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Pieter H. M. Ruys

**On the existence of an equilibrium
for an economy with
public goods only**

Research memorandum

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*T equilibrium theory
T public goods*

TILBURG INSTITUTE OF ECONOMICS
DEPARTMENT OF ECONOMETRICS

TILBURG INSTITUTE OF ECONOMICS

Department of Econometrics
Tilburg, the Netherlands

Research Memorandum 32:

ON THE EXISTENCE OF AN EQUILIBRIUM FOR AN ECONOMY
WITH PUBLIC GOODS ONLY.

By Pieter H.M. Ruys

July, 1971

Preliminary and Confidential.

The author is grateful to Dr.H.N.Weddepohl for his remarks and suggestions, which have prevented some mistakes. This paper will be presented at the European Meeting of the Econometric Society, Barcelona, September 1971.

1. Introduction.

This Paper proposes to prove the existence of an equilibrium, which is also a Pareto optimum, in an economy with public goods only. The supply of a public good is supposed to be consumed fully by every consumer. An equilibrium is defined as a bundle of goods, for which the sum of individual marginal benefit-valuations or "prices" equals the social cost of that bundle. The benefit sets and a cost set are dual sets of the individual preferred sets and the production possibility set. These sets are elements of a valuation representation of the economy $E = \{(X_i, \prec_i), Z, w_i\}$ in the dual space:

$E^* = \{(P_i, \prec_i^*), Q, w_i\}$, from which the original economy E can be derived again. This dual economy for public goods has properties which are very similar to the economy for private goods as described by Debreu (1959).

The existence of an equilibrium for an economy with public and private goods is proven by Foley (1967) and Fourgeaud (1969). Fourgeaud uses a social preference preordering on the subspace of public goods. Foley proves the existence of

an equilibrium for any given Pareto optimum and initial distribution of resources. In this paper no private goods are present; assumed is that all choice sets have a least or greatest element and that indispensable goods have valuation correspondences with some monotonic behavior. And also in contrast with the other writers, all preferences preorderings belong to individual consumers and no Pareto optimum need to be given for the proof of existence. This proof is based on the construction of sets on which the Kakutani fixed point theorem can be applied.

2. The valuation representation in the dual space.

The economy E under consideration is characterised by the following definitions and seven assumptions:

An economy $E = \{(X_i, \preceq_i), Z, w_i\}$ consists of m consumers, $i \in M$, each having a nonempty consumption set X_i (in the n - dimensional vector space E^n) on which a preference preordering $\preceq_i$ is defined. There are n commodities, $j \in N$, produced by one producer according to a production possibility or supply set Z . The producer disposes over all initial resources, which is already calculated in his supply set, and the wealth or the value of consumption is distributed over the individuals according to the function w_i .

ASSUMPTION 1: (on consumers) The consumption sets X_i are contained in the nonnegative orthant Ω and such that the closure of X_i equals Ω . The preference preordering $\preceq_i$ on X_i is complete, convex, continuous and such that $x \leq y$ implies $x \preceq_i y$, for every consumer.

ASSUMPTION 2: (on producers) The supply set Z is closed, convex, upper-bounded and contains all and only elements of E^n that are smaller ($\leq$) than any nonnegative element of Z . The vector of initial resources is positive and an element of Z .

ASSUMPTION 3: (on public goods) The consumption of every consumer equals total supply, for every commodity.

The other assumptions depend on valuations of consumers and the producer, which can be derived from the given economy E as follows:

The valuation space $(E^n)^*$ is the set of all valuations or "prices" (covectors) which can assign a value (a real number) to any commodity bundle in the commodity space (E^n) . In this space will be defined a set of feasible valuations for the producer (the social cost set) and sets of feasible valuations for the consumers with an preference preordering on each of them.

The social cost set, Q , is the set of all valuations which can assign to any bundle in the supply set a value not larger than 1:

$$Q = \{q \in (E^n)^* : qz \leq 1, \text{ for } z \in Z\}$$

Assumption (2) on Z implies that Q is also closed, convex upper-bounded and contains all and only elements of $(E^n)^*$ that are smaller ($\leq$) than any nonnegative element of Z , but under the restriction that the set Q is completely contained in the nonnegative orthant. Also is true that the set dual to Q is equal again to the supply set Z . (See e.g. Shephard, 1970, p. 236)

If the producer maximizes the value of a bundle to be supplied by him (as will be assumed later) and he is constrained to a valuation which assigns to the bundle a value of at most one, he will take some element of the social cost correspondence, $q(z)$, on the boundary of Q :

$$q(z) = \{q \in Q : qz = \max p z, \text{ for } p \in Q\}$$

PROPERTY 1: $q(z)$ is an upper hemi continuous correspondence from Z in Q .

proof: If $m(q) = qz$, a continuous function in the compact set Q and if $Q(z) = Q$, a continuous (constant) correspondence from Z in Q such that $Q(z) \neq \emptyset$ for any $z \in Z$, then according to the Maximum Theorem (Berge, 1959, p. 122), $M(z) = \max \{m(q) = qz : q \in Q\}$ is a continuous function in Z and $q(z) = \{q \in Q : m(q) = M(q)\}$ is an u.h.c. correspondence from Z in Q .

To every individual preferred to y - set, $X_i(y)$, an individual benefit to y - set, $P_i(y)$, can be assigned which contains all valuations that give any bundle equivalent to or better than y for consumer i a value of at least one:

$$X_i(y) = \{x \in X_i : y \preceq_i x\},$$

$$P_i(y) = \{p \in (E^n)^* : px \geq 1, \text{ for } x \in X_i(y)\}.$$

Assumption (1) implies that both sets are closed, convex, contained in the nonnegative orthant and containing all elements that are greater ($\leq$) than any element of the set concerned. Again, the dual set of $P_i(y)$ is equal to $X_i(y)$. (See Weddepohl, 1970, p. 70). Equivalently, $P_i(y)$ can be written as:

$$P_i(y) = \{p \in (E^n)^* : M_i(p) \cap \text{Int } X_i(y) = \emptyset\}, \text{ where}$$

$$M_i(p) = \{x \in X_i : px \leq 1\}$$

The set $M_i(p)$ is called the consumers' "budget" set or choice set given the valuation p .

By definition, the consumers' individual valuation set, P_i , consists of all valuations that consumer i can give to any bundle x in his consumption set X_i , according to the defini-

tions given above:

$$P_i = \cup P_i(x), \text{ for } x \in X_i.$$

$$P_i = \{p \in (E^n)^* : M_i(p) \cap \text{Int } X_i(x) = \emptyset, \text{ for some } x \in X_i\}$$

By definition, a choice set for the i -th consumer is a correspondence from P_i in X_i : $M_i(q) = \{x \in X_i : qx \leq 1\}$, or from X_i in P_i : $M_i(y) = \{p \in P_i : py \leq 1\}$. If a choice set $M_i(p)$ has no greatest elements for $\preceq_i$, an indifference correspondence converges asymptotically to the boundary of $M(p)$, the hyperplane $L(p) = \{x \in E^n : px = 1\}$. There does not exist an equilibrium in this choice situation for the consumer and he would be willing to substitute large quantities of any commodity for the one (or more) he feels indispensable. $L(p)$ is then called a hyperplane of indispensability, where p is a covector of zero's except for the components (p_j) , commodity j being indispensable. If one wants to be sure that once a commodity is considered indispensable at a certain level, a lower level of that commodity cannot be feasible (or be an element of X_i), then only one hyperplane of indispensability can exist on the boundary of X_i and does not belong to X_i . Such a hyperplane of indispensability can be considered as a degenerate choice set. By assumption 1, it can only be a boundary of the nonnegative orthant, for every consumer. The above described situation is also realised if (See assumption 4) the consumers' preferences guarantee that all choice sets in X_i have greatest elements for $\preceq_i$. As choice sets in X_i are generated only by valuations which are elements of P_i , the assumption that some elements can a priori not belong to P_i (which could be possible according to the definition of P_i) rules out a certain class of preference preorderings on X_i , viz. those in which indifference correspondences converge asymptotically to a choice set in X_i . The assumption is equivalently stated as:

$$P_i = \{p \in (E^n)^* : M_i(p) \cap \text{Int } X_i(x) = \emptyset \text{ and}$$

$$M_i(p) \cap X_i(x) \neq \emptyset, \text{ for some } x \in X_i\}.$$

On the consumers' individual valuation set P_i , a preference relation $\prec_i^*$ is defined as follows:

$$p \prec_i^* q \iff \exists x \in X_i : p \in P_i(x) \text{ and } q \in \text{Bd } P_i(x)$$

$$p <_i^* q \iff p \prec_i^* q \text{ and not } p \succ_i^* q$$

$$p \sim_i^* q \iff p \prec_i^* q \text{ and } p \succ_i^* q$$

This preference relation will be shown a complete preordering on P_i and has all characteristics of $\prec_i$ on X_i . However, the economic relevance of $\prec_i^*$ is a derivated one: if for the i -th consumer two valuations are equivalent, then the best commodity bundles he can get in the choice sets determined by the two valuations are equivalent for him. In this sense one can say that a smaller valuation or "price" is always better: $p \leq q$ implies $p \succ_i^* q$.

The consumer will be supposed (See assumption 5) to choose a least element for $\prec_i^*$ out of the choice set $M_i(y) = \{p \in P_i : py \leq 1\}$. These elements are given by the individual benefit correspondence, $p_i(y)$, from X_i in P_i :

$$p_i(y) = \{p \in P_i : p \in M(y) \text{ and } q \prec_i^* p \text{ implies } q \notin M(y)\}.$$

Equivalently one can say that the consumer assigns valuations to y from his benefit to y -set, $P_i(y)$, that minimize the value py :

$$p_i(y) = \{p \in P_i : py = \min qy, \text{ for } q \in P_i(y)\}.$$

The choice of a least element out of $M(y)$ can be considered as a vote of marginal valuation of y , given a maximum weight of 1 to the inner product. But also this consideration does not solve completely the problems of correct revelation! The four last assumptions can now be stated, given the definitions above.

ASSUMPTION 4: (on consistent valuation) All choice sets for any consumer contain greatest (ζ_i) or least (ζ_i^*) elements.

ASSUMPTION 5: (on behaviour) All consumers choose a greatest (ζ_i) or a least (ζ_i^*) element in their choice sets. The producer maximizes the value of his supply.

ASSUMPTION 6: (on distribution) The values of individual consumption, W_i , are uniformly distributed over the consumers.

ASSUMPTION 7: (on indispensable and insatiable goods) If a commodity j is indispensable for consumer i : $[x:(x_j) = 0] \text{ implies } x \notin X_i$, then the valuation of j , $(p_i(x)_j)$, increases monotonically with a decrease of the supply of j , (x_j) . If a consumer i is insatiable for commodity j :

$[p : (p_j) = 0]$ implies $p \notin P_i$, then the demand of j , (x_j) , increases monotonically with a decrease in (p_j) .

The last assumption is needed in the proof of section 3 only for locally monotonic behaviour of the individual benefit correspondence near a limit of X_i which is not an element of X_i .

Examples of indifference sets in X_i and P_i and classes which are ruled out are given in figure 1: broken lines are excluded by assumption 4, and a brace, $\}$, indicates asymptotic convergence.

fig. 1: indifference sets in X_i and P_i .

PROPERTY 2:

A valuation representation $E^* = \{(P_i, \sim_i^*), Q, W\}$ can be derived from the economy

$E = \{(X_i, \sim_i), Z, W\}$, such that both preference relations have the same properties and that E can be derived from E^* again, given the assumptions 1, 2, 4 and 6.

proof:

The assumptions 1 and 2 on $X_i(x)$ and Z satisfy the conditions which Weddepohl (1971) proved sufficient for $(K^*)^* = K$, where the dual $K^* = \{p \in (E^n)^* : x \in \text{Int } K \text{ implies } px \neq 1\}$, which is equivalent to the definitions used here for consumers, resp. the producer. Assumption 6 justifies the use of the real number 1 in the definition of the dual for all consumers.

That $\prec_i^*$ on P_i is a complete preference preordering can be shown as follows:

$$p \prec_i^* p \iff \exists x : p \in \text{Bd } P_i(x) \text{ and } p \in P_i(x),$$

follows from the definition and implies reflexivity.

If $p \prec_i^* q$ and $q \prec_i^* r$, then $\exists x, y \in X_i : p \in P_i(x), q \in \text{Bd } P_i(x), q \in P_i(y)$ and $r \in \text{Bd } P_i(y)$.

From $q \in [\text{Bd } P_i(x) \cap P_i(y)]$, follows:

$$\text{Bd } P_i(y) \not\subset P_i(x) \text{ unless } P_i(y) = P_i(x)$$

Therefore two alternatives are open:

$$\text{Bd } P_i(x) \subset P_i(y) \quad \text{or} \quad \text{Bd } P_i(y) \cap P_i(x) \neq \emptyset.$$

The last alternative will be proven to imply the first, from which follows that $p \in P_i(y)$ and $p \prec_i^* r$, or that $\prec_i^*$ is transitive.

For, suppose that $\text{Bd } P_i(x) \cap P_i(y) \neq \emptyset$ and

$\text{Bd } P_i(x) \not\subset P_i(y)$, or: some x and y have a

valuation in common, but still $x \succ_i y$ (See

case C in figure 1). Then there exists a $p \in P_i$ such that:

$$M(p) \cap \text{Int } X_i(y) = \emptyset \quad \text{and} \quad M(p) \cap X_i(y) \neq \emptyset \quad \text{and}$$

$$M(p) \cap \text{Int } X_i(x) = \emptyset \quad \text{and} \quad M(p) \cap X_i(x) \neq \emptyset \quad \text{and}$$

$$q \leq p \quad \text{implies} \quad M(p) \cap X_i(x) \neq \emptyset.$$

($\leq$ stands for $\subseteq$, but not $=$)

Therefore, for any $z \in X_i$, $z >_i y$ implies $z \prec_i x$.

Also, for a sequence $\{q\}$ such that $q \leq p$,

$$q \rightarrow p \quad \text{implies} \quad [M(p) \cap X_i(x)] \rightarrow \emptyset \quad (2.1)$$

Transitivity of $\prec_i$ on X_i assures that:

$$x \prec_i y \quad \text{implies} \quad [M(p) \cap X_i(x)] \supset [M(p) \cap X_i(y)],$$

$$\text{for any } p \in P_i \quad (2.2)$$

Assumption 4 guarantees that for all q :

$0 \leq q \leq p$, there exists a z , where $z >_i y$,

such that $M(q) \cap X_i(z) \neq \emptyset$.

Therefore, a correspondence $x_i(p)$ from P_i in X_i

can be defined:

$$x_i(p) = \{x \in X_i : M(q) \cap \text{Int } X_i(z) = \emptyset \quad \text{and}$$

$$M(q) \cap X_i(z) \neq \emptyset, \text{ for some}$$

$$z \in X_i\}$$

Continuity of $\prec_i$ on X_i requires that:

$q \rightarrow p$ implies $x_i(q) \rightarrow x_i(p)$, where $x_i(p) = M(p) \cap X_i(y) \neq \emptyset$, and $0 \leq q \leq p$ implies $z \succ_i y$, for z defined in $x_i(q)$ (2.3)

From (2.1) and (2.3) follows that there exists a q : $q \leq p$, close enough to p , such that $[M(q) \cap X_i(x)] \not\supseteq [M(q) \cap X_i(z)]$, where $z \succ_i x \succ_i y$. This contradicts transitivity of $\prec_i$ (See 2.2).

Completeness of $\prec_i^*$, finally, is implied by the definition of $P_i = \cup P_i(x)$, for $x \in X_i$. If $p \in \text{Bd } P_i(x)$ and $q \notin P_i(x)$ then there exists $y \in X_i$, such that $q \in \text{Bd } P_i(y)$ and $P_i(x) \subset P_i(y)$.

3. Equilibrium in the public economy E.

In an economy E in which all seven assumptions are met, a state is fully described by the $m + 2$ tuple of points $\{x, (p_i), q\}$. As the produced bundle x is supplied to all consumers to its full extent, one commodity is sufficient. For this same reason, all individual marginal valuations will probably be different. The necessary and sufficient conditions which Samuelson (1954) derived for a social optimum imply that the sum of individual benefit valuations must equal social cost for a specific commodity. This result is used here to define an equilibrium for the economy:

A public equilibrium of the economy E is a state $\{\tilde{x}, (\tilde{p}_i), \tilde{q}\}$ such that;

- (1) The valuation $\tilde{p}_i$ of the supply $\tilde{x}$ is a least element of the consumers' choice set $M_i(\tilde{x})$ for $\tilde{p}_i^*$;
- (2) $\tilde{q}$ maximizes $q \tilde{x}$ over Q ;
- (3) $\tilde{x} \in Z$ and $\frac{1}{m} \sum \tilde{p}_i = \tilde{q}$.

An example of an equilibrium situation is given in figure 2:

fig.2: An equilibrium for a public economy

Other definitions which will be used additionally in this section are:

the social consumption set, $X = \cap X_i$;

the social valuation set, $P = \cup P(x)$, where $P(x) = \frac{1}{m} \sum P_i(x)$;

the social benefit correspondence, $p(x) = \frac{1}{m} \sum p_i(x)$;

the excess benefit correspondence, $r(x) = p(x) - q(x)$;

the unit hyperplane, $L(x) = \{p \in (E^n)^* : p \cdot x = 1\}$.

A public equilibrium $\{\tilde{x}, (\tilde{p}_i), \tilde{q}\}$ implies that $L(\tilde{x})$ is a separating hyperplane of Q and $P(\tilde{x})$, where $Q \cap P(\tilde{x}) \neq \emptyset$, and such that all valuations $\tilde{p}_i$ are optimal relative to $\tilde{x}$ (they are least elements in $M_i(\tilde{x})$ for $\prec_i^*$ or minimize $p \cdot \tilde{x}$ in $P_i(\tilde{x})$.) It also implies that the excess benefit correspondence is zero for $\tilde{x}$: $r(\tilde{x}) = \{0\}$.

It is easily checked that a public equilibrium is a Pareto optimum. Let $y \succ_i \tilde{x}$ for all consumers and $y \succ_i \tilde{x}$ for at least one consumer. Then $P(y) \supset P(\tilde{x})$ and $P(y) \cap \text{Int } Q \neq \emptyset$. Therefore, $L(y) \cap Q \neq \emptyset$. But as $Z = Q^* = \{x \in E^n : L(x) \cap \text{Int } Q = \emptyset\}$, $y \notin Z$ and therefore not feasible.

The existence of an equilibrium in E can now be proven. The proof consists of 3 parts:

- (1) Closed sets $\bar{X}_i$ and compact sets $\bar{P}_i$ are constructed to define an $\tilde{E}$ economy.
- (2) The existence of an equilibrium in the $\tilde{E}$ economy will be proven, applying the Kakutani fixed point theorem.
- (3) The set of $\tilde{E}$ - equilibria will be shown equal to the set of E - equilibria.

THEOREM:

The economy E has an equilibrium.

proof: (1) The sets X_i and P_i in E are neither closed nor bounded. Closed and convex sets $\bar{X}_i \subset X_i$ are constructed by taking the difference.

$$\bar{X}_i = X_i \setminus N(\underline{X}_i),$$

where $N(\underline{X}_i)$ is an open neighborhood in Ω of $\underline{X}_i$, and $\underline{X}_i$ is the set of limit points of X_i that are no elements of X_i (or indispensable goods for the i -th consumer).

This neighborhood is constructed as the union of open halfspaces in E^n , determined by some valuations in the closure $\bar{P}_i$, which must be large enough to prevent the exclusion of equilibrium solutions:

$N(\underline{X}_i) = \{x \in \Omega : p \cdot x < 1, \text{ for } p \in \bar{P}_i\}$, where the set $\bar{P}_i \subset \{p \in \Omega : p \cdot x = 0, \text{ for } x \in \underline{X}_i\}$ is defined as follows:

According to the definition of $p(x)$, it can be considered as the sum of m covectors $[\frac{1}{m} p_i(x)]$ or as the average of m covectors $p_i(x)$, as is shown in figure 3.

fig.3: decomposition of $p(x)$.

If for a consumer i , $[\frac{1}{m} p_i(x)] \notin Q$, or $p_i(x) > m.q.$, for some $q \in \text{Bd } Q$, we are sure that $p(x) = \frac{1}{m} \sum p_i(x) \notin Q$.

fig.4: construction of $N(\underline{X}_i)$.

Commodity j is indispensable for the consumer (like commodity 2 in figure 4) if $(x_j) = 0$ implies $x \in \underline{X}_i$. Also is known that $x = 0$ is no element of X_i , or $0 \in \underline{X}_i$, for any consumer. This follows from $X_i(0) = \Omega$ and, for all finite $p \in \Omega$, $M(p) \cap \text{Int } \Omega \neq \emptyset$, precluding a choice set for $x = 0$, which excludes $x = 0$ from X_i by assumption 4.

To prevent that an equilibrium solution (with $p(x) \in Q$) is contained in $N(\underline{X}_i)$, we first determine the largest component of Q , (q_k) , and choose λ such that it is larger than $m.(q_k)$. If any $p_i(x)$ has a component equal to or larger than λ , we know that the social

benefit correspondence $p(x)$ is not an element of Q .

For each indispensable commodity j , choose

$\underline{p}_j = \lambda e_j$, where e_j is the covector of zero elements except for the j -th unit element. If $\underline{x} = L(\underline{p}_j)$, then for $x \in \{x : \underline{p}_j x < 1\}$ is the j -th component always smaller: $(x_j) < (\underline{x}_j)$.

This implies by assumption 7, that for

$x \in \{x : \underline{p}_j x < 1\}$, always $(p_i(x)_j) > (p_i(\underline{x})_j) = \lambda$, and $p(x) \notin Q$. For the zero supply, the neighborhood is constructed by choosing $\underline{p} = \lambda e$, where e is the covector of unit elements. It is easily checked that $\{x : \underline{p} x < 1\}$ contains no efficient points, by the choice of λ .

$N(\underline{X}_i) = \{x \in \Omega : p x < 1, \text{ for } p \in \underline{P}_i\}$, where

$\underline{P}_i = \{p \in \Omega : p = \lambda e_j \text{ or } p = \lambda e, \text{ where}$

$\lambda : \{x : x < \lambda e\} \supset m. Q \text{ and where}$

$e_j : e_j x = 0, \text{ for } x \in \underline{X}_i\}$.

The same procedure to obtain a closed set in P_i cannot be repeated, as x cannot be decomposed and no boundary can a priori be given in Z , below which no equilibrium solutions are present. But just as it was important to get a closed lower bound for X_i , it is important to get a closed upper bound for P_i . The lower bound closure of P_i may be achieved by simply taking the closure of P_i , $\bar{P}_i$, as elements from $\underline{P}_i = \bar{P}_i \setminus P_i$ will not be called anyhow by a consumer who receives only a limited supply of a commodity for which he is insatiable.

Compact sets $\tilde{P}_i \subset \bar{P}_i$ can now be constructed, using the property: $P_i = \cup p_i(x)$, for $x \in X_i$.

This property follows from $p_i(x) \subset \text{Bd } P_i(x)$ and $\text{Bd } P_i(x) = \bigcup p_i(y)$, for some $y \in \text{Bd } X_i(x)$ and given assumption 4. As $P_i = \bigcup \text{Bd } P_i(x)$, for $x \in X_i$, therefore $P_i = \bigcup p_i(x)$, for some $x \in \bigcup \text{Bd } X_i(y)$, $y \in X_i$, or from some $x \in X_i$.

For every $x \in \bar{X}_i$ such that $x \neq 0$, $p_i(x)$ is unbounded above, but closed, bounded below and ordered for $\leq$. Therefore $p_i(x)$ has a minimum. For every strictly positive $x \in \bar{X}_i$, $p_i(x)$ is bounded above, because the existence of a positive lower bound for all positive $x \in \bar{X}_i$ implies the existence of a positive upper bound for all p such that $p \cdot x = 1$. Call $\bar{p}$ the least upper bound of the minima of $p_i(x)$; if $p_i(x)$ is not ordered for $\leq$, the minimum $p_i(x)$ is considered equal to $p_i(x)$. We now can define $\tilde{P}_i$ as:

$$\tilde{P}_i = \{p \in \bar{P}_i : p \leq \bar{p}, \text{ where } \bar{p} = \text{l.u.b. of } \{\min. p_i(x) \text{ for } x \in \bar{X}_i\}\}.$$

With the construction of closed sets $\bar{X}_i$ and compact sets $\tilde{P}_i$, both convex, nonempty and containing all equilibrium points, the first part of the proof is accomplished.

- (2) The correspondence $p_i(x)$ from $\bar{X}_i$ in $\tilde{P}_i$ is upper hemi continuous, as follows from the Maximum Theorem (Berge, 1959, p. 122): as $m(p) = -p \cdot x$ is a continuous real function in $\tilde{P}_i$; $\tilde{P}_i(x) = P_i(x) \cap \tilde{P}_i$ is a continuous correspondence from $\bar{X}_i$ in $\tilde{P}_i$ such that $\tilde{P}_i(x) = \emptyset$ for any $x \in \bar{X}_i$ (by construction of $\tilde{P}_i$); $M(x) = \max \{m(p) : p \in \tilde{P}_i(x)\}$ is a continuous

real function in $\bar{X}_i$; therefore $p_i(x) = \{p : p \in \tilde{P}_i(x), m(p) = M(x)\} = \{p : p \cdot x = \min.$

$q \cdot x, \text{ for } q \in \tilde{P}_i(x)\}$ is an u.h.c. correspondence from $\bar{X}_i$ in $\tilde{P}_i$.

The social benefit correspondence $p(x)$ from $\bar{X} = \cap \bar{X}_i$ in $\tilde{P} = \frac{1}{m} \sum \tilde{P}_i$ is a sum or an average of u.h.c. correspondences and thus also u.h.c., $\bar{X}$ being closed, $\tilde{P}$ compact and both convex and nonempty.

The excess benefit correspondence $r(x) = p(x) - q(x)$ from $(\bar{X} \cap Z)$ in $(\tilde{P} - Q)$ is a difference of two u.h.c. correspondences and thus also u.h.c. (See figure 5).

fig.5: The feasible supply set, $\bar{X} \cap Z$, and the excess benefit set, $\tilde{P} - Q$.

A correspondence $x(r)$ from $(\tilde{P} - Q)$ in $(\bar{X} \cap Z)$ is defined as follows:

$$x(r) = \{x : r \cdot x = \min r \cdot y, \text{ for } y \in (\bar{X} \cap Z)\}.$$

Also this correspondence is u.h.c., as easily can be checked applying the same reasoning as

in the proof of property 1. The correspondence is also nonempty and convex (as it is an intersection of two convex sets: $(\bar{X} \cap Z)$ and a hyperplane).

Therefore, on the Cartesian product set, $S = (\bar{X} \cap Z) \times (\tilde{P} - Q)$, which compact, convex and nonempty, the product correspondence from S in $S: s = r \times x$, is u.h.c., convex and nonempty for every element of S . According to Kakutani, these conditions are sufficient for the existence of a fixed point $\tilde{s} = (\tilde{x}, \tilde{r})$. This fixed point has the following properties. From the definitions it follows that $p(\tilde{x}) \cdot \tilde{x} \geq q(\tilde{x}) \cdot \tilde{x}$, or $r(\tilde{x}) \cdot \tilde{x} \geq 0$, for all $\tilde{x} \in (X \cap Z)$; by the definition of $r(x)$ must be true that $\tilde{r} \cdot x \geq \tilde{r} \cdot \tilde{x}$, for all $\tilde{x} \in (X \cap Z)$. As the aureole closure of $(X \cap Z) = \Omega$, $r(\tilde{x}) \in \{r : r \cdot x \geq 0, \text{ for all } x \in \Omega\} = \Omega$. Also, $p(\tilde{x}) \in L(\tilde{x})$ and $q(\tilde{x}) \in L(\tilde{x})$, where $\tilde{x} \geq 0$. Therefore $p(\tilde{x}) \not\leq q(\tilde{x})$, or $r(\tilde{x}) \not\leq 0$. It follows that $r(\tilde{x}) = 0$, implying that there an $\tilde{x} \in (\bar{X} \cap Z)$ exists such that $p(\tilde{x}) = q(\tilde{x})$, or that $\tilde{x}$ is an equilibrium for $\tilde{E}$.

- (3) Finally, it will be shown that the set of equilibria for $\tilde{E} = \{(\bar{X}_i, \tilde{x}_i), (\tilde{P}_i, \tilde{x}_i^*), Z, Q, w_i\}$ equals the set of equilibria for $E = \{X_i, \tilde{x}_i, (P_i, \tilde{x}_i^*), Z, Q, w_i\} = \{(X_i, \tilde{x}_i), Z, w_i\}$.

An $\tilde{E}$ -equilibrium implies an E -equilibrium, as $\bar{X}_i \subset X_i$ and $\tilde{P}_i \subset P_i$, for all i , and boundary points of $\bar{X}_i$ and $\tilde{P}_i$ are excluded as equilibrium solution (unless they are also boundary points of X_i and P_i):

$\text{Bd } N(\underline{X}_i) = \{x \in X_i : p \cdot x = 1, \text{ for } p \notin (m \cdot Q)\}$

and $\tilde{p}_i \in \bar{P}_i \setminus P_i$ implies that the supply of a commodity j , for which a consumer is insatiable, is infinite; contra $\tilde{x} \in Z$.

An E-equilibrium implies an $\tilde{E}$ -equilibrium, as $\tilde{x} \in [(X_i \cap Z) \setminus \bar{X}_i]$ implies that $\tilde{x} \in N(\underline{X}_i)$ and

$p_i(\tilde{x}) \notin m \cdot Q$, or $p(\tilde{x}) > q(\tilde{x})$. If $\tilde{p}_i$ is a least element of $M_i(\tilde{x})$ and $\tilde{p}_i \in (P_i \setminus \tilde{P}_i)$, then

$\tilde{p}_i \in p_i(\tilde{x})$ with $\tilde{x} \in N(\underline{X}_i)$ and again $p(\tilde{x}) > q(\tilde{x})$.

Both contra $r(\tilde{x}) = 0$.

The conditions (2) and (3) for a public equilibrium are satisfied in both cases $\tilde{E}$ and E , either because the production circumstances are identical, or by definition.

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