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THE CLOSEDNESS OF THE FREE-DISPOSAL HULL OF A PRODUCTION SET*

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Our purpose in this note is to give a new method of proving the closedness of the free-disposal hull of a production set, Y , which is based on using the lattice structure of the commodity space. It may be of interest that in our approach the convexity of Y is not needed for the result.

Consider a multi-product industry with output bundles modelled as points of an ordered topological vector space, L . The problem we deal with arises when, as is usual in practical examples, the industry's production possibilities (or, equivalently, its production costs) are originally specified only for nonnegative product bundles. This is done in terms of a set, Y , consisting of those commodity bundles, (y, a) , each of which represents a nonnegative output bundle, $y \in L_+$, that the industry can produce from an input bundle, $-a \in R^N$. (The possibility that in some production plans some components of a are positive, i.e., that the industry can on occasion produce some of the commodities it uses as inputs, is not excluded. Also, it is only for simplicity that the inputs are taken to consist of a finite number, N , of homogeneous commodities, and this is not essential.) With free disposal added, the industry's production set is equal to the set

$$(1) \quad Y - L_+ \times R_+^N,$$

which is called the free-disposal hull of Y . For equilibrium existence proofs, one needs to show that this set is closed (on the assumption that Y is closed). Since L_+ is closed, this is a special case of the question about the closedness of the sum of closed sets, and a result of Khan and Vohra (1987) can be used for this purpose if: (i) L is the continuous dual of an ordered Banach space (or, more generally, of an ordered Frechet space, i.e., of a complete, metrizable, ordered locally convex space), (ii) Y is convex, and (iii) a (topological) boundedness assumption -- of the kind usually made for equilibrium existence proofs to ensure the weak* compactness of the set of all allocations -- is made (see the Remark below). Other known sufficient conditions, mentioned by Khan and Vohra (1987), can also be useful for proving the closedness of

the free-disposal hull: for example, if L is the space of measures, $\mathcal{M}$, on a compact set, with the weak* topology (for the duality with the space of continuous functions), then the nonnegative cone, $\mathcal{M}_+$, is weakly* locally compact, and the criterion of Dieudonne (1966, Proposition 1) can be applied. When L is a topological vector lattice, we propose another approach, based on representing the free-disposal hull of Y as the set, Y^+ , defined by

$$(2) \quad Y^+ = \{(y, a) \in L \times \mathbb{R}^N \mid (y^+, a) \in Y\}.$$

The closedness of Y^+ follows from that of Y since the operation of taking the nonnegative part, i.e., the mapping $y \rightarrow y^+$, is continuous.

Proposition 1: Assume that L is a topological lattice. If Y is closed, then Y^+ is also closed.

The required representation of the free-disposal hull, given in Formula (5) below, holds if Y has the following monotonicity properties:

$$(3) \quad \text{If } (y, a) \in Y \text{ and } 0 \leq y' \leq y, \text{ then } (y', a) \in Y;$$

and

$$(4) \quad \text{If } (y, a) \in Y \text{ and } a' \leq a, \text{ then } (y, a') \in Y.$$

Proposition 2: Assume that L is a vector lattice. Then:

- (i) If Y has the properties (3)-(4), then $Y^\dagger = Y^\dagger - L_+ \times R_+^N$.
- (ii) If $Y \subset L_+ \times R^N$, then $Y \subset Y^\dagger$.
- (iii) Assume that Y has the properties (3)-(4) and that $Y \subset L_+ \times R^N$. Then

$$(5) \quad Y^\dagger = Y - L_+ \times R_+^N.$$

Also,

- (iv) If Y is a cone, then $Y^\dagger$ is a cone.
- (v) If Y is convex and has the property (3), then $Y^\dagger$ is convex.

Proof: Given their assumptions, Part (i) holds by the monotonicity of the mapping $y \rightarrow y^+$. Part (ii) follows directly from the definition of $Y^\dagger$, given in Formula (2). To prove Part (iii), note that the set on the right-hand side is contained in the set on the left-hand side by Parts (i) and (ii). The reverse inclusion follows directly from the definition of $Y^\dagger$ and from the fact that $y \leq y^+$. Part (iv) holds because $(\lambda y)^+ = \lambda y^+$ for every nonnegative scalar, λ . Given its assumptions, Part (v) holds by the convexity of the mapping $y \rightarrow y^+$. Q.E.D.

Example: The space of essentially bounded real-valued functions on a sigma-finite measure space, Ξ , denoted by L^∞ , is a topological vector lattice when taken with the Mackey topology, $\tau(L^\infty, L^1)$, for the duality with the space of integrable functions, L^1 . This is a special case of a result stated by Aliprantis and Burkinshaw (1978, Chapter 6, Exercise 4 on p. 163); in that reference, the term "a locally convex-solid Riesz space" is used to mean "a locally convex lattice". It follows that Propositions 1 and 2 can be applied to a production set, Y , in the commodity space $L^\infty \times R^N$ with the Mackey topology. This space is used by Horsley and Wróbel (1989, 1990a, 1990b) to set up a model of competitive equilibrium pricing for time-differentiated commodities

(which are usually also differentiated over events of delivery) such as electricity and water, when these commodities are supplied either by price-taking, profit-maximizing industries or by public utilities pricing their products at exact marginal cost. The points of Ξ describe the characteristics of the differentiated commodity, such as the date and/or event. In peak-load pricing problems, e.g., for electricity and water, part of the capital cost is proportional to the highest output level, $\text{ess sup}_{\xi \in \Xi} y(\xi)$, and this is a Mackey lower semicontinuous function of y . The lower semicontinuity is essential for showing that the production set Y is Mackey closed. By Propositions 1 and 2, Mackey closedness of $Y^\dagger$ follows from that of Y . If Y is convex, then its Mackey closedness is equivalent to its weak*-closedness (i.e., the $\sigma(L^\infty \times \mathbb{R}^N, L^1 \times \mathbb{R}^N)$ -closedness), and it can also be shown in the way discussed below.

Remark: Take L to be the continuous dual, E^* , of an ordered Frechet space (i.e., of a complete, metrizable, ordered locally convex space), (E, τ) , with the weak* topology, $\sigma(L, E)$, and with the dual nonnegative cone (which is the polar, $(-E_+)^*$, of the nonpositive cone in E). Under the assumptions: (i) that the set $Y \cap (L_+ \times \mathbb{R}_+^N - B)$ is weakly* bounded for every weakly* bounded subset, B , of $L \times \mathbb{R}^N$, and (ii) that Y is convex, the $\sigma(L, E)$ -closedness of $Y - L_+ \times \mathbb{R}_+^N$ follows from a result of Khan and Vohra (1987, Theorem). That result is applicable because, by the Krein-Smulian theorem, every Frechet space, F , is strictly hypercomplete (i.e., a convex subset of F^* is $\sigma(F^*, F)$ -closed if its intersection with every weakly* bounded and closed set in F^* is $\sigma(F^*, F)$ -closed): see, e.g., Edwards (1965, Theorem 8.10.5 on p. 539), or, for the case of a Banach space, see Dunford and Schwartz (1958, Theorem V.7.5 on p. 429). This is used with F equal to the product of E and $\mathbb{R}^N$, taken with the product of τ and the usual topology of $\mathbb{R}^N$. Also, note that a subset of the continuous dual, F^* , of a Frechet space is weakly* bounded if and only if it is equicontinuous, by the Banach-Steinhaus theorem: see, e.g., Edwards (1965, Theorem 7.1.3 on p. 464). If F is a Banach space,

then it follows that a subset of F^* is equicontinuous if and only if it is bounded in the dual norm.

The industry's production possibilities are often specified in terms of the production cost, $C(y, q)$, of any nonnegative output, $y \in L_+$, at any input prices, $q \in \mathbb{R}^N$. For any fixed q , we study C as a function of y alone. The extension of C to the whole of L that describes the addition of free disposal is defined by

$$(6) \quad C^\dagger(y) = C(y^+).$$

since, if C is the cost function derived from a production set, Y , then $C^\dagger$ is the cost function derived from the free-disposal hull, $Y^\dagger$, of Y . This is because

$$(7) \quad C^\dagger(y) = C(y^+) = \inf \{-\langle a, q \rangle \mid (y^+, a) \in Y\} = \inf \{-\langle a, q \rangle \mid (y, a) \in Y^\dagger\}.$$

The extension of C defined by Formula (6) is studied and used by Horsley and Wrobel (1988, 1989, 1990a, 1990b).

Proposition 3: Assume that L is a topological vector lattice.

(i) If $C(y)$ is lower semicontinuous in y , then so is $C^\dagger(y)$.

(ii) If $C(y)$ is nondecreasing in y , then

$$(8) \quad \text{epi } C^\dagger = \text{epi } C + L_+ \times \mathbb{R}_+,$$

where $\text{epi } C$ denotes the epigraph of C (as a function of y).

(iii) If $C(y)$ is convex and nondecreasing in y , then $C^\dagger(y)$ is convex (and nondecreasing) in y .

Proof: Part (i) follows by the continuity of the mapping $y \rightarrow y^+$. To prove Part (ii) note that, since C is nondecreasing,

$$\begin{aligned} (9) \quad C^+(y) &= C(y^+) = \inf \{C(x) \mid x \geq 0, x \geq y\} \\ &= \inf \{C(x) + \delta(z \mid X_-) \mid x \geq 0, z = y - x\}, \end{aligned}$$

where $\delta(\cdot \mid L_-)$ is the (convex) indicator function of L_- , i.e., $\delta(z \mid L_-) = 0$ for $z \in L_-$ and $\delta(z \mid L_-) = +\infty$ for $z \in L \setminus L_-$. Since the last infimum in Formula (9) is attained (for $x = y^+$ and $z = -y^-$), Formula (8) follows. Part (iii) follows from the result on the convexity of composite mappings given, e.g., by Ioffe and Levin (1972, Lemma 1 of §8 on p. 39), since the mapping $y \rightarrow y^+$ is convex. Q. E. D.

Parts (i) to (iii) of Proposition 3 correspond to Proposition 1, Part (iii) and Part of Proposition 2, respectively. They are equivalent to their counterparts when the production set, Y , is defined as the hypograph of $-C$, i.e.,

$$(10) \quad Y = \{(y, a) \in L_+ \times \mathbb{R} \mid a \leq -C(y)\}.$$

This is the case if inputs are measured in aggregate, on the assumption -- made in much of the literature on industrial pricing -- that the prices, q , of the given industry's inputs are fixed. In this case all inputs can be aggregated into a homogeneous numeraire commodity (by positing that, to all intents and purposes, a unit of commodity n is equivalent to q_n units of the numeraire commodity, for $n = 1, 2, \dots, N$), and the commodity space is then $L^\infty(\Xi) \times \mathbb{R}$.

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FOOTNOTES:

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