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Calculation of an approximate solution of the infinite time-varying LQ-problem

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DEPARTMENT OF ECONOMICS
RESEARCH MEMORANDUM

CALCULATION OF AN APPROXIMATE SOLUTION
OF THE INFINITE TIME-VARYING
LQ-PROBLEM

Jacob C. Engwerda

FEW 545

R35
Linear Programming
Optimal Control
Algorithm

Refereed by Prof.dr. J.M. Schumacher

Calculation of an approximate solution of the
infinite time-varying LQ-problem

by

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Abstract

In this note we present an algorithm to approximate the steady state solution of the discrete time-varying LQ regulator problem.

Given a beforehand permitted approximation error and some system characteristics, the algorithm yields a solution which has the property that the norm of the difference between this solution and the steady state solution does not exceed this approximation error.

I Introduction

In the past much research has been done to find sufficient conditions for existence of a solution of the linear quadratic optimization problem w.r.t. discrete time-varying systems (see e.g. Kwakernaak et al. [1971], Hager et al. [1976], Anderson et al. [1981], Bittanti et al. [1986] and Engwerda [1990]).

Usually, however, for time-varying systems no attention is paid to the question how this solution can be calculated. In this paper we show for the approach taken by Engwerda in [1990] how the solution of the time-varying algebraic Riccati equation can be estimated by the recurrence Riccati equation. That is, given an a priori specified bound for the approximation error ϵ , we present an algorithm which calculates in finite time a matrix which approximates the solution of the algebraic Riccati equation in norm with an error less than ϵ .

The paper is organized as follows.

In section II we introduce the optimal control problem together with basic notions that are used throughout the paper. Section III shows how the approximate optimal solution can be obtained. The basics of the method are illustrated in a simple example.

II Problem statement and basic notions

Since the system we consider is time-varying it is convenient to introduce some notation before we introduce the control problem.

Let N be any positive number, $A(i)$, $B(i)$, $C(i)$ matrices with appropriate dimensions and $x(i)$, $y(i)$, $\bar{x}(i)$, $v(i)$ vectors at time i .

Then $A(k+N, k) := A(k+N-1) \dots A(k)$ if $N > 1$ and $A(k, k) := I$;

$$S[k, k-N] := [B(k) | A(k+1, k)B(k-1) | \dots | A(k+1, k-N+1)B(k-N)];$$

$$W[k, k+n] := [C(k) | \dots | (C(k+N)A(k+N, k))];$$

$$v[k, \ell] := (v^T(k), \dots, v^T(\ell))^T;$$

$$v[k, \cdot] := (v^T(k), v^T(k+1), \dots)^T;$$

$x(k, k_0, \bar{x}, u)$ is the state of the system at time k resulting from the initial state $\bar{x}$ at time k_0 if the input $u[k_0, k-1]$ is applied;

$$y(k, k_0, \bar{x}, u) := C(k)x(k, k_0, \bar{x}, u);$$

$$\|A\| := \text{largest singular value of matrix } A$$

$$\|\underline{x}\| := \text{Euclidean norm of } \underline{x}.$$

In Engwerda (1990b), it was shown that we may assume without loss of generality that a linear, finite-dimensional, time-varying difference equation is described by

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ x_3(k+1) \end{bmatrix} = \begin{bmatrix} A_{11}(k) & A_{12}(k) & A_{13}(k) \\ 0 & A_{22}(k) & A_{23}(k) \\ 0 & 0 & A_{33}(k) \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix} + \begin{bmatrix} B_1(k) \\ B_2(k) \\ 0 \end{bmatrix} u(k)$$

$$y(k) = (0 \quad C_1(k) \quad C_2(k))x(k); \quad x(k_0) = \bar{x} =: \begin{bmatrix} \bar{x}_3 \\ \bar{x}_2 \\ \bar{x}_1 \end{bmatrix}, \quad (1)$$

where

$\Sigma_1 : x_1(k+1) = A_{11}(k)x_1(k) + B_1(k)u(k)$ is reachable at any time $k \geq 0$.

$\Sigma_2 : x_2(k+1) = A_{22}(k)x_2(k) + B_2(k)u(k) + A_{23}(k)A_{33}(k,k_0)x_3(k_0)$

$$y(k) = C_2(k)x_2(k)$$

is both reachable and observable at any time $k \geq k_0$.

$\Sigma_3 : x_3(k+1) = A_{33}(k)x_3(k)$.

Here $x(k) := \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix} \in \mathbb{R}^n$ is the state of the system, $u(k) \in \mathbb{R}^m$ the applied control and $y(k) \in \mathbb{R}^r$ the output at time k .

Remarks

- 1) A state $\bar{x}$ is called reachable at k_0 (from zero) if there exists a control sequence $u[N, k_0-1]$ such that $x(k_0, N, 0, u) = \bar{x}$. A system is called reachable at time k_0 if all states of the system are reachable at time k_0 .
- 2) A state $\bar{x}$ is called unobservable at time k_0 , if $y(k, k_0, \bar{x}, 0) = 0$ for any $k \geq k_0$. A system is called observable at k_0 if $\bar{x} = 0$ is the only unobservable state at k_0 .
- 3) The dimensions of the states $x_1(k)$, $x_2(k)$, $x_3(k)$ can change when time passes. Consequently, also the dimensions of all matrices in (1) may vary in time. □

Moreover, we recall the next definitions:

Definition 1A

- * The initial state $\bar{x}$ of system (1) is called
 - exponentially stable at k_0 if there exist constants $0 \leq \alpha < 1$ and $M > 0$ such that $\|x(k, k_0, \bar{x}, 0)\| \leq M\alpha^{(k-k_0)}\|\bar{x}\|$ for all $k > k_0$.
 - exponentially stabilizable at k_0 if there exists a control sequence $u[k_0, \dots]$, with the property that $u(\cdot)$ converges exponentially fast to zero, and positive constants α and M such that $\|x(k, k_0, \bar{x}, u)\| \leq M\alpha^{(k-k_0)}\|\bar{x}\|$ for all $k > k_0$.
 - exponentially detectable at k_0 if there exists an integer $N > 0$ such that $\bar{x}$ modulo $X_e^-(A(\cdot, k_0))$ is determined from any $y[k_0, k_0+N-1]$ and

$u[k_0, k_0+N-2]$. Here $X_e^-(A(., k_0))$ is the linear subspace containing all exponentially stable states at time k_0 .

- * System (1) is called exponentially stable (respectively exponentially stabilizable, exponentially detectable) at k_0 if all initial states of the system have the corresponding property at k_0 . $\square$

To derive sufficient conditions for exponential stabilizability and detectability of system (1) two other notions play a crucial role in the analysis of Engwerda (1990b). These are periodic smooth controllability and observability. Roughly speaking, a system is called smoothly controllable (observable) if there exists a finite time period such that whenever such a time period has passed, the system has been at least once controllable (observable) during that period. Formally:

Definition 1B

System $\begin{cases} x(k+1) = A(k)x(k) + B(k)u(k) \\ y(k) = C(k)x(k) \end{cases}$ is called periodically smoothly controllable at k_0 if there exist positive constants s and k_1 such that for all $k > 1$ there exists an integer $k_2(k)$ in the interval $[k_0 + (k-1)k_1, k_0 + kk_1]$ for which $S[k_2 - k_1, k_2]S^T[k_2 - k_1, k_2] \geq s^2 I$.

Similarly we call this system periodically smoothly observable at k_0 if there exist positive constants ω and k_1 such that for all $k > 1$ there exists an integer $k_2(k)$ in the interval $[k_0 + (k-1)k_1, k_0 + kk_1]$ for which $W[k_2, k_2 + k_1]W^T[k_2, k_2 + k_1] \geq \omega^2 I$. $\square$

Using these notions we have (see Engwerda (1990b, corollary 5)).

Theorem 2

System (1) is both uniformly exponentially stabilizable and uniformly exponentially detectable if

- i) Σ_1 is uniformly exponentially stable
- ii) Σ_2 is both periodically smoothly controllable and observable at k_0
- iii) Σ_3 is uniformly exponentially stable. $\square$

Moreover, it was noted that in case the system is time-invariant these three conditions are necessary too.

Next, consider the LQ optimal control problem:

$\min_{u[k_0, k_0+N-1]} J_N$, subject to system (1),

$$\text{where } J_N := \sum_{k=k_0}^{k_0+N-1} \{ \|y(k)\|^2 + \|u(k)\|_{R(k)}^2 \} + \|y(k_0+N)\|^2, \quad (2)$$

and the assumptions: $R(k) \geq \beta I$ for some $\beta > 0$, for all k ; conditions i), ii), iii) from theorem 2; all matrices are bounded by a constant M . It is well known that the following result holds for this control problem.

Theorem 3

With $Q(k) := C^T(k)C(k)$ we have that

$$u_N(k) = -F_N(k)x(k) \text{ solves problem (2),}$$

where $F_N(k) := (R(k) + B^T(k)K_N(k+1)B(k))^{-1}B^T(k)K_N(k+1)A(k)$, and $K_N(k)$ satisfies the backwards recurrence equation

$$K_N(k) = A^T(k)(K_N(k+1) - K_N(k+1)B(k)(R(k) + B^T(k)K_N(k+1)B(k))^{-1}B^T(k)K_N(k+1)A(k) + Q(k)$$

$$K_N(N) = Q(N). \quad (\text{RRE})$$

Moreover, the minimal cost is given by $J_N(\bar{x}) = \bar{x}^T K_N(k_0) \bar{x}$. $\square$

Remark 4

Note that, due to the structure of system (1), the solution of (RRE) has the structure:

$$K_N(k) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & K_{22} & K_{23} \\ 0 & K_{23}^T & K_{33} \end{bmatrix}_N(k),$$

Consequently $F_N(k) = (0|F_2|F_3)_N(k)$.

Some elementary calculation shows that K_{22} satisfies a recurrence equation similar to (RRE) that can be rewritten as:

$$K_{22}(k) = (A_{22} - B_2 F_2)^T(k) K_{22}(k+1) (A_{22} - B_2 F_2)(k) + (C_2^T C_2 + F_2^T R F_2)(k),$$

$$\text{with } K_{22}(N) = C_2^T(N) C_2(N) \quad (\text{RRE}') \quad (2')$$

$$\text{and } F_2(k) = (R(k) + B^T(k) K_{22}(k+1) B(k))^{-1} B^T(k) K_{22}(k+1) A_{22}(k). \quad \square$$

In the next section we will consider the infinite planning horizon optimization problem, i.e.

$$\min_{u[k_0, \dots]} \lim_{N \rightarrow \infty} J_N \quad (2')$$

w.r.t. the assumptions made under (2).

We will show that an optimal solution is given by

$$u(k) = -F(k)x(k), \quad (3)$$

where $F(k) = (R(k) + B^T(k) K(k+1) B(k))^{-1} B^T(k) K(k+1) A(k)$, and $K(k)$ is given by $\lim_{N \rightarrow \infty} K_N(k)$, with $K_N(k)$ as defined in (RRE).

More in particular we will show that it is possible to specify beforehand, for any approximation error ϵ , a planning horizon $\bar{N}$ such that $\|K(k) - K_{\bar{N}}(k)\| < \epsilon$. So, for any desired precision, we can compute a solution which approximates the optimal solution (3) for the infinite planning horizon LQ optimization problem.

III Calculation of an approximate solution

We first show that the minimum in problem (2') exists. We have the following basic result.

Lemma 5

There exists a control sequence $\tilde{u}[k_0, \dots]$ such that $\lim_{N \rightarrow \infty} J_N(\bar{x}(k_0), \tilde{u}) \leq$

$$M_1 \left\| \begin{bmatrix} 0 \\ \bar{x}_2 \\ \bar{x}_3 \end{bmatrix} \right\|, \text{ which moreover has the property that } \|\tilde{u}(k)\| \leq M_2 \|x_3(k_2(2t))\|,$$

where t is such that $k_2(2t) \leq k \leq k_2(2(t+1))$, $\forall k \geq k_2(2)$ (here k_2 is as defined in definition 1B).

The constants M_1 and M_2 can be calculated from the system characteristics (see proof below).

Proof

We prove this lemma by constructing such a control sequence (see also Engwerda (1990b)).

Consider the control sequence

$$\tilde{u}[k_0, k_2(2)-k_1-1] = 0$$

$$\begin{aligned} \tilde{u}[k_2(2)-k_1, k_2(2)] &= -S_2'^T(2)(S_2'(2)S_2'^T(2))^{-1}(e(k_2(2)) + \\ &\quad x_2(k_2(2)+1, k_0, \bar{x}_2, 0)), \end{aligned} \tag{i}$$

where

$$S_2'(2) = S_2[k_2(2), k_2(2)-k_1],^* \text{ and}$$

$$e(k_2(2)) := \sum_{i=k_0}^{k_2(2)} A_{22}(k_2(2)+1, i+1)A_{23}(i)x_3(i)$$

* S_i = matrix S w.r.t. subsystem Σ_i .

(i.e. the sum of all components of the third state entering the second subsystem Σ_2)

With this input

$$x(k_2(2)+1) := \begin{bmatrix} x_1(k_2(2) + 1) \\ 0 \\ x_3(k_2(2)+1, k_0, \bar{x}_3, 0) \end{bmatrix}.$$

We now show by induction that it is possible to regulate $x_2(k_2(2k)+1)$ to zero for any k .

Let therefore t be any integer larger than one. Consider the time interval $[k_0+(2t-3)k_1, k_0+2tk_1]$. The sum of all components of the third state entering the second subsystem from $k_2(2(t-1)) + 1$ until $k_2(2t)$ is

$$e(k_2(2t)) := \sum_{i=k_2(2(t-1))+1}^{k_2(2t)} A_{22}(k_2(2t)+1, i+1) A_{23}(i) x_3(i). \quad (ii)$$

Since by induction hypothesis $x_2(k_2(2(t-1))+1)$ is zero, application of the input

$$\tilde{u}[k_2(2(t-1))+1, k_2(2t)-k_1-1] = 0$$

$$\tilde{u}[k_2(2t)-k_1, k_2(2t)] = -S_2^T(2t)(S_2'(2t)S_2^T(2t))^{-1}e(k_2(2t)),$$

where $S_2'(2t) = S_2[k_2(2t), k_2(2t)-k_1]$, yields $x_2(k_2(2t)) = 0$.

So, the induction argument is complete.

Note that $\|S_2^T(2t)(S_2'(2t)S_2^T(2t))^{-1}\| \leq \frac{1}{s}$ for all t . Consequently, $\|\tilde{u}[k_2(2(t-1))+1, k_2(2t)]\| \leq \frac{1}{s} \|e(k_2(2t))\|$ for all t . As Σ_3 is exponentially stable, we get from (i) that $\|e(k_2(2t))\| \leq \tilde{M}_2 \|x_3(k_2(2(t-1))+1)\|$ for some constant $\tilde{M}_2$, which can be calculated from the system parameters, for all $t > 1$.

So $\|\tilde{u}(k)\| \leq M_2 \|x_3(k_2(2(t-1))+1)\|$, for all $k_2(2(t-1))+1 \leq k \leq k_2(2t)$, $t = 2, 3, \dots$

Moreover, from (i) it is obvious that $\|\tilde{u}(k)\| \leq M'_2 \left\| \begin{bmatrix} 0 \\ \bar{x}_2 \\ \bar{x}_3 \end{bmatrix} \right\|$ for some constant

M'_2 which can be calculated from the system parameters, for all $k_0 \leq k < k_2(2)$.

Using this control sequence and the exponential stability of Σ_3 it is clear that there exists a constant M_1 , which can be calculated from just

the system parameters, such that $\lim_{N \rightarrow \infty} J_N(\bar{x}(k_0), \tilde{u}) \leq M_1 \left\| \begin{bmatrix} 0 \\ \bar{x}_2 \\ \bar{x}_3 \end{bmatrix} \right\|$. $\square$

From this lemma we immediately deduce that

Corollary 6

Under the assumptions (2), we have that as well $\min_{u[k_0, k_0+N-1]} J_N \leq M_1 \left\| \begin{bmatrix} 0 \\ \bar{x}_2 \\ \bar{x}_3 \end{bmatrix} \right\|$

as $\min_{u[k_0, \dots]} \lim_{N \rightarrow \infty} J_N \leq M_1 \left\| \begin{bmatrix} 0 \\ \bar{x}_2 \\ \bar{x}_3 \end{bmatrix} \right\|$.

In particular this implies that $\lim_{N \rightarrow \infty} K_N(k)$ exists. $\square$

In our analysis for obtaining an estimate for $\|K_N(k_0) - K(k_0)\|$, we will need an estimate for $\|x^0(k_0+N)\|^2$. Here $x^0(k_0+N)$ is the state of system (1) if the optimal control sequence $u[k_0, k_0+N-1]$ from theorem 3 is used. To that end we first show that the periodic smooth observability property of Σ_2 is preserved if this optimal control sequence is used to regulate the system.

Lemma 7

Let the system

$$x(k+1) = A(k)x(k) + B(k)u(k); \quad x(0) = \bar{x}$$

$$y(k) = C(k)x(k)$$

be periodically smoothly observable at time $k = 0$ with parameters ω and k'_1 .

Then, for any matrix sequence $K[0, \dots]$, the system

$$\begin{aligned} x(k+1) &= (A + KC)(k)x(k) + B(k)u(k); \quad x(0) = \bar{x} \\ \tilde{L}_y : \\ y(k) &= C(k)x(k), \end{aligned}$$

is periodically smoothly observable at time $k = 0$ with parameters ω and k'_1 too.

Proof

Follows directly from the observation that the observability Gramian for $\tilde{L}_y$ equals

$$W[0, k]X[0, k]X^T[0, k]W^T[0, k],$$

where X is an upper triangular matrix with identity matrices on the diagonal. $\square$

Remark 8

Consider the solution to the finite planning horizon LQ-problem (2) (cf. theorem 3 and remark 4).

Now, let $D := (C_2^T | F_2^T R^{\frac{1}{2}})^T$.

Then, $A_{22} - B_2 F_2 = A_{22} - [0 | B_2] R^{-\frac{1}{2}} D$.

Since periodic smooth observability of (A_{22}, C_2) implies that (A_{22}, D) has the same property, the previous lemma 7 yields that the closed-loop pair $(A_{22} - B_2 F_2, D)$ is periodically smoothly observable too.

Moreover, note that $D^T D = C_2^T C_2 + F_2^T R F_2$, a result which will be used in the next lemma. $\square$

A consequence of this result is that we can conclude that the final state $x_2(k_0 + N)$, obtained by applying the optimal control sequence $u_N(k) = -F_N(k)x(k)$ at time $k = k_0, \dots, k_0 + N - 1$ to the system, converges exponentially fast to zero if the planning horizon N expands. This result is proved

in the next lemma. Its proof is along the lines the proof in Anderson et al. (1981) of the so called extended lemma of Lyapunov.

Lemma 9

Let (A_{22}, C_2) be periodically smoothly observable and $K_{22}(\cdot)$ a bounded positive semi-definite matrix sequence satisfying (RRE)', i.e.

$$(A_{22} - B_2 F_2)^T(k) K_{22}(k+1) (A_{22} - B_2 F_2)(k) - K_{22}(k) = -(C_2^T(k) C_2(k) + F_2^T(k) R(k) F_2(k))$$

$$K_{22}(N) = C_2^T(N) C_2(N).$$

Then, there exist constants $\tilde{M}$ and α , that can be estimated from the system parameters and that are independent of N , such that

$$\|x_2(k_0 + N)\|^2 \leq \tilde{M} e^{-\alpha(N - k_0)} \|\bar{x}_2\|,$$

where $x_2(k_0 + N)$ is the state of the system $x_2(k+1) = (A_{22} - B_2 F_2)x_2(k)$, $x_2(k_0) = \bar{x}_2$, at time $k = k_0 + N$.

Proof

Associate with $x_2(k+1) = (A_{22} - B_2 F_2)x_2(k)$ the 'Lyapunov-like' function

$$V(x_2, k, \epsilon) = x_2^T(K_{22}(k) + \epsilon I)x_2.$$

Setting $V_k := V(x_2(k), k, \epsilon)$, we observe that

$$\begin{aligned} V_k - V_{k+1} &= x_2^T(k) (C_2^T(k) C_2(k) + F_2^T(k) R(k) F_2(k)) x_2(k) + \\ &\quad \epsilon x_2^T(k) (I - (A_{22} - B_2 F_2)^T(k) (A_{22} - B_2 F_2)(k)) x_2(k). \end{aligned}$$

Since (A_{22}, C_2) is periodically smoothly observable we have from remark 8 that $(A_{22} - B_2 F_2, (C_2^T | F_2^T R^{\frac{1}{2}})^T)$ has this property too with the same parameters.

So, for all $k > 1$ there exists an integer $k_2(k)$ in the interval $[k_0 + (k-1)k_0, k_0 + k_0]$ for which $W_2[k_2, k_2 + k_1] W_2^T[k_2, k_2 + k_1] \geq \omega^2 I$. Here $W_2 :=$ matrix W w.r.t. $(A_{22} - B_2 F_2, (C_2^T | F_2^T R^{\frac{1}{2}})^T)$.

Therefore,

$$\begin{aligned}
 V_{k_2} - V_{k_2+k_1+1} &= x_2^T(k_2) (W_2[k_2, k_2+k_1] W_2^T[k_2, k_2+k_1] + \epsilon I) x_2(k_2) \\
 &\quad - \epsilon x_2^T(k_2) (A_{22} - B_2 F_2)^T(k_2+k_1+1, k_2) (A_{22} - B_2 F_2)(k_2+k_1+1, k_2) x_2(k_2) \\
 &\geq (\omega^2 + \epsilon) x_2^T(k_2) x_2(k) - \epsilon \gamma x^T(k_2) x(k_2), \tag{i}
 \end{aligned}$$

where γ is a bound for $(A_{22} - B_2 F_2)^T(k_2+k_1+1, k_2) A_{22}(k_2+k_1+1, k_2)$, which, due to the boundedness assumption on the system matrices and corollary 6, can be taken independent of k_2 . Taking ϵ sufficiently small (e.g. $\epsilon = \max(0, \frac{\omega^2}{2(\gamma-1)})$) we find that (i) is greater than $\tilde{\omega} x_2^T(k_2) x_2(k_2)$ for some constant $\tilde{\omega}$ independent of k_2 and $x_2(k_2)$. Consequently, using the definition of V_{k_2} , we get that

$$V_{k_2} - V_{k_2+k_1+1} \geq \tilde{\omega} x_2^T(k_2) x_2(k_2) \geq \omega' V_{k_2} \text{ for some } \omega' > 0 \tag{ii}$$

Note that due to corollary 6 again, this bound ω' can be estimated a priori from the system parameters.

Inequality (ii) shows that $0 \leq V_{k_2(k)+k_1+1} \leq (1-\omega') V_{k_2(k)}$ for any $k > 1$.

Now, since $W_2[k_2, k_2+k_1+j] W_2^T[k_2, k_2+k_1+j] \geq \omega I$ for every $j \in \mathbb{N}$ whenever $W_2[k_2, k_2+k_1] W_2^T[k_2, k_2+k_1]$ has this property, we may assume without loss of generality that the times $k_2(k)+k_1+1$ and $k_2(k+1)$ coincide.

So there exists a subsequence V_{n_i} where $n_i \in [k_0+(i-1)k_1, k_0+ik_1]$ such that

$$V_{n_i} \leq a b^i \text{ for some } a > 0, \text{ depending on } \bar{x}_2,$$

$$\text{and } 0 < b < 1, \text{ independent of } \bar{x}_2.$$

Since $V_{n_i} \geq \epsilon x^T(n_i) x(n_i)$, $\|x(n_i)\|$ decays at the same rate. As $x_2(k) = (A_{22} - B_2 F_2)(k-1, n_i) x(n_i)$ for all $n_i \leq k \leq n_{i+1}$ we have that

$$\|x_2(k)\|^2 \leq \tilde{M} e^{-\alpha(k-k_0)} \|\bar{x}_2\|^2 \text{ for all } k_0 \leq k \leq k_0+N,$$

where the positive constants α , $\tilde{M}$ can be determined a priori from the system parameters and are in particular independent of the choice of N . $\square$

This last lemma is basic in proving our main theorem.

Theorem 10

The control sequence (3) minimizes the infinite planning horizon optimization problem (2').

Moreover, for any $\epsilon > 0$ there is an N such that $\|K_i(k) - K(k)\| < \epsilon$, for all $i \geq N$, where N can be calculated from the system parameters.

Proof

From corollary 6 we have that $\lim_{N \rightarrow \infty} K_N(k)$ exists for any k . Consequently, $\lim_{N \rightarrow \infty} u_N(k)$ in theorem 3 exists. Denote this limit by $\bar{u}(k)$. Then, according to Bellman's principle we have that

$$\lim_{N \rightarrow \infty} J_N(\bar{x}(k_0), \bar{u}) \leq \min_{u[k_0, \dots]} \lim_{N \rightarrow \infty} J_N.$$

Since J_N consists of a sum of positive functions, elementary analysis shows that we also have

$$\lim_{N \rightarrow \infty} J_N(\bar{x}(k_0), \bar{u}) \geq \min_{u[k_0, \dots]} \lim_{N \rightarrow \infty} J_N.$$

Thus we conclude that $\bar{u}[k_0, \dots]$ minimizes $\lim_{N \rightarrow \infty} J_N$.

Now,

$$\begin{aligned} & \bar{x}(k_0)K(k_0)\bar{x}(k_0) - \bar{x}(k_0)K_N(k_0)\bar{x}(k_0) = \\ & \min_{u[k_0, \dots]} \lim_{N \rightarrow \infty} J_N(\bar{x}(k_0), u) - \min_{u[k_0, k_0+N-1]} J_N(\bar{x}(k_0), u) \leq \\ & \min_{u[k_0, k_0+N-1]} J_N(\bar{x}(k_0), u) + J_\infty(\bar{x}(k_0+N), \tilde{u}) - \min_{u[k_0, k_0+N-1]} J_N(\bar{x}(k_0), u) = \\ & J_\infty(\bar{x}(k_0+N), \tilde{u}), \end{aligned} \tag{i}$$

where $\bar{x}(k_0+N)$ is the state of the system obtained by applying the state-feedback control which minimizes $J_N(\bar{x}(k_0), u)$, and $\tilde{u}[k_0+N, \dots]$ is a control sequence similar to the one we constructed in lemma 5.

Using the results of lemma 9 and the exponential stability of Σ_1 and Σ_3 we can estimate $\|x(k_0+N)\|^2 \leq \tilde{M} e^{-\alpha(N-k_0)} \|\bar{x}(k_0)\|^2$ for some positive constants α and $\tilde{M}$ which can be obtained a priori from the system parameters. Consequently,

$$\bar{x}^T(k_0)K(k_0)\bar{x}(k_0) - \bar{x}^T(k_0)K_N(k_0)\bar{x}(k_0) \leq M_1 \tilde{M} e^{-\alpha(N-k_0)} \|\bar{x}(k_0)\|^2,$$

which yields the announced result. $\square$

Summarising, the following two basic steps are required to calculate an approximate solution for $K(k_0)$.

Algorithm 11

- 1) Find an estimate for $\min_{u[k_0, \dots]} \lim_{N \rightarrow \infty} J_N(\bar{x}(k_0), u)$ which just depends on $\bar{x}(k_0)$ and is independent of the time k_0 .
- 2) Find an estimate for $x_2^0(k_0+N)$, which results if the control $u^0[k_0, k_0+N-1]$ is applied minimizing $J_N(\bar{x}(k_0), u)$. This estimate must satisfy the requirement that $\|x_2^0(k_0+N)\| \leq M e^{-\alpha(N-k_0)} \|x_2(k_0)\|$, for some positive constants α and M independent of N . $\square$

In the above analysis we showed that given our assumptions (2) on the system it is always possible to obtain these estimates. However, since the estimates that can be deduced from our analysis are rather rough, is a cumbersome exercise and, probably, can be much improved if one is confronted with a "real" system we choose not to present actual estimates.

We conclude with a simple example illustrating the two basic steps in the algorithm. Since we like to have the opportunity to compare the results of our analysis with the exact solution of (ARE), we chose a time-invariant example.

Example 12

Consider the scalar system

$$x(k+1) = x(k) + u(k); x(0) = \bar{x}$$

(4)

$$y(k) = x(k),$$

together with the cost criterion: $J_N := \sum_{k=0}^{N-1} \{y^2(k) + u^2(k)\} + y^2(N)$.

Simple calculations show that the solution to the algebraic Riccati equation corresponding to the problem $\min_{u[0, \dots]} \lim_{N \rightarrow \infty} J_N$, subject to (4) is $K = \frac{1+\sqrt{5}}{2}$.

Now consider the control sequence $u(0) = -\bar{x}$, $u[1, \dots] = 0$. Using this control sequence it is clear that

$$\min_{u[0, \dots]} \lim_{N \rightarrow \infty} J_N(\bar{x}, u) \leq 2\bar{x}^2.$$

So, $K_N(k) \leq 2$ for all $k, N \in \mathbb{N}$ with $0 \leq k \leq N \leq \infty$.

Furthermore, it is clear that the system is periodically smoothly observable with parameters $k_1 = 1$ and $\omega = 1$ and that it is periodically smoothly controllable with the same parameters.

To find an estimate for the state $x^0(N)$ if the optimal control from theorem 3 is applied, we use lemma 9.

From (i) in this lemma we have that

$$V_k - V_{k+2} \geq (1 + \epsilon - \gamma\epsilon)x^2(k), \quad (\text{ii})$$

where γ is such that $(A - BF_N)^T(k+2, k)(A - BF_N)(k+2, k) \leq \gamma I$. It is easily seen that $(A - BF_N(k)) = \frac{K_N(k+1)}{1 + K_N(k+1)}$. Consequently, $\gamma = 1$ is an appropriate bound.

Since $V_k = x^T(k)(K_N(k) + \epsilon I)x(k)$, we get from (i) that

$$x^2(k) \geq \frac{1}{2+\epsilon} V_k. \text{ So, from (ii) we obtain}$$

$$V_{k+2} \leq \left[\frac{1+\epsilon}{2+\epsilon} \right] V_k, \text{ with } V_0 \leq (2+\epsilon)\bar{x}^2.$$

From this last inequality we deduce that $V_{2k} \leq \left[\frac{1+\epsilon}{2+\epsilon} \right]^k (2+\epsilon)\bar{x}^2$, $k = 1, 2, \dots$

Since $x(2k+1) = \frac{K_N(2k+2)}{1+K_N(2k+2)} x(2k) \leq x(2k)$ we conclude that $x^2(k) \leq \frac{2+\epsilon}{\epsilon} \left(\frac{1+\epsilon}{2+\epsilon}\right)^{\lfloor \frac{k}{2} \rfloor} x^2$. Here $\lfloor \frac{k}{2} \rfloor :=$ largest integer smaller or equal than $\frac{k}{2}$.

So, choosing $\epsilon = 1$, we conclude that $x^{0^2}(N) \leq 3 \left(\frac{2}{3}\right)^{\lfloor \frac{N}{2} \rfloor} x^2$. Using inequality (i) from theorem 10 we finally have:

$$\bar{x}^T K \bar{x} - \bar{x}^T K_N(0) \bar{x} \leq 2x^{0^2}(N) \leq 6 \cdot \left(\frac{2}{3}\right)^{\lfloor \frac{N}{2} \rfloor} x^2,$$

which yields the estimate $\|K - K_N(0)\|^2 \leq 6 \cdot \left(\frac{2}{3}\right)^{\lfloor \frac{N}{2} \rfloor}$.

So given an a priori desired approximation error ϵ we conclude that if N is chosen such that $\lfloor \frac{N}{2} \rfloor > \frac{\ln \epsilon / 6}{\ln 2/3}$, we have $\left\| \frac{1+\sqrt{5}}{2} - K_N(0) \right\|^2 < \epsilon$, where $K_N(k)$ satisfies the backwards recurrence equation: $K_N(N) = 1$; $K_N(k) = K_N(k+1) - \frac{K_N^2(k+1)}{1+K_N(k+1)} + 1$. □

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