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**On consistent solutions
for strategic games**

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FEW 732

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ON CONSISTENT SOLUTIONS FOR STRATEGIC GAMES

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Abstract

Nash equilibria for strategic games were characterized by Peleg and Tijs (1996) as those solutions satisfying the properties of consistency, converse consistency and one-person rationality.

There are other solutions, like the ε -Nash equilibria, which enjoy nice properties and appear to be interesting substitutes for Nash equilibria when their existence cannot be guaranteed. They can be characterized using an appropriate substitute of one-person rationality. More generally, we introduce the class of "personalized" Nash equilibria and we prove that it contains all of the solutions characterized by consistency and converse consistency.

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We are grateful to H. Norde for helpful comments.

1 Introduction

The starting point for this work was an attempt to see whether the axiomatizations for Nash equilibria given in Peleg and Tijs (1996) and in Peleg, Potters and Tijs (1996), could be adapted to get axiomatizations for ε -equilibria also.

In Peleg and Tijs (1996) it was proved that (OPR) (One Person Rationality), (CONS) (Consistency) and (COCONS) (Converse Consistency) can be used to provide an axiomatic characterization for Nash equilibria (briefly: NE). We refer to that paper for motivation and references to related work. Further references, in which the above mentioned results have been extended, are: Norde, Potters, Reijnierse and Vermeulen (1996), Peleg and Sudhölter (1994) and Shinotsuka (1994).

Looking for a characterization of ε -NE, it was clear that the key point was to replace (OPR) by $(\varepsilon\text{-OPR})$. By (OPR), in Peleg and Tijs (1996) it was named the requirement that for 1-person games the solution to be characterized coincides with payoff maximization. Clearly, $(\varepsilon\text{-OPR})$ should mean that for 1-person games we look for ε -maximizers instead of maximizers.

Being interested also in other kinds of approximate NE, like the $(\varepsilon\text{-}k)$ -NE introduced in Lucchetti, Patrone and Tijs (1986), and studied also in Jurg and Tijs (1993) and Norde and Potters (1995), we looked at a unified way to achieve this kind of axiomatic characterizations.

The solution is simple: it is sufficient to introduce a (possibly personalized) "choice rule" p for all of the potential players, and to substitute a conveniently defined $(\mathcal{P}\text{-OPR})$ for (OPR). In this way, we get the axiomatic characterizations of NE and of the various kinds of approximate NE in which we were interested: the distinction between the various solution concepts is simply done by means of an appropriate choice of the choice rules p .

To be more detailed, by means of a set $\mathcal{P}$ of choice rules p we define $\mathcal{P}\text{-NE}$. Then, $\mathcal{P}\text{-NE}$ is characterized by $(\mathcal{P}\text{-OPR})$, (CONS) and (COCONS). Furthermore, it is proved that under mild conditions every solution satisfying (CONS) and (COCONS) is in fact a $\mathcal{P}\text{-NE}$, for an appropriate family $\mathcal{P}$ of choice rules p : so, $\mathcal{P}\text{-NE}$ appear to be all of the

solutions which can be considered if we want to respect consistency (direct and converse).

Also a characterization of $\mathcal{P}$ -NE is provided by means of ($\mathcal{P}$ -OPR), (NEM) (non emptiness) and (CONS), along the lines of Peleg, Potters and Tijs (1996).

2 $\mathcal{P}$ -Nash equilibria

The main tool to define $\mathcal{P}$ -NE is the “choice rule” that was mentioned in the introduction. Actually, it will be assumed that different players may have different choice rules: i.e., we are prepared to consider “personalized” choice rules, which is reflected in the name of $\mathcal{P}$ -NE.

Definition 2.1 A choice rule is a pair $(\mathcal{U}, p)$, where:

- $\mathcal{U}$ is a nonempty set of real valued functions
- p is a map that to every $u \in \mathcal{U}$, $u : A \rightarrow \mathbf{R}$, associates a subset $p(u)$ of A .

Remark 2.1 We (implicitly) assume that every function $u \in \mathcal{U}$ is defined on a nonempty set. We do not assume that $p(u) \neq \emptyset$. □

The definition says, in words, that the rule $(\mathcal{U}, p)$ “chooses”, for a given function u , a subset of A , the domain of u . Usually, we shall refer to the rule simply by p . The set $\mathcal{U}$ will be called the domain of the rule.

For ease of reference, we shall use also the notation $p(A, u)$, where A stands for the domain of the function u . Clearly superfluous, but useful to give “a name” to the domain of u .

Before providing some examples, we introduce the following compatibility condition:

Definition 2.2 The choice rule $(\mathcal{U}, p)$ satisfies the compatibility condition if:

- (CC) given (A, u) and (B, v) , s.t. $u, v \in \mathcal{U}$ and A, B are their respective domains, assume that there is a bijection $f : A \rightarrow B$ s.t. $v \circ f = u$: then, $p(v) = f(p(u))$.

Clearly condition (CC) excludes a too wild behavior for the choice rule: it means a sort of “independence of irrelevant details”, and will provide to be very useful in the characterizations we shall prove in the following sections.

Notice that (CC) has nothing to do with utility transformations that leave the preferences unaffected. About preferences, let us point out that we could have considered choice rules based on preferences, instead of (utility) functions: however, for definiteness, we shall not pursue this point of view.

Example 2.1 In the cases (a)-(g), there is no particular restriction on the choice of the domain $\mathcal{U}$ for p . For definiteness, one could think of $\mathcal{U}$ as being $\mathcal{U}_F$, the set of all real-valued functions defined on all finite sets.

$$(a) \ p(A, u) = \operatorname{argmax}_A u$$

$$(b) \ p(A, u) = A$$

$$(c) \ p(A, u) = \{a \in A : u(a) \geq \sup_A u - \varepsilon\}, \text{ where } \varepsilon \geq 0 \text{ is given, independent of } u$$

$$(d) \ p(A, u) = \{a \in A : u(a) \geq k\}, \text{ where } k \in \mathbf{R} \text{ is given, independent of } u$$

$$(e) \ p(A, u) = \{a \in A : u(a) \geq \sup_A u - \varepsilon \text{ or } u(a) \geq k\}, \text{ where } \varepsilon \text{ and } k \text{ are as before}$$

$$(f) \ p(A, u) = \{a \in A : u(a) = \max_A u, \text{ if any; else } u(a) \geq \sup_A u - \varepsilon, \text{ if any; else } u(a) \geq k\}, \text{ where } \varepsilon \text{ and } k \text{ are as before}$$

Notice that, for $\varepsilon > 0$, in (e) and (f) we have $p(A, u) \neq \emptyset$ for every $u \in \mathcal{U}$ (also for $\mathcal{U} \neq \mathcal{U}_F$)

$$(g) \ p(A, u) = \{a \in A : u(a) \text{ is an even integer} \}$$

$$(h) \text{ Assume } \mathcal{U} = \mathcal{U}_F:$$

$$p(A, u) = \{a \in A : u(a) \text{ is equal to the mean value of } u\}$$

$$(i) \text{ Assume } \mathcal{U} \text{ is the set of all real valued functions defined on subsets of } \mathbf{N}:$$

$$p(A, u) = \{a \in A : a \text{ is even} \}$$

□

It should be clear that in the examples (a)-(h) condition (CC) is satisfied, while this is not true for (i). Notice that in examples (b) and (i) p depends actually only on the domain of the function u .

Now we are ready to give the definition of $\mathcal{P}$ -NE. Let $G = (N, (A_i)_{i \in N}, (u_i)_{i \in N})$ be a (finite person) game in strategic form: N is the set of players, and will be assumed to be finite; A_i is the strategy space of player $i \in N$ and $u_i : A \rightarrow \mathbf{R}$ is the payoff for player i ,

where $A = \prod_{i \in N} A_i$.

Definition 2.3 Let G be as above. Assume that for every $i \in N$ it is given a choice rule $(\mathcal{U}_i, p_i)$. Let $\bar{a} = (\bar{a}_i)_{i \in N} \in A$. Define $\bar{A}_i = A_i \times \{\bar{a}_{-i}\}$ and let $\bar{u}_i$ be $u_i|_{\bar{A}_i}$. Assume that

$$(2.1) \quad \bar{u}_i \in \mathcal{U}_i \quad \text{for every } i \in N$$

We shall say that $\bar{a}$ is a $\mathcal{P}$ -NE if $\bar{a} \in p_i(\bar{A}_i, \bar{u}_i)$ for every $i \in N$.

Remark 2.2 As usual $\bar{a}_{-i}$ denotes the element $\bar{a}$ from which it has been deleted the i -th coordinate, so that (slight abuse of notation) $\bar{a} = (\bar{a}_i, \bar{a}_{-i})$. Notice that, when $N = \{i_0\}$, $\bar{A}_{i_0} = A_{i_0} = A$. □

Remark 2.3 There is another way to define $\mathcal{P}$ -NE, as pointed out by Norde. Given $\bar{a} \in A$, define $\hat{u}_i : A_i \rightarrow \mathbf{R}$ by $\hat{u}_i(a_i) = u_i(a_i, \bar{a}_{-i})$, and assume that $\hat{u}_i \in \mathcal{U}_i$ for every $i \in N$. Then, $\bar{a}$ is a $\mathcal{P}$ -NE if $\bar{a} \in p_i(A_i, \hat{u}_i)$ for every $i \in N$. Following this approach, which is also coherent with the definition of reduced game (see Definition 3.1), one can get rid of (CC). Doing so, one should be careful not to confuse $\hat{u}_i$ with $\bar{u}_i$, since the “preferred elements” chosen by means of p_i need not to be the same. □

Example 2.2 Let G be a game. If the class $\mathcal{U}$ is conveniently chosen (e.g.: it contains all of the u_i and all of their restrictions to subsets of A , so that condition (2.1) is satisfied), then $\mathcal{P}$ -NE corresponding to the case in which all of the players use the choice rule described in (a) of Example 2.1 are just the Nash equilibria, while the case in which all use the rule (c) gives rise to ε -NE. □

Example 2.3 Let G be a semi-infinite bimatrix game. Assume that $\mathcal{U}$ is chosen as in the previous example. Then, the definition of weakly determined game, given in Lucchetti, Patrone and Tijs (1986), can be given using the rules described in cases (c) and (d) of Example 2.1. The results established in Jurg and Tijs (1993) and Norde and Potters (1995) can be rephrased saying that the games considered there have a $\mathcal{P}$ -NE for every $\varepsilon > 0$ and $k \in \mathbf{R}$, where:

- player I uses the rule given in (a)

- player II uses the rule given in (e) □

Example 2.4 In the game given by the following table, (T, L) is the unique $\mathcal{P}$ -NE (we consider only pure strategies), if both players adopt the rule described in Example 2.1, (g). □

$I \setminus II$	L	R
T	$(0, -2)$	$(2, 1)$
B	$(1, 0)$	$(1, 1)$

Notice that in Example 2.3 the players use different rules. Another obvious instance of this is the case of a saddle point: it is a $\mathcal{P}$ -NE for a game where both players look at the same payoff function, but player I maximizes and player II minimizes.

3 Consistency and converse consistency

Let $G = (N, (A_i)_{i \in N}, (u_i)_{i \in N})$ be a game. Let $S \subseteq N, S \neq \{\emptyset, N\}$ and $\bar{x} = (\bar{x}_i)_{i \in N} \in A$. We shall say that $G^{S, \bar{x}} = (S, (A_i)_{i \in S}, (u_i^{\bar{x}})_{i \in S})$ is the reduced game of G , w.r.t. S and $\bar{x}$, if $u_i^{\bar{x}} : A_S = \prod_{i \in S} A_i \rightarrow \mathbf{R}$ is defined as follows:

$$u_i^{\bar{x}}(y_S) = u_i(y_S, \bar{x}_{N \setminus S})$$

Definition 3.1 Let $\mathcal{G}$ be a class of games. We shall say that $\mathcal{G}$ is closed under reduction if the following holds:

(CLOS) for every $G = (N, (A_i)_{i \in N}, (u_i)_{i \in N}) \in \mathcal{G}$, for every $\bar{x} \in A$ and for every S s.t. $S \subseteq N, S \neq \{\emptyset, N\}$, we have $G^{S, \bar{x}} \in \mathcal{G}$.

Definition 3.2 Given a class $\mathcal{G}$ of games, we shall say that ϕ is a solution (on $\mathcal{G}$) if, for every $G = (N, (A_i)_{i \in N}, (u_i)_{i \in N}) \in \mathcal{G}$, it is $\phi(G) \subseteq A$.

We shall say that a solution ϕ satisfies consistency if:

(CONS) for every $G \in \mathcal{G}$, for every $\bar{x} \in A$ and for every S ($S \subseteq N, S \neq \{\emptyset, N\}$) s.t. $G^{S, \bar{x}} \in \mathcal{G}$:
[if $\bar{x} \in \phi(G)$, then $\bar{x}_S \in \phi(G^{S, \bar{x}})$].

We say that a solution ϕ , defined on a class $\mathcal{G}$ of games satisfying (CLOS), satisfies converse consistency if:

$$(\text{COCONS}) \quad \left| \begin{array}{l} \text{for every } G \in \mathcal{G} \text{ s.t. } \text{card}(N) \geq 2, \text{ and for every } \bar{x} \in A : \\ \text{[if, for every } S \text{ s.t. } S \subseteq N, S \not\subseteq \{\emptyset, N\}, \bar{x}_S \in \phi(G^{S, \bar{x}}), \text{ then } \bar{x} \in \phi(G)]. \end{array} \right.$$

It is now time to introduce ($\mathcal{P}$ -OPR). Before doing so, we shall put some restrictions on the class of games that we consider.

We shall assume that there is given $\mathcal{N}$, a set of potential players, and that for every $i \in \mathcal{N}$ there is given a choice rule $(\mathcal{U}_i, p_i)$.

We shall consider classes of games $\mathcal{G}$ s.t. for every $G = (N, (A_i)_{i \in N}, (u_i)_{i \in N}) \in \mathcal{G}$, $N \subseteq \mathcal{N}$.

On such classes of games, we shall say that a solution ϕ satisfies personalized one-person rationality if the following holds:

$$(\mathcal{P}\text{-OPR}) \quad \left| \begin{array}{l} \text{for every } i \in \mathcal{N}, \text{ and for every game } G = (\{i\}, A, u) \in \mathcal{G} \text{ we have that} \\ u \in \mathcal{U}_i \text{ and } \phi(G) = p_i(A, u) \end{array} \right.$$

We can now state the following

Theorem 3.1 *Let $\mathcal{G}$ be a class of games satisfying (CLOS). Assume that, for every $i \in \mathcal{N}$, the choice rule satisfies (CC). Assume, moreover, that for every $G \in \mathcal{G}$ and for every $\bar{a} \in A$ it is satisfied condition (2.1) of Definition 2.3.*

Then, a solution ϕ on $\mathcal{G}$ is the $\mathcal{P}$ -NE if and only if ϕ satisfies ($\mathcal{P}$ -OPR), (CONS) and (COCONS).

It is straightforward to verify that $\mathcal{P}$ -NE satisfies ($\mathcal{P}$ -OPR), (CONS) and (COCONS). Notice that, to achieve (CONS) and (COCONS), conditions (CLOS) and (2.1) guarantee that the appropriate choice rules can be applied, while (CC) implies that the choice rules work in an appropriate way.

For the converse, the key argument is the following lemma.

Lemma 3.1 *Let $\mathcal{G}$ be a family of games satisfying (CLOS). Let ϕ_1, ϕ_2 be solutions on $\mathcal{G}$ s.t. ϕ_1 satisfies ($\mathcal{P}$ -OPR) and (CONS), while ϕ_2 satisfies ($\mathcal{P}$ -OPR) and (COCONS). Then, $\phi_1 \subseteq \phi_2$.*

Proof of Theorem 3.1. Since $\mathcal{P}$ -NE satisfies all of the three properties, this Lemma gives us, for ϕ in Theorem 3.1, both $\phi \subseteq \mathcal{P}$ -NE and $\mathcal{P}$ -NE $\subseteq \phi$. Hence, the proof of Theorem 3.1. $\square$

Proof of Lemma 3.1. By induction on the number of players. For $\text{card}(N) = 1$, it is guaranteed by $(\mathcal{P}$ -OPR); we have even equality of the sets.

Assume that $\phi_1(H) \subseteq \phi_2(H)$ for every game H s.t. $\text{card}(N(H)) \leq k$. We shall prove that for every game G s.t. $\text{card}(N(G)) = k + 1$ we have $\phi_1(G) \subseteq \phi_2(G)$.

Let $\bar{x} \in \phi_1(G)$. By (CLOS), $G^{S, \bar{x}} \in \mathcal{G}$ for every S s.t. $S \subseteq N, S \neq \emptyset, N$. By (CONS), $\bar{x}_S \in \phi_1(G^{S, \bar{x}})$. So, by the induction hypothesis, $\phi_1(G^{S, \bar{x}}) \subseteq \phi_2(G^{S, \bar{x}})$. But (COCONS) guarantees that $\bar{x} \in \phi_2(G)$. $\square$

The proofs of Theorem 3.1 and of Lemma 3.1 show that $(\mathcal{P}$ -OPR) lies, so to say, in the background. It has some kind of parametric rôle. In other words, everything is pointing to the fact that a solution satisfying (CONS) and (COCONS) should be some kind of $\mathcal{P}$ -NE.

Actually, the following result is an instance of the principle sketched above. It is stated in a special class to avoid too many technical details. Furthermore, we shall confine to solutions and choice rules which satisfy a compatibility condition. This condition is essentially the same for both. To be more precise, for the solution ϕ we shall ask for the following property, that is essentially an adaptation of (CC) for games.

Definition 3.3 Given a class $\mathcal{G}$ of games, and a solution ϕ on $\mathcal{G}$, we shall say that ϕ satisfies the compatibility condition if:

$$(CC\gamma) \quad \left\{ \begin{array}{l} \text{given } G = (N, (A_i)_{i \in N}, (u_i)_{i \in N}) \in \mathcal{G} \text{ and } H = (N, (B_i)_{i \in N}, (v_i)_{i \in N}) \in \mathcal{G}, \\ \text{if} \\ [\text{for every } i \in N \text{ there is a bijection } f_i : A_i \longrightarrow B_i \text{ s.t. for every } j \\ v_j(f_1(a_1), \dots, f_i(a_i), \dots, f_N(a_N)) = u_j(a_1, \dots, a_i, \dots, a_N)], \\ \text{then:} \\ [(\bar{a}_1, \dots, \bar{a}_i, \dots, \bar{a}_N) \in \phi(G) \text{ if and only if} \\ (f_1(\bar{a}_1), \dots, f_i(\bar{a}_i), \dots, f_N(\bar{a}_N)) \in \phi(H)]. \end{array} \right.$$

Notice that G and H have the same set of players. Apart from this fact that we just stressed, condition (CC γ) is far from being new. It can be found, e.g., in Rosenthal (1974).

Furthermore, condition $(CC\gamma)$ is satisfied by all of the main solutions concepts for strategic games (NE, correlated equilibria, ε -NE, perfect equilibria, etc.)

Example 3.1 Consider the following two games:

$I \setminus II$	L	C	R
T	(2, 2)	(1, 1)	(3, 0)
M	(1, 1)	(1, 1)	(4, 1)
B	(0, 5)	(1, 4)	(5, 3)

$I \setminus II$	ξ	η	ζ
α	(3, 0)	(2, 2)	(1, 1)
β	(5, 3)	(0, 5)	(1, 4)
γ	(4, 1)	(1, 1)	(1, 1)

We have:

$$f_I(T) = \alpha, f_I(M) = \gamma, f_I(B) = \beta$$

$$f_{II}(L) = \eta, f_{II}(C) = \zeta, f_{II}(R) = \xi$$

And so, since (T, L) and (M, C) are NE for G , (α, η) and (γ, ζ) are NE for H . □

Theorem 3.2 Let $\mathcal{G}$ be the class of all finite games s.t. $N \subseteq \mathcal{N}$. Let ϕ be a solution on $\mathcal{G}$, satisfying (CONS) and (COCONS).

If ϕ satisfies $(CC\gamma)$, then ϕ determines, by one-person games, a unique family of choice rules $(p_i)_{i \in \mathcal{N}}$ defined on $\mathcal{U}_F$ and satisfying (CC). Moreover, ϕ is the $\mathcal{P}$ -NE determined by these choice rules.

Proof. Let $i \in \mathcal{N}$, A be a finite set and let $u : A \rightarrow \mathbf{R}$. Define $p_i(A, u) = \phi(\{i\}, A, u)$. Such p_i satisfies (CC).

By definition of p_i , clearly ϕ satisfies $(\mathcal{P}$ -OPR) w.r.t. this family of choice rules.

Since by assumption ϕ satisfies also (CONS) and (COCONS), thanks to Theorem 3.1 we have that $\phi = \mathcal{P}$ -NE. □

Let us notice at this point that, since perfect NE do not satisfy (CONS), they cannot be $\mathcal{P}$ -NE. The same is true for other refinements of NE which do not satisfy (CONS): see Example 2.4 of Peleg and Tijs (1996).

One word on the independence of the properties ($\mathcal{P}$ -OPR), (CONS) and (COCONS). Again, examples 2.16 and 2.17 from Peleg and Tijs (1996) show that neither (CONS) nor (COCONS) are implied by the other two. For what concerns ($\mathcal{P}$ -OPR), the previous Theorem gives an answer (at least under the compatibility conditions). Let us notice that for the solution ϕ provided in Peleg and Tijs to show that (OPR) is not implied by (CONS) and (COCONS), we have actually that $\phi = \mathcal{P}$ -NE, for p given as in Example 2.1 (b).

Referring to Peleg and Tijs (1996) once more, we point out that it is possible to extend to this setting also their results on the extensive form games: in particular, defining $\mathcal{P}$ -SPE (that is, $\mathcal{P}$ -subgame perfect equilibrium).

We end this section providing an example of a solution which is not a $\mathcal{P}$ -NE. This solution could be seen as a refinement of the $(\varepsilon-k)$ -NE which correspond to the $\mathcal{P}$ -NE induced by the choice rule described in Example 2.1 (e).

Example 3.2 Let be given $\varepsilon \geq 0$ and $k \in \mathbf{N}, k > 1$. Given $G = (N, (A_i)_{i \in N}, (u_i)_{i \in N})$, we define

$$\phi(G) = \begin{cases} \{\bar{a} : \text{for every } i \in N, u_i(\bar{a}) \geq \sup_{A_i} u_i(a_i, \bar{a}_{-i}) - \varepsilon\} & \text{if nonempty,} \\ \{\bar{a} : \text{for every } i \in N, u_i(\bar{a}) \geq \sup_{A_i} u_i(a_i, \bar{a}_{-i}) - \varepsilon \text{ or } u_i(\bar{a}) \geq k\} & \text{otherwise.} \end{cases}$$

We provide a counterexample to show that ϕ does not satisfy (CONS).

$$\begin{aligned} \text{Let } N &= \{I, II, III\}; A = \{T, B\} \times \mathbf{N} \times \mathbf{N}; u_I(x, y, z) = \begin{cases} 1 & \text{if } x = T \text{ and } y = 1 \\ 0 & \text{otherwise} \end{cases}; \\ u_{II}(x, y, z) &= \begin{cases} 1 & \text{if } x = T \\ y & \text{if } x = B \end{cases}; u_{III}(x, y, z) = z. \end{aligned}$$

Clearly $\bar{x} = (B, k, k)$ is in $\phi(G)$ (notice that the payoff of III prevents the existence of ε -NE).

However, for the reduced game $G^{S, \bar{x}}$, with $S = \{I, II\}$, $(B, k) \notin \phi(G^{S, \bar{x}})$, because $G^{S, \bar{x}}$ has a NE (the point $(T, 1)$). □

4 The ancestor property

Given a class $\mathcal{G}$ of games and a solution ϕ on $\mathcal{G}$, Peleg, Potters and Tijs (1996) have introduced a directed graph, denoted by $\text{Graph}(\mathcal{G}, \phi)$, whose vertices are couples (G, x) with $G \in \mathcal{G}$ and $x \in \phi(G)$.

Two nodes (H, y) and (K, z) in the graph are connected if K is a reduced game of H , given y . That is:

$$N(K) \subseteq N(H), \quad N(K) \not\subseteq \{\emptyset, N(H)\}$$

$$K = H^{N(K), y}$$

$$z = y|_{N(K)}$$

Also the following property was introduced in Peleg, Potters and Tijs (1996).

Definition 4.1 *The graph $\text{Graph}(\mathcal{G}, \phi)$ has the ancestor property if:*

$$(AP) \quad \left| \begin{array}{l} \text{for every } (G, x) \in \text{Graph}(\mathcal{G}, \phi), \text{ there is a game } H \in \mathcal{G} \text{ s.t. for every} \\ y \in \phi(H) \text{ the vertex } (H, y) \text{ is connected with } (G, x). \end{array} \right.$$

It is easy to notice that Theorem 1 of Peleg, Potters and Tijs (1996) holds also if we drop any reference to (OPR). That is, we can prove the following theorem. Before that, a piece of notation: if a solution ϕ on a class $\mathcal{G}$ is nonempty valued, we shall say that ϕ satisfies (NEM).

Theorem 4.1 *Let $\mathcal{G}$ be a class of games, and let ϕ be a solution on $\mathcal{G}$, satisfying (NEM) and (CONS).*

If ϕ is a solution on $\mathcal{G}$ which satisfies (AP), then ϕ is minimal (on $\mathcal{G}$, w.r.t. inclusion) in the class of solutions on $\mathcal{G}$ which satisfy (NEM) and (CONS).

Proof. Let $\bar{\phi}$ be given, which satisfies (NEM) and (CONS), s.t. $\bar{\phi} \subseteq \phi$. We shall prove that $\bar{\phi} = \phi$.

Let $G \in \mathcal{G}$ and $x \in \phi(G)$. Consider H , whose existence is guaranteed by (AP). We have, for (NEM), that $\bar{\phi}(H) \neq \emptyset$. So, let $\bar{y} \in \bar{\phi}(H)$. Now, $\bar{y}|_{N(G)} \in \bar{\phi}(H^{N(G), \bar{y}})$ by (CONS) applied to $\bar{\phi}$. But $\bar{y} \in \bar{\phi}(H)$, hence $\bar{y} \in \phi(H)$. So, $\bar{y}|_{N(G)} = x$ by (AP).

Hence, we have proved that $x \in \bar{\phi}(H^{N(G), \bar{y}})$, but $H^{N(G), \bar{y}} = G$ by the definition of $\text{Graph}(\mathcal{G}, \phi)$. $\square$

Now, we can prove the following result, which extends to $\mathcal{P}$ -NE the considerations made by Peleg, Potters and Tijs (1996) after their Theorem 1.

Theorem 4.2 *Let $\mathcal{G}$ be a class of games satisfying (CLOS) and s.t. $\mathcal{P}$ -NE satisfies (NEM) and (AP) on $\mathcal{G}$.*

If $\hat{\phi}$ is a solution on $\mathcal{G}$ that satisfies (NEM), (CONS) and ($\mathcal{P}$ -OPR), then $\hat{\phi} = \mathcal{P}$ -NE.

Proof. Since $\hat{\phi}$ satisfies ($\mathcal{P}$ -OPR) and (CONS), while $\mathcal{P}$ -NE satisfies ($\mathcal{P}$ -OPR) and (COCONS), we have that $\hat{\phi} \subseteq \mathcal{P}$ -NE by Lemma 3.1.

But $\mathcal{P}$ -NE on $\mathcal{G}$ satisfies (CONS) and (NEM). Since we assumed also that $\mathcal{P}$ -NE satisfies (AP) on $\mathcal{G}$, thanks to Theorem 4.1 we can conclude that $\mathcal{P}$ -NE is minimal on $\mathcal{G}$ w.r.t. (NEM) and (CONS). So, $\mathcal{P}$ -NE = $\hat{\phi}$ $\square$

We shall now see how to extend Theorem 3-a of Peleg, Potters and Tijs (1996). To simplify, we shall assume that every player in $\mathcal{N}$ has its choice rule with domain $\mathcal{U}_{\mathcal{F}}$. We shall, furthermore, assume that all of these rules satisfy (CC). We need a further restriction on these rules.

Definition 4.2 *A choice rule $(\mathcal{U}, p)$ is said to separate points if:*

(SP) $\left| \begin{array}{l} \text{there exist } \beta, \gamma \in \mathbf{R} \text{ s.t. for every function } u : \Omega \longrightarrow \mathbf{R} \text{ (} u \in \mathcal{U} \text{) s.t.} \\ u(\Omega) = \{\beta, \gamma\}, \text{ it is } p(\Omega, u) = \{\omega \in \Omega : u(\omega) = \gamma\} \end{array} \right.$

The idea behind this definition is clear. Let us add that (SP) is not very useful without (CC). Referring to Example 2.1, notice that on $\mathcal{U}_{\mathcal{F}}$ rules (a), (c), (d), (e), (f) and (g) satisfy (SP), while the rules (b) and (h) do not. All of them satisfy (CC). For rule (c), when $\varepsilon > 0$, it is sufficient to take $\beta = 0$ and $\gamma = 2\varepsilon$. Notice that the functions u considered in (SP) fail to be continuous on “nice” topological spaces (e.g., an interval of the reals), so that this road is barred if one is interested in extending the result that we shall give to a context like those studied in Peleg and Sudhölter (1994) or in Norde, Potters, Reijnierse and Vermeulen (1996).

Let us prove our result:

Theorem 4.3 *Let be given an infinite set $\mathcal{N}$ of potential players. Assume that for every $i \in \mathcal{N}$ we are given a choice rule $(\mathcal{U}_i, p_i)$ s.t. $\mathcal{U}_i = \mathcal{U}_F$ and satisfying (CC) and (SP).*

Let $\mathcal{G} = \mathcal{G}^{f, \mathcal{P}-NE}$ be the class of all finite games, with players in $\mathcal{N}$, which have $\mathcal{P}$ -NE.

Then, $\mathcal{P}$ -NE satisfies (AP) on $\mathcal{G}$.

Proof. Let $(G, \bar{x}) \in \mathcal{G}$. We shall essentially repeat the construction in Peleg, Potters and Tijs (1996) of a game H to prove (AP).

We add one player to $N(G)$. Take a player $j \in \mathcal{N}$ s.t. $j \notin N$. So, $N(H) = N(G) \cup \{j\}$. The strategy spaces of H are: for $i \in N(G)$, A_i is as in G . For player j , $A_j = \{T, B\}$.

The payoff functions are defined as follows:

for player j :

$$U_j(\bar{x}, B) = \gamma_j$$

$$U_j(\bar{x}, T) = \beta_j$$

$$U_j(x, B) = \beta_j \text{ for } x \in A \text{ s.t. } x \neq \bar{x}$$

$$U_j(x, T) = \gamma_j \text{ for } x \in A \text{ s.t. } x \neq \bar{x}$$

for player i :

$$U_i(x, B) = u_i(x)$$

$$U_i(x, T) = \beta_i \text{ if } x_i \neq \bar{x}_i$$

$$U_i(x, T) = \gamma_i \text{ if } x_i = \bar{x}_i$$

It can be verified that $(\bar{x}, B)$ is the unique $\mathcal{P}$ -NE for H (the arguments are as in Peleg, Potters and Tijs (1996)). So, $H \in \mathcal{G}^{f, \mathcal{P}-NE}$. It is straightforward to see that $(\bar{x}, B)_{N(G)} = \bar{x}$ and that $H^{N(G), (\bar{x}, B)} = G$. □

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