	1300	20.0	2.0	986	1253	20.0	0.6	4.3×10^{-4}
6.0	2.4	2454	940	30.0	4.3	2467	918	30.0	0.5	1.0×10^{-4}
8.0	3.1	3885	686	38.0	6.3	3880	639	38.2	0.5	5.4×10^{-3}
10.0	3.6	5275	459	45.2	8.2	5255	503	45.2	0.6	1.1×10^{-3}
12.0	3.9	6604	406	51.9	10.0	6585	437	52.1	0.5	3.7×10^{-3}
14.0	4.2	7912	401	58.6	11.8	7896	400	59.2	0.5	1.0×10^{-2}
16.0	4.4	9217	405	65.2	13.3	9219	377	65.5	0.5	5.2×10^{-3}
18.0	4.6	10524	401	71.8	14.9	10536	360	71.9	0.5	2.6×10^{-3}
20.0	4.7	11847	407	78.3	16.6	11862	348	78.4	0.6	6.9×10^{-4}
22.0	4.8	13177	409	84.9	17.9	13184	339	85.7	0.5	9.2×10^{-3}
24.0	4.9	14523	348	91.4	19.8	14516	331	92.3	0.5	9.1×10^{-3}
26.0	5.0	15849	356	98.1	21.0	15848	326	98.9	0.6	7.5×10^{-3}
28.0	5.0	17184	342	104.7	22.7	17182	321	105.3	0.5	5.8×10^{-3}

Figure 8: Relative errors in cost of order strategies 1 and 2 for model family (18)

Table 3: Influence of minimum order size and packing size on total cost

minimum order size	packing size	optimal order strategy				relative cost increase
		s	q	K	t	
64	1	6846	231	219.7	21.1	0.0×10^{-0}
	4	6844	232	220.2	17.6	2.4×10^{-3}
	16	6852	224	220.2	15.5	2.4×10^{-3}
	64	6820	256	221.2	14.3	6.8×10^{-3}
256	1	6827	346	220.7	7.6	4.6×10^{-3}
	4	6836	256	221.1	7.0	6.8×10^{-3}
	16	6836	256	221.1	5.8	6.8×10^{-3}
	64	6820	256	221.2	4.9	6.8×10^{-3}
1024	1	6659	1038	248.8	2.3	1.3×10^{-1}
	4	6656	1040	249.0	2.3	1.3×10^{-1}
	16	6660	1040	249.0	2.2	1.3×10^{-1}
	64	6628	1024	250.4	2.1	1.4×10^{-1}

From Table 2 it is seen that the cost increase owing to the increase in the expected lead time is enormous. In particular, if the difference between the short lead times (delivery from a nearby depot) and the long lead times (delivery from a remote depot) is very large, the involved total cost is very high. So we conclude that enormous profits may be gained by reducing this difference. (For example, by contracting an expensive but reliable supplier.)

For the inventory situations considered in Table 2, we see that a high uncertainty in the lead times results in a small optimal order quantity and a (very) large optimal order level. Clearly, the model with simultaneous order arrivals allows more flexibility in meeting the difficulties concerned with highly varying lead times. In particular, by ordering more frequently during the long lead times prevents unnecessary high inventory and backlogging levels during the short lead times. The extra ordering cost involved is amply compensated by the reduced inventory and backlogging costs, if the long lead times occur only incidentally. This is the case for the models considered in Table 2.

Finally, we investigate the influence of the minimum order size and the packing size on the total cost. Because the search for the optimum is restricted to integer order strategies only, such an investigation can be carried out easily by expressing the demand d in packing units and adapting the cost coefficients c_1 and c_2 accordingly. The following model is considered:

$$d = 100,$$

$$c = [0.05, 0.95, 50, 25],$$

$$l = [1, 2, 3, 4, 5, 56, 63, 70, 77, 84],$$

$$p = [0.25, 0.2, 0.15, 0.1, 0.05, 0.025, 0.05, 0.1, 0.05, 0.025].$$

It is easily verified that $E\{L\} = 19.25$ and $c_L^2 = 2.36$. The obtained results are shown in Table 3. From this table it is seen that, for the sizes considered, the influence of the packing size on the total cost is negligible. Furthermore, the time to compute the optimal solution decreases with increasing packing size, which is of

course to be expected. In conclusion, requiring that deliveries take place in pack sizes containing more than one unit results in a negligible increase in cost and a possibly considerable decrease in computation time.

It is apparent from Table 3 that, if the minimum allowable order quantity is too large, the resulting increase in cost may be considerable. A minimum order quantity is usually specified by the supplier, because the transportation of goods in too small quantities may be unprofitable. In such a situation, however, it may be advantageous for both parties to set up an arrangement that allows also order quantities below the specified limit. Under this arrangement, the supplier receives a financial compensation, when the transportation quantity turns out too small. By introducing higher ordering costs (in particular a higher start-up ordering cost) for order quantities below the specified limit, the difference in transportation costs can easily be incorporated into the model with simultaneous order arrivals.

7 Generalizations

In practice it often occurs that quantity discounts (for instance on the purchase price) are given. The most common situation is that there exist several intervals on which different 'all units' discounts hold. (Thus, each discount applies to *all* units in a certain interval.) The model with simultaneous order arrivals can be easily modified to allow for quantity discounts. For each region with a constant discount tariff, the cost function is of the form (7) but with different cost coefficients. Hence, the optimization procedure, discussed in Section 4, can be used to obtain the optimum in each discount region. The overall optimum can then be found by computing the total cost, including the total purchase cost, associated with each discount optimum. Because in general the number of different discount tariffs is small, the incorporation of quantity discounts into the model with simultaneous order arrivals gives rise to a small number of additional discontinuities of the involved cost function. Consequently, the effort required to compute the overall optimum only slightly increases with respect to the situation without quantity discounts. Because the jumps in the cost function owing to the different discount tariffs may be large, an approximation of the involved cost function by a continuous cost function is pointless. So it is only meaningful to apply the approximate method, discussed in Section 5, to each discount region separately. An overall approximate solution can then be found by computing the total costs associated with each approximate discount optimum. In consequence, the time required to compute the overall approximate solution increases proportionally with the number of different discount tariffs.

We next mention some possible generalizations of the lead time mechanism. In practical inventory situations the successive lead times are frequently dependent of each other. Our approach is based on incorporating this dependency into the model explicitly. In the present study such an analysis is carried out for the model in which simultaneous order arrivals are allowed. A possible generalization is to describe the successive times of delivery in an ordering cycle by a (delayed) renewal process, that is, an order placed during the lead time of the preceding order does not arrive simultaneously with the preceding order, but requires an additional lead time $\bar{L}$, where the distribution of $\bar{L}$, in general, differs from that of L . (Note that L denotes the lead time of the first order in an ordering cycle.) It is easily seen that the involved inventory process is again regenerative, so that by computing the expected length and cost during a regeneration cycle an expression for the long-run expected cost per unit of time can be obtained. However, it is not yet clear whether the resulting cost function is tractable for optimization with respect to s and q . Another possible generalization is to maintain the simultaneous arrival mechanism

but to let a new (follow-up) order cause a certain fixed delay of all outstanding orders. Several other generalizations of the lead time mechanism are also possible. The generalization that will be considered in a future research mainly depends on the relevancy to a practical inventory situation.

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