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Publication date:
1989

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Citation for published version (APA):

Bossert, W., & Stehling, F. (1989). *On the Uniqueness of Cardinaly Interpreted Utility Functions*. (CentER Discussion Paper; Vol. 1989-19). CentER.

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No. 8919

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CARDINALLY INTERPRETED UTILITY FUNCTIONS

by Walter Bossert and
Frank Stehling

May, 1989

**ON THE UNIQUENESS OF
CARDINALLY INTERPRETED UTILITY FUNCTIONS**

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ABSTRACT

In this paper, we reexamine the problem of uniqueness of cardinally interpreted utility functions. It is shown by some realistic examples that the frequently claimed uniqueness up to increasing affine transformations is invalid in many plausible cases. The main result of the paper provides a necessary and sufficient condition on the range of a cardinal utility function for its uniqueness up to affine transformations, if the differences between two values of this function cannot be arbitrarily small.

1. Introduction

Utility functions have become a useful tool in various branches of economic theory as well as in other social sciences (see, for instance, Chipman [1960], Chipman et al. [1971], Fishburn [1988] for discussions). It is often convenient to work with real-valued representations of preference orderings over sets of alternatives instead of using these orderings themselves, since this translation of preference orderings over arbitrary sets into the relation " $\succeq$ " over the set of real numbers allows for the application of standard analytical methods. Furthermore, using utility functions enables us to formulate various assumptions about the information contained in utility numbers (or utility vectors, if we are concerned with group decisions rather than individual decisions).

If we interpret a utility function representing some preference ordering as being ordinal, the only information contained in that function is ordinal information, i.e., such an ordinally interpreted utility function can only be used to make statements like "alternative x is considered at least as good as alternative y ", but neither are the utility values attached to some alternatives meaningful, nor can we say anything about the ranking of utility differences. This is the case, because ordinally interpreted utility functions over arbitrary nonempty sets of alternatives are unique only up to arbitrary increasing transformations.

Cardinally interpreted utility functions, on the other hand, allow – by definition – for the comparison of utility differences in the sense that statements such as " x is more preferred to y than z is preferred to w " are meaningful. It is easy to see that for cardinally interpreted utility functions not any arbitrary increasing transformation contains the same information as the given utility function itself, if this function has at least three distinct indifference classes.

Hence, for an ordinal utility function U , the set of all informationally equivalent utility functions is given by the set of increasing transformations of U , whereas for cardinally interpreted utility functions the set of all informationally equivalent utility functions must be a subset of the set of all increasing transformations. This set of informationally equivalent functions for cardinally interpreted utility functions will be the object to be studied in this paper.

It is often asserted in the literature that this subset is given by all increasing affine transformations of a given cardinally interpreted utility function, i.e., that cardinal utility functions are unique up to increasing affine transformations. However, the validity of this assertion depends crucially on the utility function under consideration, in particular, on its range. As will be demonstrated below, it is possible to find quite realistic examples violating this uniqueness property.

The fact that this uniqueness does not hold in general has been noted in earlier contributions as well; however, a complete answer to the question when this uniqueness holds and when it is violated is, to the best of our knowledge, still missing in the literature (see also Roberts [1979, p. 141]). It is the purpose of this paper to contribute to filling that gap by providing a necessary and sufficient condition for this uniqueness in the case where utility differences cannot be arbitrarily small. It will also be shown that this condition is not sufficient in general, i.e., if utility differences can, indeed, be arbitrarily small. Hence, some possible cases remain unresolved. However, in particular, all cases where the set of alternatives is finite are covered by our result. Since in reality most of the sets of alternatives we can consider in fact are finite, we feel that our result can be applied to a rather broad variety of decision problems.

The paper is organized as follows: Section 2 contains the basic definitions, a few remarks on the notation used in this paper, and some preliminary results. Section

3 provides examples of cardinally interpreted utility functions some of which are unique up to increasing affine transformations and some of which are not. Our main result – a characterization of the set of cardinally interpreted utility functions which are unique up to increasing affine transformations, if there exists a minimal (positive) difference between any two elements of the range of the utility function – is stated and discussed in Section 4. Section 5 summarizes and concludes the paper. The proof of our main theorem can be found in the Appendix.

2. Notation , Definitions , and Preliminary Results

We will denote the set of positive (resp. all) real numbers by $\mathbb{R}_{++}$ (resp. $\mathbb{R}$). $\mathbb{N}$ stands for the set of positive integers. The expression $\sup(A)$ ($\inf(A)$) is used to denote the supremum (infimum) of $A \subseteq \mathbb{R}$. Let Ω be an arbitrary nonempty set of alternatives. A binary relation over Ω is a nonempty subset R of $\Omega \times \Omega$; for notational convenience, we will write xRy for $(x,y) \in R$. The asymmetric and symmetric components of R (denoted by P and I , resp.) are defined by

For all $x,y \in \Omega$:

$$\begin{aligned} xPy & \text{ if and only if } xRy \text{ and not } yRx, \\ xIy & \text{ if and only if } xRy \text{ and } yRx. \end{aligned}$$

R is an ordering if and only if R is reflexive, transitive, and complete.² A preference ordering over Ω is an ordering R with the interpretation

xRy if and only if $x \in \Omega$ is considered at least as good as $y \in \Omega$.

A real-valued function $U: \Omega \rightarrow \mathbb{R}$ is a representation of R if it satisfies

For all $x, y \in \Omega$: xRy if and only if $U(x) \geq U(y)$.

Note that if some U represents R , then so does any increasing transformation of U ; furthermore, increasing transformations of U are the only functions representing the same preference ordering as U itself. This result can be found, e.g., in Krantz et al. [1971, p. 42], see Theorem 1 below. To be able to state this theorem formally, we need some further definitions. Given a utility function $U: \Omega \rightarrow \mathbb{R}$, define

$$W(U) := \{u \in \mathbb{R} \mid \exists x \in \Omega \text{ such that } U(x) = u\}.$$

Then a function $\phi: W(U) \rightarrow \mathbb{R}$ is increasing (on $W(U)$) if and only if, for all $u, v \in W(U)$, $u > v$ implies $\phi(u) > \phi(v)$.

Now we can state

Theorem 1:

Let Ω be a nonempty set of alternatives, and let $U: \Omega \rightarrow \mathbb{R}$ be a utility function representing some preference ordering R over Ω . Then a function $V: \Omega \rightarrow \mathbb{R}$ represents the same preference ordering R if and only if there exists an increasing function $\phi: W(U) \rightarrow \mathbb{R}$ such that $V(x) = \phi(U(x))$ for all $x \in \Omega$.

Proof: See Krantz et al. [1971, p. 42].

If we interpret U just as a function translating the ordering R over Ω into the ordering " $\geq$ " over $\mathbb{R}$, Theorem 1 tells us that such a function is unique up to arbitrary increasing transformations. In that case, we say that U is interpreted as an ordinal utility function (for R).

On the other hand, a function $U: \Omega \rightarrow \mathbb{R}$ representing an ordering R over Ω is interpreted as a cardinal utility function, if comparisons of utility differences are meaningful. That is, if we interpret U as a cardinal utility function, we can compare the intensities of preference between pairs of alternatives. It is clear that, in general, cardinally interpreted utility functions are not unique up to increasing transformations, since a statement such as

$$U(x) - U(y) \geq U(z) - U(w)$$

for some $x, y, z, w \in \Omega$ is not necessarily preserved, if we apply an arbitrary increasing transformation to U . In case $W(U)$ contains less than three elements, though, cardinality cannot be distinguished from ordinality; we will rule out this rather degenerate case in what follows by assuming that $W(U)$ contains at least three elements.

Interpreting U as a cardinal utility function requires that we restrict the set of possible transformations to those preserving the cardinal information contained in U . It has become common in the literature to say that U is interpreted as a cardinal utility function if and only if it is unique up to increasing affine transformations. (An increasing function $\phi: W(U) \rightarrow \mathbb{R}$ is affine if and only if there exist $a \in \mathbb{R}_{++}$, $b \in \mathbb{R}$ such that $\phi(u) = a \cdot u + b$ for all $u \in W(U)$.) However, as will be demonstrated below, this is not true in general; the validity of this assertion crucially depends on the structure of $W(U)$. On the other hand, it is easy to show that any increasing affine transformation of U contains the same cardinal information as U ; for some U , though, those are not the only transformations possessing that property. To illustrate that, we first formulate the corresponding information invariance condition for admissible transformations and then translate this condition into a Cauchy functional equation. The general solution of this equation depends crucially on the structure of $W(U)$.

In light of the above discussion, admissible transformations $\phi: W(U) \rightarrow \mathbb{R}$ for cardinally interpreted utility functions U have to satisfy

For all $x, y, z, w \in \Omega$:

$$\begin{aligned} U(x) - U(y) \geq U(z) - U(w) \quad &\text{if and only if} \\ \phi(U(x)) - \phi(U(y)) \geq \phi(U(z)) - \phi(U(w)). \end{aligned} \quad (1)$$

This condition can be reformulated as

For all $s, t, u, v \in W(U)$:

$$u - v \geq s - t \quad \text{if and only if} \quad \phi(u) - \phi(v) \geq \phi(s) - \phi(t).$$

It can be seen that this condition is equivalent to the requirement that the difference $\phi(u) - \phi(v)$ has to be a function f of the expression $u - v$, i.e.:

$$\text{For all } u, v \in W(U): \quad \phi(u) - \phi(v) = f(u - v) \quad (2)$$

where f is, by Theorem 1, an increasing real-valued function with the domain

$$D(U) := \{d \in \mathbb{R} \mid \exists u, v \in W(U) \text{ such that } u - v = d\}.$$

It is sufficient to restrict our attention to the set

$$D^{++}(U) := \{d \in \mathbb{R}_{++} \mid \exists u, v \in W(U) \text{ such that } u - v = d\},$$

since, by (2), we have $f(0) = 0$ and $f(-d) = -f(d)$ for all $d \in D(U)$, i.e., if we know the values of f on $D^{++}(U)$, we know the values of f for the whole of its domain $D(U)$. We can, without loss of generality, assume that $0 \in W(U)$,³ and hence, ϕ can be recovered from f by setting $\phi(u) = f(u) + \phi(0)$. Clearly, ϕ is an increasing

affine function if and only if f is linear, i.e., $f(d) = a \cdot d$ for all $d \in D(U)$ with a positive constant a . Now observe that we must have

$$f(u - v) + f(v - t) = f(u - t) \quad \text{for all } u, v, t \in W(U) \quad (3)$$

where we can, without loss of generality, assume that $u > v > t$. Alternatively, we can formulate (3) using a concatenation operation: Define a binary operation $B(U) \subseteq D^{++}(U) \times D^{++}(U)$ by

$$(d, d') \in B(U) \quad \text{if and only if} \\ \exists u, v, t \in W(U) \ (u > v > t) \text{ such that } u - v = d, v - t = d'.$$

Then (3) can be written as

$$f(d) + f(d') = f(d + d') \quad \text{for all } (d, d') \in B(U), \quad (4)$$

i.e., f has to satisfy a Cauchy functional equation (see Aczél [1966, p. 31] or Eichhorn [1978, p. 3]). However, (4) need not hold for all $d, d' \in D^{++}(U)$, but only for those points $d, d' \in D^{++}(U)$ which can be concatenated according to (3). Consequently, the solution of (4) does not only depend on the domain $D^{++}(U)$, but also on the structure of $B(U)$.

If $W(U)$ is "nicely behaved" (we will be more precise below), the only solutions of (4) are linear functions, which implies that ϕ must be affine. For instance, if $W(U)$ is a nonempty, nondegenerate interval, it follows that ϕ is an affine function (see, e.g., Aczél [1987, pp. 20]). However, as will be demonstrated in the following section, it is not hard to find realistic examples for $W(U)$ that allow for further solutions of (4), so that the frequently claimed uniqueness of cardinally interpreted utility functions up to increasing affine transformations is not valid in

these cases.

3. Some Examples

This section is devoted to a few intuitive examples to point out that the problem addressed here indeed shows up in many practical applications.

Example 1:

As a simple example, suppose Ω consists of the possible grades obtainable for a university course, e.g., $\Omega = \{\text{first class, second class, failed}\}$, or $\Omega = \{1, 2, F\}$, for short. An obvious choice for the preference ordering of a student over Ω is 1P2PF. Then a plausible utility function $U: \Omega \rightarrow \mathbb{R}$ with cardinal interpretation could be the one defined by

$$U(1) = 1, \quad U(2) = 0, \quad U(F) = -5,$$

since the difference between passing and failing is much more severe than the difference between a first class and a second class. Obviously, we have

$$U(1) - U(F) > U(2) - U(F) > U(1) - U(2).$$

Now consider the following representation $V: \Omega \rightarrow \mathbb{R}$ of the same ordering:

$$V(1) = 1, \quad V(2) = 0, \quad V(F) = -10.$$

Again,

$$V(1) - V(F) > V(2) - V(F) > V(1) - V(2),$$

i.e., we have the same ranking of utility differences (which means that V contains the same cardinal information as U), but obviously, V cannot be expressed as an affine transformation of U .

At first sight, one might be tempted to think that the reason for this result is the "discrete" nature of Ω ; this is, however, not the case. To see that, observe the following two examples:

Example 2:

Let $\Omega = [0,1]$. This set of alternatives could, for instance, be interpreted as the possible shares of some good which an individual possesses. We assume that this individual's preferences ordering over Ω is given by

$$\text{For all } x, y \in \Omega: \quad xRy \text{ if and only if } x \geq y. \quad (5)$$

Now suppose that R is represented by

$$U(x) = \begin{cases} -1 & \text{for } x = 0 \\ x & \text{for } x \in (0,1]. \end{cases}$$

A person with this cardinally interpreted utility function finds it extremely unpleasant not to possess anything at all; consequently, the utility difference between having a positive share and having nothing at all is greater than any utility difference between two positive shares. The function $V: \Omega \rightarrow \mathbb{R}$ defined by

$$V(x) = \begin{cases} -2 & \text{for } x = 0 \\ x & \text{for } x \in (0,1] \end{cases}$$

contains the same cardinal information as U ; again, though, V is not an affine transformation of U .

Example 3:

Let $\Omega = \mathbb{N}$, and let R be given by (5) (see Example 2). Choose the representation

$$U(x) = x \quad \text{for all } x \in \Omega,$$

and suppose that U is interpreted as a cardinal utility function. It can be shown that the only functions containing the same cardinal information as U must be increasing affine transformations of U . Since this is a special case of our result in the following section, we do not provide a detailed proof here.

The uniqueness of U in the above example is due to the fact that the elements of $W(U)$ are "regularly spaced". If we disturb this regularity slightly in a certain way, this uniqueness vanishes:

Example 4:

As in Example 3, let $\Omega = \mathbb{N}$ and let R be given by (5). We leave it to the reader to check that the representations U and V given by

$$U(x) = \begin{cases} 3/2 & \text{for } x = 1 \\ x & \text{for } x \in \Omega \setminus \{1\}, \end{cases}$$

$$V(x) = \begin{cases} 5/4 & \text{for } x = 1 \\ x & \text{for } x \in \Omega \setminus \{1\}, \end{cases}$$

contain the same cardinal information but are not affine transformations of each

other.

It was mentioned before that $W(U)$ being an interval is sufficient for U to be unique up to affine transformations. However, counterexamples do exist, if $D^{++}(U)$ is an interval (but not $W(U)$). Consider, as a final example,

Example 5:

Let $W(U) = (0,1] \cup [2,3)$. Then $D^{++}(U)$ is given by the interval $(0,3)$, but we can define a transformation $\phi: W(U) \rightarrow \mathbb{R}$ by

$$\phi(u) = \begin{cases} u & \text{for } u \in (0, 1] \\ u + 1 & \text{for } u \in [2, 3). \end{cases}$$

Clearly, ϕ is not an affine function, but $V: \Omega \rightarrow \mathbb{R}$, defined by

$$V(x) = \phi(U(x)) \quad \text{for all } x \in \Omega$$

contains the same cardinal information as U in the sense of condition (1).

4. A Necessary and Sufficient Condition

In this section, we provide a necessary and sufficient condition for U to be unique up to increasing affine transformations, if we require $D^{++}(U)$ to possess a minimal element, i.e., the utility differences cannot be arbitrarily small. As will be demonstrated later, our condition is not sufficient, if this additional requirement is dropped.

Now we can state

Theorem 2:

Let Ω be a nonempty set and let $U: \Omega \rightarrow \mathbb{R}$ be a cardinally interpreted utility function representing some preference ordering over Ω . Let $W(U)$ contain at least three distinct elements, and let U be such that $D^{++}(U)$ contains a minimal element $d_{\min}$. Then U is unique up to increasing affine transformations if and only if

$$\text{For all } d \in D^{++}(U) \setminus \{d_{\min}\}: \exists (d', d'') \in B(U) \text{ such that } d = d' + d''. (6)$$

Proof: See the Appendix.

Condition (6) requires that all possible (positive) differences greater than $d_{\min}$ must be concatenable using two (smaller) differences $(d', d'') \in B(U)$. Loosely speaking, (6) is sufficient for uniqueness, because the Cauchy equation (4) must be satisfied for all $d, d' \in D^{++}(U)$ such that $(d + d') \in D^{++}(U)$. Necessity comes from the fact that a violation of (6) allows us some "flexibility" in constructing an increasing function f satisfying (4). For illustration, we conclude this section by explaining why the examples of the foregoing section where $D^{++}(U)$ contains a minimal element satisfy resp. violate (6).

In Example 1, we have $D^{++}(U) = \{1, 5, 6\}$. Clearly, the difference $d=5$ cannot be obtained as a concatenation of some pair $(d', d'') \in B(U)$. On the other hand, it is easy to verify that the utility function of Example 3 satisfies (6). Finally, observe that in Example 4, we have $D^{++}(U) = \{n/2 \mid n \in \mathbb{N}\}$, and we cannot find $(d', d'') \in B(U)$ such that $d' + d'' = 1 \in D^{++}(U)$, where $d_{\min} = 1/2$.

5. Concluding Remarks

The main concern of this paper was to further investigate the problem of deriving the set of informationally equivalent utility functions for some given function U , assuming that U has a cardinal interpretation. By providing some examples, we demonstrated that it is not true in general that cardinally interpreted utility functions are unique up to increasing affine transformations; even more, quite plausible counterexamples violating this uniqueness property can be found.

Whereas this and similar frequently occurring inaccuracies were also pointed out by other authors (see, e.g., Falmagne [1975, 1981]), a complete answer to the question when this uniqueness holds and when it does not has not been given in the literature yet. The present paper attempted at contributing to filling this gap by providing a sharp dividing line in terms of a necessary and sufficient condition on the range of the utility function under consideration, if utility differences cannot be arbitrarily small.

It should be noted that we were not dealing with the problem of the existence of cardinal utility functions (or, more general, of algebraic difference systems; see, e.g., Krantz et al. [1971]), a problem which was analyzed in numerous contributions; rather, we were assuming a cardinally interpreted utility function to be given and examined its uniqueness properties. Obviously, there are links between this type of literature and the present paper. In providing sufficient conditions for the existence of cardinally interpreted utility functions (or difference scales), the uniqueness properties of these representations often follow as a corollary to the existence result. However, the sufficiency conditions for the existence are typically too strong to be necessary for uniqueness up to affine transformations at the same time.

In light of our Theorem 2, the main conclusion to be drawn from this paper is, in our opinion, the necessity of taking care in making use of uniqueness properties of cardinally interpreted utility functions. The necessary and sufficient condition presented in this paper provides a helpful device for checking the suitability of employing the usually assumed uniqueness property, at least in the case where minimal differences exist.

In the case where $D^{++}(U)$ can contain arbitrarily small elements, the uniqueness problem analyzed in this paper still is unresolved. If utility differences can be arbitrarily small, condition (6) alone is not sufficient for the uniqueness up to affine transformations any more: For example, if U is such that $W(U) = (0,1) \cup (2,3)$, we have $D^{++}(U) = (0,1) \cup (1,3)$, and it is not hard to verify that (6) is satisfied. However, we can define

$$\phi(u) = \begin{cases} u & \text{for } u \in (0,1) \\ u + 1 & \text{for } u \in (2,3), \end{cases}$$

and the resulting transformation of U (which is not affine) contains the same cardinal information as U . Consequently, further conditions are needed to establish a result in the same spirit as our Theorem 2 in this case.

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APPENDIX: Proof of Theorem 2

(a) "If": Suppose that U is a cardinally interpreted utility function, that $D^{++}(U)$ contains a minimal element $d_{\min} =: d_1$, and that (6) is satisfied. First, we show that the "next-to-smallest" element in $D^{++}(U)$ must be $2 \cdot d_{\min} =: d_2$, i.e., the set

$$\{d \in D^{++}(U) \mid d > d_{\min}\}$$

has a minimal element given by $2 \cdot d_{\min}$. (6) requires, for all

$d \in D^{++}(U) \setminus \{d_{\min}\}$, the existence of $(d', d'') \in B(U)$ such that $d = d' + d''$. Since $d', d'' \geq d_{\min}$, it follows $d \geq 2 \cdot d_{\min}$ for all $d \in D^{++}(U) \setminus \{d_{\min}\}$. If $2 \cdot d_{\min}$ would not be contained in $D^{++}(U)$, we would contradict (6), since this would imply the existence of a nonconcatenable $d \in D^{++}(U) \setminus \{d_{\min}\}$.

Now it follows by induction, from (6) and from the existence of $d_{\min}$, that $D^{++}(U)$ must consist of (not necessarily all) multiples of $d_{\min}$ which we denote by $d_3, d_4, \dots$ in increasing order. If we set

$$f(d_1) = c = a \cdot d_1 \quad (a = c/d_1),$$

we obtain, using (4),

$$f(d_2) = f(d_1 + d_1) = 2 \cdot f(d_1) = 2 \cdot a \cdot d_1 = a \cdot d_2.$$

We can now conclude part (a) of the proof by induction. Suppose it is true that

$$f(d_k) = a \cdot d_k \quad \text{for all } k = 1, \dots, m-1 \quad (m \in \mathbb{N}).$$

Then we must have, by (6) and (4),

$$f(d_m) = f(d_i + d_j) \text{ for some } (d_i, d_j) \in B(U), d_i, d_j < d_m.$$

It follows

$$f(d_m) = f(d_i + d_j) = f(d_i) + f(d_j) = a \cdot (d_i + d_j) = a \cdot d_m,$$

and the proof of sufficiency is complete.

(b) "Only if": Suppose there exists a nonconcatenable $\hat{d} \in D^{++}(U) \setminus \{d_{\min}\}$, i.e., a $\hat{d} \in D^{++}(U) \setminus \{d_{\min}\}$ such that

$$\nexists (d', d'') \in B(U) \text{ such that } \hat{d} = d' + d''.$$

Define $\hat{n} \in \mathbb{N}$ as the smallest positive integer satisfying

$$\hat{n} \cdot d_{\min} < d^0 \leq (\hat{n} + 1) \cdot d_{\min} \text{ for some nonconcatenable } d^0 \in D^{++}(U) \setminus \{d_{\min}\}.$$

Clearly, $\hat{n}$ is well-defined, and it follows by the minimality property of $\hat{n}$ that all $d \in D^{++}(U)$ with $d \leq \hat{n} \cdot d_{\min}$ must be concatenable. Furthermore, with the same reasoning as in part (a), all those $d \in D^{++}(U)$ must be multiples of $d_{\min}$. We now define an increasing function $f: D(U) \rightarrow \mathbb{R}$ satisfying (4) which is not linear. This will establish the existence of a non-affine admissible transformation $\phi: W(U) \rightarrow \mathbb{R}$ which completes the proof of necessity of (6). Of course, it is sufficient to define f on $D^{++}(U)$; the values of f on the rest of its domain can be retrieved unambiguously given that information.

First, define

$f(d) = d$ for all $d \in D^{++}(U)$ such that $d \leq \hat{n} \cdot d_{\min}$

and

$f(d) = (\hat{n} \cdot d_{\min} + d)/2$ for all $d \in D^{++}(U) \cap (\hat{n} \cdot d_{\min}, (\hat{n}+1) \cdot d_{\min})$.

If $(\hat{n}+1) \cdot d_{\min}$ is in $D^{++}(U)$ and concatenable with some $(d', d'') \in B(U)$, define

$$f((\hat{n}+1) \cdot d_{\min}) = f(d') + f(d'').$$

If $(\hat{n}+1) \cdot d_{\min}$ is in $D^{++}(U)$ and nonconcatenable, define

$$f((\hat{n}+1) \cdot d_{\min}) = (\hat{n} \cdot d_{\min} + (\hat{n}+1) \cdot d_{\min})/2.$$

By definition of $\hat{n}$, all elements of $D^{++}(U) \cap (\hat{n} \cdot d_{\min}, (\hat{n}+1) \cdot d_{\min})$ must be nonconcatenable. If $D^{++}(U) \cap (\hat{n} \cdot d_{\min}, (\hat{n}+1) \cdot d_{\min})$ is empty, $(\hat{n}+1) \cdot d_{\min}$ is nonconcatenable by assumption. Note that this guarantees the nonlinearity of f .

For $d \in D^{++}(U)$, $d > (\hat{n}+1) \cdot d_{\min}$, we define f successively, where we have to distinguish several cases in order to ensure that increasingness as well as (4) is satisfied for these elements of $D^{++}(U)$ as well. Starting with $k=1$, construct f for all $d \in D^{++}(U) \cap ((\hat{n}+k) \cdot d_{\min}, (\hat{n}+k+1) \cdot d_{\min})$ successively for all $k=1, 2, \dots$ in the following way:

(i) If d is concatenable, i.e., $\exists (d', d'') \in B(U)$ such that $d = d' + d''$, define

$$f(d) = f(d') + f(d'').$$

(ii) If d is nonconcatenable, we obtain two subcases. To simplify notation, we define

$$L(d) := \{\bar{d} \in D^{++}(U) \mid \bar{d} < d \text{ and } [\bar{d} \text{ is concatenable or } \bar{d} = d_{\min}]\},$$

$$G(d) := \{\bar{d} \in D^{++}(U) \mid (\hat{n}+k+1) \cdot d_{\min} \geq \bar{d} > d \text{ and } \bar{d} \text{ is concatenable with } (d', d'') \in B(U)\},$$

$$L_f(d) := \{f(\bar{d}) \mid \bar{d} \in L(d)\},$$

$$G_f(d) := \{f(\bar{d}) \mid \bar{d} \in G(d)\}.$$

As the first subcase, suppose that we have

(ii.1) $\sup(L(d)) = d$. Then define

$$f(d) = \sup(L_f(d)).$$

(ii.2) If $\sup(L(d)) < d$, we again have to consider two possibilities:

(ii.2.a) If $G(d)$ is empty, define

$$f(d) = (\sup(L_f(d)) + d)/2.$$

Finally, in the case

(ii.2.b) $G(d)$ is nonempty, define

$$f(d) = d \cdot [\inf(G_f(d)) - \sup(L_f(d))] / [\inf(G(d)) - \sup(L(d))] +$$

$$[\sup(L_f(d)) \cdot \inf(G(d)) - \sup(L(d)) \cdot \inf(G_f(d))] / [\inf(G(d)) - \sup(L(d))].$$

The function f is well-defined, increasing, and satisfies (4). As a straightforward consequence, the corresponding $\phi(U): \Omega \rightarrow \mathbb{R}$ contains the same cardinal information as U . By definition, f is not linear and hence ϕ is not affine. This completes the proof of Theorem 2. $\square$

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FOOTNOTES

¹Parts of the work on this paper were carried out while Walter Bossert was visiting the Center for Economic Research at Tilburg University. The hospitality and the support of Center and its members are gratefully acknowledged.

²Reflexivity, transitivity, and completeness are defined as follows:

Reflexivity: xRx for all $x \in \Omega$,

Transitivity: $((xRy \text{ and } yRz) \text{ implies } xRz)$ for all $x, y, z \in \Omega$,

Completeness: $(x \neq y \text{ implies } (xRy \text{ or } yRx))$ for all $x, y \in \Omega$.

³If $0 \notin W(U)$, we can consider the function $\hat{U}: \Omega \rightarrow \mathbb{R}$ defined by

$$\hat{U}(x) = U(x) - U(\hat{x}) \text{ for all } x \in \Omega,$$

where $\hat{x} \in \Omega$ is arbitrarily chosen. Clearly, $0 \in W(\hat{U})$, $\hat{U}$ contains the same cardinal information as U , and we can proceed with $\hat{U}$ instead of U .

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