Tilburg University

On the existence and computation of an equilibrium in an economy with constant returns to scale production

van der Laan, G.; Talman, A.J.J.; Kremers, J.A.W.M.

Publication date:
1992

[Link to publication in Tilburg University Research Portal](#)

Citation for published version (APA):

van der Laan, G., Talman, A. J. J., & Kremers, J. A. W. M. (1992). *On the existence and computation of an equilibrium in an economy with constant returns to scale production*. (Research memorandum / Tilburg University, Department of Economics; Vol. FEW 577). Unknown Publisher.

General rights

Copyright and moral rights for the publications made accessible in the public portal are retained by the authors and/or other copyright owners and it is a condition of accessing publications that users recognise and abide by the legal requirements associated with these rights.

- Users may download and print one copy of any publication from the public portal for the purpose of private study or research.
- You may not further distribute the material or use it for any profit-making activity or commercial gain
- You may freely distribute the URL identifying the publication in the public portal

Take down policy

If you believe that this document breaches copyright please contact us providing details, and we will remove access to the work immediately and investigate your claim.

ECO

CBM

R

7626

RSITY

7626

1992

577

IVE

UNIVERSITEIT

BRABANT

POSTBOX 90153

5000 LE TILBURG

THE NETHERLANDS

* C I N O 1 8 7 3 *

R11

Models

Equilibrium Theory
Economics of Scale

DEPARTMENT OF ECONOMICS
RESEARCH MEMORANDUM

ON THE EXISTENCE AND COMPUTATION OF AN
EQUILIBRIUM IN AN ECONOMY WITH
CONSTANT RETURNS TO SCALE PRODUCTION

Gerard van der Laan, Dolf Talman,
Hans Kremers

FEW 577

Communicated by Prof.dr. P.H.M. Ruys

On the existence and computation of an equilibrium in an economy with constant returns to scale production*

Gerard van der Laan[†], Dolf Talman[‡], and Hans Kremers[‡]

October, 1992

*This research is part of the VF-program "Competition and Cooperation".

[†]Department of Econometrics, Free University, De Boelelaan 1105, 1081 HV Amsterdam, The Netherlands

[‡]Department of Econometrics, Tilburg University, P.O. Box 90153, 5000 LE Tilburg, The Netherlands

Abstract

In this paper we consider the problem of finding an equilibrium in an economy with nonlinear constant returns to scale production activities. To find an equilibrium we propose an adjustment process in which the prices of the commodities and the activity levels of production adjust simultaneously. The process starts at a price vector at which each production activity has non positive profit. We show that the process follows a path which connects the starting point with an equilibrium of the economy. From this it follows that the existence of a price vector at which each production activity has non positive profit implies the existence of an equilibrium. The equilibrium can be computed by using a simplicial algorithm or by solving a sequence of Linear Variational Inequality Problems.

Key words: constant returns to scale production, equilibrium, adjustment process, simplicial approximation

1 INTRODUCTION

In this paper we consider the equilibrium problem in an economy with nonlinear constant returns to scale production activities. In Van den Elzen, Van der Laan and Talman [7] an adjustment process has been introduced to find an equilibrium in an economy with linear production activities. This adjustment process starts at an arbitrarily chosen price vector in the unit price simplex, such that all firms have negative profits per unit production. Therefore the activity levels of all firms are set initially equal to zero. Along the path followed by the process prices and activity levels are adjusted according to the sign vector of the net excess demand and keeping the levels of the activities equal to zero as long as the profits are negative. It is shown that the adjustment process converges to a pair of equilibrium prices and activity levels. In Kremers and Talman [9] an iterative algorithm to solve the equilibrium problem has been given. This algorithm solves the problem through solving a sequence of Linear Variational Inequality Problems on the set of feasible prices.

In this paper we generalize the adjustment process of Van den Elzen, Van der Laan and Talman [7] to find an equilibrium in an economy with nonlinear constant returns to scale production. The production side of such an economy can be described by an activity matrix whose entries depend continuously on the prices. A sufficient condition for the existence of an equilibrium is that at any price vector there can be no production without input. Under this condition there exists a price vector at which all production technologies have negative profits per unit output. Starting at such a price vector we propose an adjustment process which generically converges to an equilibrium of the economy. This process increases initially the prices of the commodities with positive excess demand and decreases the prices of the commodities with negative excess demand. Firms only produce when they make zero profits.

In Section 2 we describe the model and we state conditions under which there exists a price vector for which all production technologies have negative profits. In Section 3 we describe the adjustment process and prove that the process converges to an equilibrium. The Sections 4 and 5 are concerned with the computation of an equilibrium. A summary and conclusions are given in Section 6.

2 THE MODEL

We consider an economy with a finite number of consumers, m firms having constant returns to scale production technologies, indexed by $i = 1, \dots, m$, and $n + 1$ commodities, indexed by $j = 1, \dots, n + 1$.

The consumers are assumed to maximize their utilities under their budget constraints. Each consumer is endowed with an $(n + 1)$ -dimensional strictly positive vector of commodities. At some price vector $p \in \mathbb{R}_+^{n+1} \setminus \{0\}$, the budget of a consumer is the value of his endowment. Let $d(p)$ be the aggregate demand of the consumers at price vector p and let $z(p)$ be the excess demand, i.e. $z(p)$ is the aggregate demand $d(p)$ minus the total initial endowments. Under standard assumptions on the utility functions of the consumers and bounded consumption sets, utility maximization yields an excess demand function z being homogeneous of degree zero in the prices, i.e., $z(\lambda p) = z(p)$ for all $\lambda > 0$, and which

satisfies Walras' law, i.e., for all p , $p^T z(p) = 0$. Moreover, z is a continuous function of p on $\mathbf{R}_+^{n+1}$, see e.g. Varian [15].

Concerning the production technologies we assume that each firm produces just one output. Let $\pi: \{1, \dots, m\} \rightarrow \{1, \dots, n+1\}$ be a function which assigns the output commodity $\pi(i) \in \{1, \dots, n+1\}$ to the production technology $i \in \{1, \dots, m\}$. For $a \in \mathbf{R}^{n+1}$, let a^h be the n -vector $(a_1, \dots, a_{h-1}, a_{h+1}, \dots, a_{n+1})^T$. Then firm i produces commodity $\pi(i)$ according to a continuous production function $f^i: \mathbf{R}_+^n \rightarrow \mathbf{R}$, i.e., $y_{\pi(i)} = f^i(y^{-\pi(i)})$ is the amount of commodity $\pi(i)$ produced by firm i when $y^{-\pi(i)}$ is the n -vector of inputs of the other commodities $j \neq \pi(i)$. Observe that for some given commodity j , the set $\{i \mid \pi(i) = j\}$ might be empty or contain more than one element. We assume that there is constant returns to scale production.

Assumption P

For all $i = 1, \dots, m$, $f^i(\lambda y^{-\pi(i)}) = \lambda f^i(y^{-\pi(i)})$ for all $\lambda > 0$.

From production theory (e.g. see Varian [15]) we know that cost-minimizing behaviour of producer i yields a cost function $c^i: \mathbf{R}_+^n \rightarrow \mathbf{R}_+$ per unit of output of commodity $\pi(i)$ satisfying the next properties.

1. c^i is non decreasing in p_j , $j \neq \pi(i)$,
2. c^i is homogeneous of degree 1 in $p^{-\pi(i)}$,
3. c^i is concave in p_j , $j \neq \pi(i)$,
4. c^i is continuous in $p^{-\pi(i)}$ for $p^{-\pi(i)} \in \mathbf{R}_+^n$.

Furthermore, we assume differentiability of the cost functions.

Assumption C

For all $i = 1, \dots, m$, c^i is differentiable in $p^{-\pi(i)} \in \mathbf{R}_+^n$.

With Euler's law it follows from property (2) that for every $p^{-\pi(i)} \in \mathbf{R}_+^n$ that

$$c^i(p^{-\pi(i)}) = \sum_{j \neq \pi(i)} p_j \frac{\partial c^i(p^{-\pi(i)})}{\partial p_j}. \quad (1)$$

Since production goes to infinity if profits per unit of output are positive, we have that in equilibrium prices have to satisfy the nonpositive profit conditions, i.e.,

$$p_{\pi(i)} \leq c^i(p^{-\pi(i)}), \quad i = 1, \dots, m. \quad (2)$$

Together with (1) condition (2) can be written as

$$p^T a^i(p) \leq 0, \quad i = 1, \dots, m,$$

where $a_{\pi(i)}^i(p) = 1$ and $a_j^i(p) = -\partial c^i / \partial p_j$, $j \neq \pi(i)$. From Hotelling's Lemma we know that $y_j^i(p) = \partial c^i / \partial p_j$, $j \neq \pi(i)$, is the cost-minimizing amount of input of good j needed to

produce one unit of output of commodity $\pi(i)$. So, $a_j^i(p)$, $j \neq \pi(i)$, is the negative supply of commodity j by producer i per unit of output of commodity $\pi(i)$ at prices p . Following others (e.g. see Morishima [13]) we have now written the nonpositive profit condition as a generalization of the nonpositive profit condition $p^T a^i \leq 0$ for a linear production technology with activity vector a^i . The only difference is that the amount of input of good $j \neq \pi(i)$ per unit of output of good $\pi(i)$ depends on p . Observe that according to property (2) of the cost functions, the input $a_j^i(p)$ is homogeneous of degree zero in p . This formulation is well-known in AGE (Applied General Equilibrium) models, e.g. see Mathiesen [12] or Fischer, Froberg, Keyzer and Parikh [8].

Summarizing we have that for positive prices the economy is characterized by the excess demand function z and the input-output functions a^i , $i = 1, \dots, m$, with for all $p \in \mathbf{R}_+^{n+1}$, $p^T z(p) = 0$ and $p^T a^i(p)$ the profit of firm i per unit production of commodity $\pi(i)$, $i = 1, \dots, m$. Moreover, z and a^i , $i = 1, \dots, m$, are homogeneous in p of degree zero, and continuous on $\mathbf{R}_+^{n+1}$.

Let x be a nonnegative m -vector $(x_1, \dots, x_m)^T$ of production levels, i.e., at production level x_i , producer i produces x_i units of output of commodity $\pi(i)$. Hence at price vector p and production level x_i the input-output vector or supply vector of producer i is $x_i a^i(p)$. With $A(p)$ the $(n+1) \times m$ matrix $[a^1(p), \dots, a^m(p)]$, the net supply of the production side at price vector p and production level vector x is $A(p)x$. Now an equilibrium for this economy is a price vector p^* and a production level vector x^* , such that the excess demand of the consumption sector is less than or equal to the net supply of the production sector and each production technology satisfies the nonpositive profit condition. Let the net excess demand $f: \mathbf{R}_+^{n+1} \setminus \{0\} \times \mathbf{R}_+^m \rightarrow \mathbf{R}^{n+1}$ be defined as the excess demand of the consumption sector at p minus the net supply of the production sector at (p, x) , i.e., $f(p, x) = z(p) - A(p)x$.

Definition 2.1 A pair $(p^*, x^*) \in \mathbf{R}_+^{n+1} \setminus \{0\} \times \mathbf{R}_+^m$ is an equilibrium if

1. $f(p^*, x^*) \leq 0$,
2. $p^{*T} A(p^*) x^* \leq 0$.

From 1) it follows that at an equilibrium $p^{*T} f(p^*, x^*) = p^{*T} z(p^*) - p^{*T} A(p^*) x^* = -p^{*T} A(p^*) x^* \leq 0$, whereas 2) implies that $p^{*T} A(p^*) x^* \leq 0$. Hence $p^{*T} A(p^*) x^* = 0$, so that we have complementarity between x^* and $p^{*T} A(p^*)$, i.e., $x_i^* = 0$ if $p^{*T} a^i(p^*) < 0$ and $p^{*T} a^i(p^*) = 0$ if $x_i^* > 0$. Furthermore, it follows that $p^{*T} f(p^*, x^*) = 0$, so that we also have complementarity between p^* and $f(p^*, x^*)$. In equilibrium commodities can only be in excess demand if the corresponding prices are zero. Since both z and a^i , $i = 1, \dots, m$, are homogeneous of degree zero in the prices we have that if (p^*, x^*) is an equilibrium pair, then also $(\lambda p^*, x^*)$ is an equilibrium pair for any $\lambda > 0$. Hence we can normalize the price vectors to the n -dimensional unit simplex

$$S^n = \{p \in \mathbf{R}_+^{n+1} \mid \sum_j^{n+1} p_j = 1\}.$$

From condition 2) of Definition 2.1 it follows that the set of feasible prices in S^n is the set

$$T^n = \{p \in S^n \mid p^T A(p) \leq 0\}.$$

The concavity of the cost functions implies that T^n is a convex subset of S^n . Since T^n is the set of prices satisfying the nonpositive profit condition, a necessary condition for the existence of an equilibrium is that the set T^n is not empty. Therefore we make the following no free production assumption.

Assumption F

For any $p \in S^n$, $A(p)x \geq 0$ and $x \geq 0$ implies that $x = 0$.

Assumption F says that for any price vector there can not be production without input.

Theorem 2.2 The set T^n is not empty.

Proof. Let $\mu: S^n \rightarrow S^n$ be a correspondence defined by

$$\mu(p) = \{q \in S^n \mid q^T A(p) \leq 0\}, p \in S^n.$$

Assumption F says that $A(p)x = 0$ if $A(p)x \geq 0$ and $x \geq 0$. According to Farkas' lemma, this implies that $\mu(p)$ is not empty. For given p , $A(p)$ is a matrix of fixed coefficients. Hence, $\mu(p)$ is a subset of S^n defined by linear constraints and so $\mu(p)$ is convex. Furthermore, $\mu(p)$ is closed and hence also compact. Finally, since the functions a^i , $i = 1, \dots, m$, are continuous in p , it follows that μ is upper semi-continuous in p . Hence μ satisfies the conditions of Kakutani's fixed point theorem. From this theorem we have that there exists a fixed point $p^* \in \mu(p^*) = \{q \in S^n \mid q^T A(p^*) \leq 0\}$. Hence $p^* \in T^n$. This proves that T^n is not empty.

Q.E.D.

The next lemma shows that assumption F is not only sufficient to prove that T^n is not empty, but also that the relative interior of T^n , defined by

$$\text{int}(T^n) = \{p \in S^n \mid p_j > 0, j = 1, \dots, n+1 \text{ and } p^T a^i(p) < 0, i = 1, \dots, m\}$$

has dimension n and hence $\text{int}(T^n)$ is not empty. We will use this lemma to prove the existence of an equilibrium by defining a path which connects an arbitrarily chosen price vector in $\text{int}(T^n)$ with an equilibrium price vector. Let $\text{int}(\mu(q)) = \{p \in S^n \mid p_j > 0, j = 1, \dots, n+1 \text{ and } p^T a^i(q) < 0, i = 1, \dots, m\}$ denote the relative interior of $\mu(q)$.

Lemma 2.3 $\text{Int}(T^n)$ has dimension n .

Proof. Let p^* be a fixed point of μ . According to Farkas' lemma the relative interior of $\mu(p^*)$ has dimension n . Now, let p^k , $k = 1, 2, \dots$, be any sequence of price vectors in the relative interior of $\mu(p^*)$ converging to p^* . So, for all k , $p_j^k > 0$, $j = 1, \dots, n+1$, and $p^{kT} a^i(p^*) < 0$, $i = 1, \dots, m$. Since a^i is continuous in p , there exists an M such that $p^{kT} a^i(p^k) < 0$ for $k > M$. Hence p^k , $k > M$, is in the relative interior of T^n . Therefore the dimension of $\text{int}(T^n)$ is also n .

Q.E.D.

Also the next lemma will be used in the the following sections.

Lemma 2.4 Let v be a point in the relative interior of T^n . Then $v \in \text{int}(\mu(q))$ for any $q \in S^n$.

Proof. Since v is in $\text{int}(T^n)$, we have that $v_j > 0$ for all j . So, it remains to prove that for any $q \in S^n$, $v^T a^i(q) < 0$ for $i = 1, \dots, m$. Define the function $g^i: S^n \rightarrow \mathbb{R}$ by

$$g^i(p) = p_{\pi(i)} - c^i(p^{-\pi(i)}).$$

Since c^i is concave, it follows that g^i is a convex function of p . Hence, for any q we have

$$g^i(v) \geq g^i(q) + (v - q)^T Dg^i(q),$$

where $Dg^i(p) = (\frac{\partial g^i(p)}{\partial p_1}, \dots, \frac{\partial g^i(p)}{\partial p_{n+1}})^T$. By definition we have that $Dg^i(p) = a^i(p)$. Hence

$$g^i(v) \geq g^i(q) + v^T a^i(q) - q^T a^i(q).$$

Since $g^i(q) = q^T a^i(q)$ it follows that $v^T a^i(q) \leq g^i(v) = v^T a^i(v) < 0$.

Q.E.D.

Observe that by the same reasoning we have that $v \in T^n$ implies that $v^T a^i(q) \leq 0$ for all $q \in S^n$. So, if we linearize the production side of the economy in a point $q \in S^n$ by taking the input-output coefficients $a^i(q)$ at point q as fixed coefficients, then we have that $T^n \subset \{p \in S^n \mid p^T A(q) \leq 0\}$. Thus we have the following corollary.

Corollary 2.5 For any $q \in S^n$, the set of feasible prices T^n is a subset of the set of feasible prices of the economy with linear technologies $a^i(q)$, $i = 1, \dots, m$.

Example

Take $n = 2$ and suppose there is one technology given by $a(p) = (1, -\sqrt{p_3/p_2}, -\sqrt{p_2/p_3})^T$. Then T^n is given by

$$T^n = \{p \in S^2 \mid p_1 - 2\sqrt{p_2 p_3} \leq 0\}.$$

Take $q = (1 - 2b, b, b)^T$ for $0 \leq b \leq 1/2$. Then for any b , $a(q) = (1, -1, -1)^T$ and

$$T^n \subset \{p \in S^2 \mid p^T a(q) \leq 0\} = \{p \in S^2 \mid p_1 - p_2 - p_3 \leq 0\}.$$

Observe that for any $q = (1 - 2b, b, b)^T$, the set $\{p \in S^2 \mid p_1 - p_2 - p_3 = 0\}$ is tangent on T^n in the point $p = (1/2, 1/4, 1/4)^T$.

3 AN ADJUSTMENT PROCESS

In Van den Elzen, Van der Laan and Talman [7] an adjustment process has been given for an economy with linear production activities represented by $(n + 1)$ -vectors a^i , $i = 1, \dots, m$. In this section we generalize this process for an economy with nonlinear production technologies represented by the functions a^i , $i = 1, \dots, m$. We assume that Assumption

F is satisfied. Then, let v be a point in $\text{int}(T^n)$. Such a point can be found by applying a simplicial fixed point algorithm to find a fixed point of the correspondence μ . Without loss of generality we may assume that $z(v)$ does not contain zero components. For a set $U \subset \{1, \dots, m\}$, let $A(U)$ be defined by

$$A(U) = \{(p, x) \in T^n \times \mathbb{R}_+^m \mid p^T a^i(p) = 0 \text{ for } i \in U, x_i = 0 \text{ for } i \notin U\}.$$

Observe that for $(p, x) \in A(U)$, $p^T f(p, x) = p^T [z(p) - A(p)x] = p^T z(p) - \sum_{i=1}^m x_i p^T a^i(p) = 0$, i.e., the value of the excess demand $z(p)$ minus the value of the net production $A(p)x$ is equal to zero. So, for positive prices, either $f(p, x) = 0$ and we have an equilibrium, or $f(p, x)$ has both a negative and a positive component. Let the set S of feasible sign vectors be defined as the set of sign vectors in $\mathbb{R}^{n+1}$ with at least one component positive and one component negative, i.e.

$$S = \{s \in \mathbb{R}^{n+1} \mid s_j \in \{-1, 0, 1\}, j = 1, \dots, n+1, \text{ with } s_j = -1 \text{ for at least one } j \text{ and } s_k = +1 \text{ for at least one } k\}.$$

Given the starting price vector in the relative interior of T^n , we define for $s \in S$ and $U \subset \{1, \dots, m\}$ the sets $A(s, U)$ by

$$A(s, U) = \{(p, x) \in A(U) \mid p_j/v_j = \min_k p_k/v_k \text{ when } s_j = -1 \text{ and } p_j/v_j = \max_k p_k/v_k \text{ when } s_j = +1\}.$$

This definition says that for a pair $(p, x) \in A(s, U)$, $p_j = \alpha v_j$ if $s_j = -1$, $p_j = \beta v_j$ if $s_j = +1$ and $\alpha v_j \leq p_j \leq \beta v_j$ if $s_j = 0$, with $\alpha = \min_k p_k/v_k$ and $\beta = \max_k p_k/v_k$.

Now, let $|U|$ be the number of elements of U and let $|s|$ be the number of components of s equal to zero. Then the restrictions $p_j = \alpha v_j$ if $s_j = -1$ and $p_j = \beta v_j$ if $s_j = +1$ put $n - 1 - |s|$ restrictions on the prices. Moreover, $p^T a^i(p) = 0$ for $i \in U$ puts $|U|$ restrictions on the prices. So, for $(p, x) \in A(s, U)$, the number of restrictions on the prices is $n - 1 - |s| + |U|$. Since the dimension of S^n is equal to n , $n - (n - 1 - |s| + |U|) = 1 + |s| - |U|$ degrees of freedom are left for the price variables. So, in case $|U| > 1 + |s|$, no degrees of freedom are left and in general there is no price vector satisfying all the conditions, i.e., $A(s, U)$ is empty. On the other hand, for $(p, x) \in A(s, U)$ and hence $(p, x) \in A(U)$, there are $|U|$ degrees of freedom for the x -variables. Hence, if $|U| \leq 1 + |s|$ we have that $A(s, U)$ is generically a well-defined manifold of dimension $1 + |s| - |U| + |U| = 1 + |s|$. Finally, for $s \in S$ and $U \subset \{1, \dots, m\}$ we define the set $B(s, U)$ by

$$B(s, U) = \{(p, x) \in A(s, U) \mid f_j(p, x) \geq 0 \text{ when } s_j = +1, \\ f_j(p, x) \leq 0 \text{ when } s_j = -1, \\ f_j(p, x) = 0 \text{ when } s_j = 0\}.$$

So, $B(s, U)$ is the closure of the set of pairs $(p, x) \in A(s, U)$, such that the componentwise sign of the net excess demand is equal to the sign vector s , i.e., s_j is the sign of $f_j(p, x)$,

$j = 1, \dots, n+1$. Observe that $B(s, U)$ puts $|s|$ restrictions on the $(1 + |s|)$ -manifold $A(s, U)$. Assuming standard nondegeneracy, regularity and transversality conditions, this implies that the set $B(s, U)$ is either empty or a 1-manifold, i.e., a collection of smooth paths and loops. A path has either two end points, or one end point and goes to infinity, or comes from and goes to infinity. So, a bounded path has two end points. Let B be the union over all $s \in S$ and all $U \subset \{1, \dots, m\}$ of the sets $B(s, U)$. Using the same arguments as in [7] for the case of linear production, we have that an end point (p, x) of a path in $B(s, U)$ either lies in a different set $B(s', U')$ or satisfies one of the following cases:

- a) $(p, x) = (v, 0)$,
- b) $\min_k p_k/v_k = 0$,
- c) $f_j(p, x) = 0$ and $s_k \in \{-1, 0\}$ for all $k \neq j$,
- d) $f_j(p, x) = 0$ and $s_k \in \{+1, 0\}$ for all $k \neq j$.

In case an end point (p, x) of a path in $B(s, U)$ lies in some other $B(s', U')$, then (p, x) is also an end point of a path in $B(s', U')$. We have that either for just one component j , $s'_j \in \{s_{j+1}, s_{j-1}\}$, while $s'_k = s_k$ for all $k \neq j$ and $U' = U$, or $s' = s$ and for some $i \in U$, $U' = U \setminus \{i\}$ or for some $i \notin U$, $U' = U \cup \{i\}$. Linking together all paths we get that the set B is generically a collection of piecewise smooth paths and loops. Each end point (p, x) of a path in B satisfies one of the cases a)-d).

Suppose that case a) occurs. Since $(v, 0)$ is an end point of a path in B , there exist an s and U , such that $(v, 0)$ is an end point of $B(s, U)$. We have assumed that $z(v)$ does not contain zero components and that $v^T a^i(v) < 0$ for all i . Hence, s must be equal to s^0 with $s_j^0 = \text{sign } z_j(v) = \text{sign } f_j(v, 0)$, $j = 1, \dots, n+1$ and U must be the empty set. It follows that $(v, 0)$ is an end point of a path in $B(s, U)$ if and only if $s = s^0$ and $U = \emptyset$.

In case b), by definition of the set $A(s, U)$, we must have that $p_j = 0$ for all j with $s_j = -1$. By definition of $B(s, U)$ we have that $f_j(p, x) \leq 0$ for all j with $s_j = -1$. Hence $f_j(p, x) \leq 0$ for all j with $p_j = 0$. Moreover, $p_j > 0$ for all j with $f_j(p, x) > 0$. Since $p^T f(p, x) = 0$ for all $(p, x) \in A(U)$, it follows that $f_j(p, x) = 0$ for all j with $p_j > 0$. Hence $f(p, x)$ satisfies the conditions of Definition 2.1 and therefore (p, x) is an equilibrium pair.

In case c) we have by definition of the set $B(s, U)$ that $f_k(p, x) \leq 0$ for all $k \neq j$. Hence (p, x) is an equilibrium pair.

In case d) we have that $f_k(p, x) \geq 0$ for all $k \neq j$ and $f_j(p, x) = 0$. Suppose that for some k , $f_k(p, x) > 0$. Then by definition of the set $B(s, U)$ we have that $s_k = +1$ and hence by definition of the set $A(s, U)$ we must have that $p_k/v_k = \max_h p_h/v_h > 0$. This contradicts the fact that $p^T f(p, x) = 0$. Hence we must have that $f_k(p, x) = 0$ and (p, x) is an equilibrium pair.

Consequently we can conclude that B contains a path with $(v, 0)$ in $B(s^0, 0)$ as one of its end points. All other end points of a path in B satisfy one of the cases b), c) or d) and therefore is an equilibrium pair. So the path in B starting in $(v, 0)$ leads to an equilibrium if this path is bounded.

Lemma 3.1 Each path in B is bounded.

Proof. Suppose that some path in B is unbounded. Then there exists a sequence (p^k, x^k) , $k = 1, 2, \dots$, in B with some of the components of (p^k, x^k) going to infinity. Without loss of generality we may assume that for some $s \in S$ and some set U it holds that $(p^k, x^k) \in A(s, U)$ for all k . Since $p^k \in T^n$ and therefore the components of p^k are bounded, we must have that some components of the sequence x^k go to infinity. Moreover, T^n is compact and hence the sequence p^k has a cluster point q in T^n . It is sufficient to consider a (sub)sequence of points (p^k, x^k) converging to q . Since $(p^k, x^k) \in A(s, U)$ for all k we have that $q^T a^i(q) = 0$ for all $i \in U$. Because $(p^k, x^k) \in B(s, U)$ we also have that $f_j(p^k, x^k) \geq 0$ when $s_j = +1$, $f_j(p^k, x^k) \leq 0$ when $s_j = -1$, and $f_j(p^k, x^k) = 0$ when $s_j = 0$. Hence there exist numbers $\mu_j^k \geq 0$, for all j with $s_j \neq 0$, such that

$$f(p^k, x^k) - \sum_{\{j|s_j \neq 0\}} \mu_j^k s_j e(j) = z(p^k) - \sum_{i \in U} x_i^k a^i(p^k) - \sum_{\{j|s_j \neq 0\}} \mu_j^k s_j e(j) = 0,$$

with $e(j)$ the j -th unit vector in $\mathbf{R}^{n+1}$. Now, suppose that some components of x^k go to infinity and hence some of the components of the nonnegative solution (p^k, x^k, μ^k) go to infinity. Since p^k converges to q , we must have that some components of (x^k, μ^k) go to infinity. With q the limit point of the sequence p^k , we have that $z(p^k)$ converges to $z(q)$ and $a^i(p^k)$ to $a^i(q)$, $i \in U$. So, if some components of the sequence (x^k, μ^k) go to infinity, then we must have nonnegative vectors x^0 and μ^0 and nonzero nonnegative vectors x^* and μ^* , such that the linear system

$$z(q) - \sum_{i \in U} x_i a^i(q) - \sum_{\{j|s_j \neq 0\}} \mu_j s_j e(j) = 0, \quad (3)$$

has a ray of solutions $x = x^0 + \lambda x^*$, $\mu = \mu^0 + \lambda \mu^*$, for all $\lambda \geq 0$. Taking equation (3) for $\lambda = 1$ and $\lambda = 0$ and subtracting the latter equation from the first we get that

$$\sum_{i \in U} x_i^* a^i(q) + \sum_{\{j|s_j \neq 0\}} \mu_j^* s_j e(j) = 0, \quad (4)$$

must hold. Since $q \in A(s, U)$, there is a number b , $0 \leq b \leq 1$, and a vector $w \in \mathbf{R}_+^{n+1}$ with $w_j > 0$ if $s_j = +1$, $w_j \geq 0$ if $s_j = 0$ and $w_j = 0$ if $s_j = -1$, such that $q = bv + w$. Since $q^T a^i(q) = (bv + w)^T a^i(q) = 0$ for $i \in U$, and, according to Lemma 2.4, $v^T a^i(q) < 0$, it follows that $w^T a^i(q) > 0$ for $i \in U$. Premultiplying the system (4) with w^T yields

$$\sum_{i \in U} x_i^* w^T a^i(q) + \sum_{\{j|s_j = 1\}} \mu_j^* w_j = 0. \quad (5)$$

Since $w^T a^i(q) > 0$ for all $i \in U$ and $w_j > 0$ for all j with $s_j = 1$, (5) can only hold when $x_i^* = 0$ for all $i \in U$ and $\mu_j^* = 0$ for all j with $s_j = 1$. Then (4) becomes

$$\sum_{\{j|s_j = -1\}} \mu_j^* s_j e(j) = 0, \quad (6)$$

which can only hold when $\mu_j^* = 0$ for all j with $s_j = -1$. This contradicts that (x^*, μ^*) is nonzero and hence each path in B is bounded.

Q.E.D.

From Lemma 3.1 we obtain that following the path in B starting at $(v, 0)$ we will reach an equilibrium pair (p^*, x^*) being the other end point of the path. This gives a constructive proof of the existence of an equilibrium under condition F.

Theorem 3.2 Under condition F the nonlinear constant returns to scale production economy has an equilibrium.

The path in B starting in $(v, 0)$ can be interpreted as an adjustment process. Prices and production levels are adjusted until an equilibrium has been reached. As long as all production technologies have negative profits, the prices are adjusted in the relative interior of T^n according to the sign vector of the excess demand, keeping the production levels equal to zero. If some production technology i has zero profit $p^T a^i(p)$, then also the level of this activity is adjusted, while prices are adjusted according to the sign vector of the net demand and keeping the profit of activity i equal to zero. For the case of a linear production technology with fixed coefficients a detailed description of this adjustment process can be found in [7].

4 THE SIMPLICIAL APPROXIMATION OF AN EQUILIBRIUM

In this section we propose a simplicial algorithm to compute an equilibrium of an economy with nonlinear constant returns to scale production. For a survey about simplicial algorithms we refer to e.g. Doup [3] or Allgower and Georg [1]. To follow the path simplicially in case of linear production, each piece of a piecewise linear approximation of the path can be traced by making a linear programming pivot step in a system of $n + m + 2$ linear equations. Starting with a price vector $v \in S^n$, such that $v^T a^i < 0$ for all linear technologies a^i , $i = 1, \dots, m$, a column representing the linear activity a^i is pivoted into the system as soon as for some i , $p^T a^i$ becomes equal to zero. This column remains in the system of equations as long as prices are generated satisfying $p^T a^i = 0$. This makes that this algorithm can not be applied in the case of nonlinear technologies, because the vector of coefficients $a^i(p)$ depends on the prices. The linear programming technique does not allow to adapt all columns representing the technologies $a^i(p)$ with $p^T a^i(p) = 0$ at each new price vector p generated by the algorithm. Therefore we propose an alternative simplicial algorithm to follow piecewise linearly the path described in the previous section. This algorithm can be seen as a modification of the simplicial algorithm introduced by Doup, Van der Laan and Talman [4] on S^n to find an equilibrium of an economy without production. The algorithm operates with a system of only $n + 2$ linear equations, which is an advantage above the algorithm shortly described above to solve the equilibrium problem with linear technologies.

The algorithm of Doup, Van der Laan and Talman is one of the most efficient variable

dimension simplicial algorithms on S^n , originally initiated by Van der Laan and Talman [11]. To solve the zero point problem of an excess demand function z on S^n , the algorithm can start in an arbitrarily chosen interior point of S^n , say v . Given the starting point v , the unit simplex S^n is subdivided in subsets $A(s)$, $s \in S$, defined by

$$A(s) = \{p \in A(U) \mid \begin{aligned} p_j/v_j &= \min_k p_k/v_k \text{ when } s_j = -1 \text{ and} \\ p_j/v_j &= \max_k p_k/v_k \text{ when } s_j = +1 \end{aligned} \}.$$

Observe that $A(s) \cap T^n$ is equal to the set of prices in the projection of $A(S, \emptyset)$ on S^n . The subset $A(s)$ has dimension $|s| + 1$. The algorithm follows by approximating simplicially a path of prices in $A(s)$ for various sign vectors $s \in S$, such that for any $p \in A(s)$ on the path it holds that $z_j(p) \geq 0$ when $s_j = +1$, $z_j(p) \leq 0$ when $s_j = -1$ and $z_j(p) = 0$ when $s_j = 0$. So the algorithm follows in S^n the path of prices described in the previous section in case $m = 0$ (no production technologies). To do so, the algorithm traces a piecewise linear path in a simplicial subdivision of S^n , which has the property that each $(|s| + 1)$ -dimensional set $A(s)$ is triangulated into a finite number of $(|s| + 1)$ -dimensional simplices. An appropriate subdivision having this property is the V -triangulation of S^n , described in Doup and Talman [5]. To adapt the Doup, Van der Laan and Talman algorithm for the case of an economy with nonlinear production technologies, each price vector $p \in S^n$ is labelled according to a vector labelling function $b: S^n \rightarrow \mathbf{R}^{n+2}$.

Labelling Rule. For $p \in S^n$, compute $p^T a^i(p)$, $i = 1, \dots, m$. Now, let h be the technology which maximizes the profit per unit production. In case there are several maximizing technologies, let h be the technology with the smallest index. So, $p^T a^h(p) \geq p^T a^i(p)$ for all $i = 1, \dots, m$ and $h \leq k$ for all k with $p^T a^k(p) = p^T a^h(p)$. Then

- $b(p) = [(-a^h(p))^T, 0]^T$ when $p^T a^h(p) \geq 0$,
- $b(p) = [z^T(p), 1]^T$ when $p^T a^h(p) < 0$.

Except for the last additional component with values 0 or 1, this labelling has been introduced already by Scarf [14] in order to compute an equilibrium for an economy with linear production activities. Now, let t be equal to $|s| + 1$. Then we have the following definition.

Definition 4.1. Let $\sigma(p^1, \dots, p^{t+1})$ be a t -simplex in the simplicial subdivision of S^n . Then the simplex σ is s -complete if the $(n + 2)$ -system of linear equations

$$\sum_{k=1}^{t+1} \lambda_k b(p^k) - \sum_{\{j|s_j \neq 0\}} \mu_j s_j e(j) = e(n + 2), \quad (7)$$

with $e(j)$ the j -th unit vector in $\mathbf{R}^{n+2}$, has a nonnegative solution λ_k^* , $k = 1, \dots, t + 1$, and μ_j^* for each $j = 1, \dots, n + 1$ with $s_j \neq 0$.

Let Z be the piecewise linear approximation of the excess demand function z with respect to the underlying simplicial subdivision, i.e. for p in a t -simplex $\sigma(p^1, \dots, p^{t+1})$, $Z(p) = \sum_k \alpha_k z(p^k)$ if $p = \sum_k \alpha_k p^k$ with $\alpha_k \geq 0$ for all k and $\sum_k \alpha_k = 1$. For an s -complete simplex $\sigma(p^1, \dots, p^{t+1})$ with solution (λ, μ) to the system (7) of linear equations, let U be the subset of $\{1, \dots, m\}$, such that for each $i \in U$, there exists a (unique) index $k(i) \in \{1, \dots, t+1\}$ for which $b(p^{k(i)}) = [(-a^i(p))^T, 0]^T$. So for every index $i \in U$, there is a vertex in σ whose label corresponds to technology i . Furthermore, let K be the subset of $\{1, \dots, t+1\}$, such that vertex p^k , $k \in K$ has label $b(p^k) = [z^T(p^k)^T, 1]^T$. System (7) can now be written as

$$\sum_{k \in K} \lambda_k \begin{pmatrix} z(p^k) \\ 1 \end{pmatrix} - \sum_{i \in U} \lambda_{k(i)} \begin{pmatrix} a^i(p^{k(i)}) \\ 0 \end{pmatrix} - \sum_{\{j | s_j \neq 0\}} \mu_j s_j e(j) = e(n+2). \quad (8)$$

From the last equation it follows that $\sum_{k \in K} \lambda_k = 1$. Hence, K contains at least one element, so that the label of at least one vertex of σ , say p^h , corresponds to the excess demand $z(p^h)$ at p^h . This implies that for at least one vertex p^h of an s -complete simplex σ we have that $p^{hT} a^i(p^h) < 0$ for all $i = 1, \dots, m$, and hence p^h lies in the relative interior of T^n when also $p_j^h > 0$ for all j . The first $n+1$ equations of (8) can now be written as

$$Z(p^*) - \sum_{i \in U} \lambda_{k(i)} a^i(p^{k(i)}) = \sum_{\{j | s_j \neq 0\}} \mu_j s_j e(j), \quad (9)$$

where $p^* = \sum_{k \in K} \lambda_k p^k$. Finally, set $x_i^* = \lambda_{k(i)}$ for $i \in U$ and $x_i^* = 0$ for $i \notin U$. Observe that for all $k \in K$, $p^{kT} a^i(p^k) < 0$ for all $i = 1, \dots, m$. So, if $p_j^* > 0$ for all j , then $p^* \in \text{int}(T^n)$ since T^n is convex.

Theorem 4.2 Let the starting point v be in the interior of S^n and let for some $s \in S$, $\sigma(p^1, \dots, p^{t+1})$ be an s -complete simplex in $A(s)$ with solution (λ, μ) to the system (7) of linear equations. Then σ yields an approximating solution to the equilibrium problem if one of the following properties holds:

- a) $\mu_j = 0$ for all j with $s_j = -1$,
- b) $\mu_j = 0$ for all j with $s_j = +1$,
- c) for all $k = 1, \dots, t+1$ with $\lambda_k > 0$, $p_j^k = 0$ for all j with $s_j = -1$.

Proof. Let U , K , p^* and x^* be defined as above. To prove that (p^*, x^*) is an approximating equilibrium, observe that $p^* \in T^n$, and hence p^* satisfies condition (2) of Definition 2.1. We now consider condition (1). Since p^* and $p^{k(i)}$ are in the same simplex σ , we have that p^* is close to $p^{k(i)}$. So $a^i(p^{k(i)})$ is close to $a^i(p^*)$ and from (9) we obtain that

$$Z(p^*) - \sum_{i \in U} x_i^* a^i(p^*) \simeq \sum_{\{j | s_j \neq 0\}} \mu_j s_j e(j), \quad (10)$$

where $\simeq$ stands for 'approximately equal to'. Since $(p^{k(i)})^T a^i(p^{k(i)}) > 0$, it follows that for some positive ϵ_1 close to zero, $p^{*T} a^i(p^*) > -\epsilon_1$ for $i \in U$. Hence, the profit per unit production $p^{*T} a^i(p^*) \simeq 0$ for the technologies $i \in U$ with positive production level x_i^* .

In case a) we have that $\sum_{\{j|s_j \neq 0\}} \mu_j s_j e(j) = \sum_{\{j|s_j = +1\}} \mu_j s_j e(j) \geq 0$. For ease of notation, let $F_j(p^*, x^*)$ be the j -th component of $Z(p^*) - \sum_{i \in U} x_i^* a^i(p^*)$. Then it follows from (10) that $F_j(p^*, x^*) \simeq 0$ if $s_j \neq +1$ and $F_j(p^*, x^*) \simeq 0$ or $F_j(p^*, x^*) \geq 0$ if $s_j = +1$. However, v is in the interior of S^n and σ lies in $A(s)$. By definition of $A(s)$ this implies that $p_j^* > 0$ for all j with $s_j = +1$. Moreover Z is the piecewise linear approximation to z and hence $p^{*T} Z(p^*) \simeq 0$, while also $p^{*T} a^i(p^*) \simeq 0$ for all $i \in U$. From this it follows that also $F_j(p^*, x^*) \simeq 0$ in case $s_j = +1$. Hence (p^*, x^*) satisfies approximately condition (1) of Definition 2.1.

In case b), we have that $\sum_{\{j|s_j \neq 0\}} \mu_j s_j e(j) = \sum_{\{j|s_j = -1\}} \mu_j s_j e(j) \leq 0$. Then $F_j(p^*, x^*) \simeq 0$ if $s_j \neq -1$ and $F_j(p^*, x^*) \leq 0$ or $F_j(p^*, x^*) \simeq 0$ if $s_j = -1$. Hence (p^*, x^*) satisfies approximately condition (1) of Definition 2.1.

Finally, in case c) we have that $F_j(p^*, x^*) \simeq 0$ if $s_j = 0$, $F_j(p^*, x^*) \simeq 0$ or $F_j(p^*, x^*) \leq 0$ if $s_j = -1$ and $F_j(p^*, x^*) \simeq 0$ or $F_j(p^*, x^*) \geq 0$ if $s_j = +1$. However, $p_j^* = 0$ in case $s_j = -1$ and hence $p_j^* F_j(p^*, x^*) \simeq 0$ if $s_j = -1$. By the same reasoning as in case a) it follows that $F_j(p^*, x^*) \simeq 0$ in case $s_j = +1$. Hence for all j , $F_j(p^*, x^*) \simeq 0$ or $F_j(p^*, x^*) \leq 0$. This completes the proof.

Q.E.D.

We now consider the piecewise linear path of prices followed by the algorithm. Firstly, observe that the left hand side of this system has $|s| + 2 + n + 1 - |s| = n + 3$ columns.

Nondegeneracy Assumption For each solution to the system (6) of linear equations, at most one of the $n + 3$ variables (λ, μ) is equal to zero.

Under this assumption the system has a line segment of solutions (λ^*, μ^*) , if any. An end point of such a line segment is called a basic solution and has exactly one of the variables equal to zero. Let $v \in \text{int}(T^n)$ be a price vector such that $z_j(v) \neq 0$ for all j and set $s^0 = \text{sign}z(v)$. Then there is a unique 1-dimensional simplex $\sigma(p^1, p^2)$ in $A(s^0)$ with v as one of its vertices, say $v = p^1$. This simplex is s^0 -complete with $\lambda_1 = 1$, $\lambda_2 = 0$ and $\mu_j = |z_j(v)|$, $j = 1, \dots, n + 1$ as one of its basic solutions. The algorithm starts by making a linear programming step with $b(p^2)$ in the corresponding system (7). In general, the algorithm traces a piecewise linear path of prices by making linear programming pivot steps in the system (7) with respect to a sequence of adjacent s -complete $(|s| + 1)$ -simplices in $A(s)$ for varying sign vectors s . At a solution (λ, μ) , let $p = \sum_k \lambda_k p^k$. By making a linear programming step in the system (7) corresponding to a simplex $\sigma(p^1, \dots, p^{t+1})$ in some subset $A(s)$, the following cases can occur.

Case 1: λ_g becomes zero for some $g \in \{1, \dots, t + 1\}$. Then p lies in the facet τ of σ opposite the vertex p^g . This facet is either a facet of exactly one other t -simplex σ' in $A(s)$, or τ lies in the boundary of $A(s)$. In the first case, a pivot step is made in (7) with $b(p')$ where p' is the unique vertex of σ' not being a vertex of τ . In the latter case, either τ lies in the boundary of S^n , or τ is a simplex in a subset $A(s')$, where s' is a sign vector such that $s'_i \neq 0$ for some i with $s_i = 0$ and $s'_j = s_j$ for all $j \neq i$. If τ lies in the boundary of S^n , then by definition of $A(s)$ and the fact that v lies in the interior of T^n , we have that $p_j^k = 0$ if

$s_j = -1$ for all $k \neq g$. Hence by c) of Theorem 4.2, we have an approximate equilibrium. If τ lies in $A(s')$ with s' as just defined, then τ is an s' -complete $(t-1)$ -simplex in $A(s')$ and the algorithm continues in $A(s')$ by pivoting the $(n+2)$ -column $-s_i e(i)$ into the system, thereby raising μ_i from zero.

Case 2: μ_k becomes zero for some k with $s_k \neq 0$. If $s_j \geq 0$ for all $j \neq k$, we have an approximate equilibrium according to a) of Theorem 4.2. If $s_j \leq 0$ for all $j \neq k$, we have an approximate equilibrium according to b) of Theorem 4.2. Otherwise, $\sigma(p^1, \dots, p^{t+1})$ is a facet of a unique $(t+1)$ -simplex σ' in $A(s')$, where $s'_k = 0$ and $s'_j = s_j$ for all $j \neq k$. Let p' be the vertex of σ' not in σ . Then σ' is s' -complete and the algorithm continues in $A(s')$ by pivoting $b(p')$ into the system.

Since all steps are unique the algorithm either terminates with an approximating equilibrium within a finite number of steps, or one of the variables λ_k corresponding to a vertex p^k with a label $b(p^k) = [-a^h(p^k)^T, 0]^T$ for some h goes to infinity. Now, let $\sigma(p^1, \dots, p^{t+1})$ be an s -complete simplex in $A(s)$ and let U, K, p^* and x^* be as defined earlier. Then, analogously to the reasoning in Theorem 4.2, it can be shown that (p^*, x^*) is approximately close to a pair (p, x) in $B(s, U)$. So, the path of prices followed by the algorithm approximately follows the projection on T^n of the adjustment path in B of prices and quantities starting in $(v, 0)$. Similarly to the proof that the path in B is bounded, it can be shown that all variables λ_k are bounded. This implies together with the finiteness of the number of simplices that the algorithm ends with an approximating equilibrium (p^*, x^*) as defined in Theorem 4.2. We have seen that p^* satisfies condition (2) of Definition 2.1. Therefore, we can measure the inaccuracy of the approximation by taking the smallest $\epsilon \geq 0$ such that $f_j(p^*, x^*) \leq \epsilon$, $j = 1, \dots, n+1$. If the accuracy is not sufficient, i.e. ϵ is too large, then the algorithm can be repeated with a finer simplicial subdivision of S^n with $p^* \in \text{int}(T^n)$ as the new starting point if $p_j^* > 0$ for all j , or a point in $\text{int}(T^n)$ close to p^* if $p_j^* = 0$ for some j . In this way a sequence of approximating equilibria can be found with inaccuracy going to zero.

5 AN ITERATIVE ALGORITHM FOR THE COMPUTATION OF AN EQUILIBRIUM

In Mathiesen [12] the equilibrium problem is approximated by solving a sequence of Linear Complementarity Problems (LCP's). Each LCP in the sequence is solved by the Lemke-Howson algorithm. In this way, a sequence of prices may be generated possibly converging to an equilibrium price vector. However, the method breaks down if the sequence of LCP's contains an LCP for which the Lemke complementary pivoting algorithm does not converge. For an economy with linear production technologies, Eaves [6] proposed an alternative formulation of the Linear Complementarity Problems. For this formulation it can be shown that the Lemke-Howson algorithm always finds a solution within a finite number of steps. So, the method always generates a sequence of approximating solutions which may converge to a solution of the equilibrium problem. Zhao and Dafermos [17] show that for an economy with linear production the solution of the equilibrium problem may be obtained by solv-

ing a sequence of quadratic programming problems. At each problem of the sequence the equilibrium can be found in two steps: at the first step the equilibrium price vector is determined as a solution of a variational inequality problem of smaller size and then the activity level vector is determined as the solution of a set of linear inequalities. If the sequence of solution converges, then it converges to a solution of the equilibrium problem. In Kremers and Talman [9], the equilibrium problem for the economy with linear production is formulated as a Variational Inequality problem on the set of feasible prices T^n . By linearizing the excess demand function, the corresponding linear variational inequality problem on T^n is solved by applying an algorithm of Kamiya and Talman [10] for solving linear variational inequality problems on a polytope. This algorithm always finds a solution within a finite number of steps. By linearizing the excess demand function at the solution of the previous problem, again a sequence of approximate solutions is generated, possibly converging to a solution of the equilibrium problem.

In this section we generalize the algorithm of Kremers and Talman to the problem with nonlinear production. Since we are linearizing the problem, we now assume that z is differentiable on $\mathbf{R}_+^{n+1}$. The next theorem characterizes the set of equilibrium price vectors of an economy with nonlinear production technologies $a^i(p)$, $i = 1, \dots, m$ (e.g. see Yamamoto [16]).

Theorem 5.1 A price vector $p^* \in T^n$ is an equilibrium price vector of the economy with excess demand function z and production technologies $a^i(p)$, $i = 1, \dots, m$, if and only if p^* is a solution to the variational inequality

$$p^T z(p^*) \leq p^{*T} z(p^*), \text{ for all } p \in T^n.$$

The theorem implies that if p^* is a solution to the variational inequality problem, there exists production levels x_i^* , $i = 1, \dots, m$, such that (p^*, x^*) satisfies the conditions of Definition 2.1. Given the solution $p^* \in T^n$, let U be the set of indices, such that $p^{*T} a^i(p^*) = 0$ for all $i \in U$ and $p^{*T} a^i(p^*) < 0$ for all $i \notin U$. Then $x_i^* = 0$ for $i \notin U$. For $i \in U$, x_i^* follows from solving the linear system $z(p^*) - \sum_{i \in U} x_i a^i(p^*) \leq 0$.

From Theorem 5.1 it follows that the equilibrium problem can be reduced to the variational inequality problem of the excess demand function z on T^n . This problem can possibly be solved by a sequence of Linear Variational Inequality Problems. To show this, let $p^k \in S^n$ be the solution to the k -th iterative in the sequence of Linear Variational Inequality Problems, $k = 1, 2, \dots$, and take p^0 arbitrarily in S^n . Furthermore, let v be a point in the interior of T^n . We will see that p^k may lie outside T^n . Therefore, let λ^k be the solution to

$$\max \lambda, \text{ such that } p^{\lambda T} a^i(p^\lambda) \leq 0 \text{ for all } i, 0 \leq \lambda \leq 1, \quad (11)$$

with $p^\lambda = \lambda p^k + (1 - \lambda)v$. Finally, let $v^k = \lambda^k p^k + (1 - \lambda^k)v$. So, $\lambda^k = 1$ and $v^k = p^k$ if p^k is in the interior of the convex set T^n . If $p^k \notin T^n$, then $\lambda^k < 1$ and v^k lies on the boundary of T^n with $v^k a^i(v^k) = 0$ for some i . We now take $v^k \in T^n$ as the initial point in the $(k + 1)$ -th iterate and linearize the problem around v^k . So we define for $p \in S^n$,

$$z^{k+1}(p) = z(v^k) + [Jz(v^k)]p,$$

where $[Jz(v^k)]$ is the Jacobian matrix of first order derivatives of z to p in v^k . Furthermore, we set $a^{(k+1,i)}(p) = a^i(v^k)$. So, at the $(k+1)$ -th iterate we consider the linear production technology given by $a^i(v^k)$, $i = 1, \dots, m$. Let $T^n(v^k)$ be defined by

$$T^n(v^k) = \{p \in S^n \mid p^T A(v^k) \leq 0\}.$$

Then, by Corollary 2.5, $T^n \subset T^n(v^k)$. Since, $v^k \in T^n$ we have that $v^k \in T^n(v^k)$. Moreover, v^k lies on the boundary of $T^n(v^k)$ if v^k lies on the boundary of T^n . Applying the algorithm in [9], we can solve the variational inequality problem of the linearized excess demand function $z^{k+1}(p)$ on the set of feasible prices $T^n(v^k)$ of the economy with linear technologies $a^i(v^k)$. The algorithm always finds a solution p^{k+1} within a finite number of steps. Since $T^n \subset T^n(v^k)$, p^{k+1} may lie outside T^n and we have to solve the one-dimensional maximization problem (11) for $k+1$ to find $v^{k+1} \in T^n$ as the approximate solution to the variational inequality problem of z on T^n . This approximate solution v^{k+1} can be used as the new initial point in the next iterate. The sequence of approximate solutions v^k , $k = 1, 2, \dots$, may converge to a solution p^* of the variational inequality problem. If the sequence indeed converges, this procedure of solving the problem by a sequence of Linear Variational Inequality Problems is probably more efficient than the simplicial algorithm described in the previous section. If not, then the simplicial algorithm should be applied to solve the equilibrium problem because such an algorithm is globally convergent.

6 SUMMARY AND CONCLUSIONS

In this paper we have considered the problem of finding an equilibrium in an economy with nonlinear constant returns to scale production activities. We have shown by a constructive argument that there exists an equilibrium under the condition that there is no production without input. Under this condition there exists a price vector such that all nonlinear production activities make negative profits. It can be shown that starting from such a price vector there exists a path leading to an equilibrium pair of prices and activity levels.

In the second part of the paper we are concerned with the problem of the computation of an equilibrium. In the literature several algorithms have been proposed for solving the equilibrium problem in case of linear production activities. Amongst these algorithms are simplicial methods and a method which solves the problem by solving a sequence of linear variational inequality problems. We adapt these methods for solving the equilibrium problem with nonlinear technologies. The simplicial method follows approximately the path defined in the constructive proof of the existence of an equilibrium. This method is globally convergent. The sequence of solutions to the linear variational inequality problems may not converge. Moreover, at each iteration the set of feasible prices (non positive profit for each technology) is a subset of the set of feasible prices in the linear approximation of the problem and hence the solution to the linear variational inequality problem does not need to be feasible. If so, we have to solve a subproblem to find a feasible starting price vector for the next iterate of the sequence.

References

- [1] E.L. Allgower and K. Georg, *Numerical Continuation Methods* (Springer-Verlag, Berlin, 1990).
- [2] G. Debreu, *Theory of Value* (Yale University Press, New Haven, 1959).
- [3] T.M. Doup, *Simplicial Algorithms on the Simplotope* (Springer-Verlag, Berlin, 1988).
- [4] T.M. Doup, G. van der Laan and A.J.J. Talman, The $(2^{n+1} - 2)$ -ray algorithm: a new simplicial algorithm to compute economic equilibria, *Mathematical Programming* 39 (1987) 241-252.
- [5] T.M. Doup and A.J.J. Talman, A new simplicial variable dimension algorithm to find equilibria on the product space of simplices, *Mathematical Programming* 37 (1987) 319-355.
- [6] B.C. Eaves, Thoughts on computing market equilibrium with SLCP, in: *The Computation and Modelling of Economic Equilibria*, ed. A.J.J. Talman and G. van der Laan (North Holland, Amsterdam, 1987), pp.1-17.
- [7] A.H. van den Elzen, G. van der Laan and A.J.J. Talman, An adjustment process for an exchange economy with linear production technologies, Research Memorandum 1989-82, Vrije Universiteit, Amsterdam (1989), to appear in *Mathematics of Operations Research*.
- [8] G. Fischer, K. Frohberg, M.A. Keyzer and K.S. Parikh, *Linked National Models: A Tool for International Food Policy Analysis* (Kluwer Academic Publishers, Dordrecht, 1989).
- [9] J.A.W.M. Kremers and A.J.J. Talman, An SLSP-algorithm to compute an equilibrium in an economy with linear production technologies, Research Memorandum 496, Tilburg University, Tilburg (1991).
- [10] K. Kamiya and A.J.J. Talman, Linear Stationary Point Problems, CentER Discussion Paper 9022, Center for Economic Research, Tilburg University, Tilburg (1990).
- [11] G. van der Laan and A.J.J. Talman, A restart algorithm for computing fixed points without an extra dimension, *Mathematical Programming* 17 (1979) 74-84.
- [12] L. Mathiesen, Computation of economic equilibria by a sequence of linear complementarity problems, *Mathematical Programming Study* 23 (1985) 144-162.
- [13] M. Morishima, *Equilibrium, Stability and Growth, A Multi-Sectoral Analysis* (Oxford University Press, London, 1964).
- [14] H. Scarf, *The Computation of Economic Equilibria* (Yale University Press, New Haven, 1973).
- [15] H.R. Varian, *Microeconomic Analysis* (W.W. Norton and Company, New York, 1984, second edition).

- [16] Y. Yamamoto, Fixed point algorithms for stationary point problems in: *Mathematical Programming, Recent Developments and Applications* ed. M.Iri and K. Tanabe (KTK Scientific Publishers, Tokyo, 1989) pp. 283-307.
- [17] L. Zhao and S. Dafermos, General economic equilibrium and variational inequalities, *Operations Research Letters* 10 (1991) 369-376.

IN 1991 REEDS VERSCHENEN

- 466 Prof.Dr. Th.C.M.J. van de Klundert - Prof.Dr. A.B.T.M. van Schaik
Economische groei in Nederland in een internationaal perspectief
- 467 Dr. Sylvester C.W. Eijffinger
The convergence of monetary policy - Germany and France as an example
- 468 E. Nijssen
Strategisch gedrag, planning en prestatie. Een inductieve studie binnen de computerbranche
- 469 Anne van den Nouweland, Peter Borm, Guillermo Owen and Stef Tijs
Cost allocation and communication
- 470 Drs. J. Grazell en Drs. C.H. Veld
Motieven voor de uitgifte van converteerbare obligatieleningen en warrant-obligatieleningen: een agency-theoretische benadering
- 471 P.C. van Batenburg, J. Kriens, W.M. Lammerts van Bueren and R.H. Veenstra
Audit Assurance Model and Bayesian Discovery Sampling
- 472 Marcel Kerkhofs
Identification and Estimation of Household Production Models
- 473 Robert P. Gilles, Guillermo Owen, René van den Brink
Games with Permission Structures: The Conjunctive Approach
- 474 Jack P.C. Kleijnen
Sensitivity Analysis of Simulation Experiments: Tutorial on Regression Analysis and Statistical Design
- 475 C.P.M. van Hoesel
An $O(n \log n)$ algorithm for the two-machine flow shop problem with controllable machine speeds
- 476 Stephan G. Vanneste
A Markov Model for Opportunity Maintenance
- 477 F.A. van der Duyn Schouten, M.J.G. van Eijs, R.M.J. Heuts
Coordinated replenishment systems with discount opportunities
- 478 A. van den Nouweland, J. Potters, S. Tijs and J. Zarzuelo
Cores and related solution concepts for multi-choice games
- 479 Drs. C.H. Veld
Warrant pricing: a review of theoretical and empirical research
- 480 E. Nijssen
De Miles and Snow-typologie: Een exploratieve studie in de meubel-
branche
- 481 Harry G. Barkema
Are managers indeed motivated by their bonuses?

- 482 Jacob C. Engwerda, André C.M. Ran, Arie L. Rijkeboer
Necessary and sufficient conditions for the existence of a positive definite solution of the matrix equation $X + A^T X^{-1} A = I$
- 483 Peter M. Kort
A dynamic model of the firm with uncertain earnings and adjustment costs
- 484 Raymond H.J.M. Gradus, Peter M. Kort
Optimal taxation on profit and pollution within a macroeconomic framework
- 485 René van den Brink, Robert P. Gilles
Axiomatizations of the Conjunctive Permission Value for Games with Permission Structures
- 486 A.E. Brouwer & W.H. Haemers
The Gewirtz graph - an exercise in the theory of graph spectra
- 487 Pim Adang, Bertrand Melenberg
Intratemporal uncertainty in the multi-good life cycle consumption model: motivation and application
- 488 J.H.J. Roemen
The long term elasticity of the milk supply with respect to the milk price in the Netherlands in the period 1969-1984
- 489 Herbert Hamers
The Shapley-Entrance Game
- 490 Rezaul Kabir and Theo Vermaelen
Insider trading restrictions and the stock market
- 491 Piet A. Verheyen
The economic explanation of the jump of the co-state variable
- 492 Drs. F.L.J.W. Manders en Dr. J.A.C. de Haan
De organisatorische aspecten bij systeemontwikkeling een beschouwing op besturing en verandering
- 493 Paul C. van Batenburg and J. Kriens
Applications of statistical methods and techniques to auditing and accounting
- 494 Ruud T. Frambach
The diffusion of innovations: the influence of supply-side factors
- 495 J.H.J. Roemen
A decision rule for the (des)investments in the dairy cow stock
- 496 Hans Kremers and Dolf Talman
An SLSPP-algorithm to compute an equilibrium in an economy with linear production technologies

- 497 L.W.G. Strijbosch and R.M.J. Heuts
Investigating several alternatives for estimating the compound lead
time demand in an (s,Q) inventory model
- 498 Bert Bettonvil and Jack P.C. Kleijnen
Identifying the important factors in simulation models with many
factors
- 499 Drs. H.C.A. Roest, Drs. F.L. Tijssen
Beheersing van het kwaliteitsperceptieproces bij diensten door middel
van keurmerken
- 500 B.B. van der Genugten
Density of the F-statistic in the linear model with arbitrarily
normal distributed errors
- 501 Harry Barkema and Sytse Douma
The direction, mode and location of corporate expansions
- 502 Gert Nieuwenhuis
Bridging the gap between a stationary point process and its Palm
distribution
- 503 Chris Veld
Motives for the use of equity-warrants by Dutch companies
- 504 Pieter K. Jagersma
Een etiologie van horizontale internationale ondernemingsexpansie
- 505 B. Kaper
On M-functions and their application to input-output models
- 506 A.B.T.M. van Schaik
Productiviteit en Arbeidsparticipatie
- 507 Peter Borm, Anne van den Nouweland and Stef Tijs
Cooperation and communication restrictions: a survey
- 508 Willy Spanjers, Robert P. Gilles, Pieter H.M. Ruys
Hierarchical trade and downstream information
- 509 Martijn P. Tummers
The Effect of Systematic Misperception of Income on the Subjective
Poverty Line
- 510 A.G. de Kok
Basics of Inventory Management: Part 1
Renewal theoretic background
- 511 J.P.C. Blanc, F.A. van der Duyn Schouten, B. Pourbabai
Optimizing flow rates in a queueing network with side constraints
- 512 R. Peeters
On Coloring j-Unit Sphere Graphs

- 513 Drs. J. Dagevos, Drs. L. Oerlemans, Dr. F. Boekema
Regional economic policy, economic technological innovation and networks
- 514 Erwin van der Krabben
Het functioneren van stedelijke onroerend-goed-markten in Nederland - een theoretisch kader
- 515 Drs. E. Schaling
European central bank independence and inflation persistence
- 516 Peter M. Kort
Optimal abatement policies within a stochastic dynamic model of the firm
- 517 Pim Adang
Expenditure versus consumption in the multi-good life cycle consumption model
- 518 Pim Adang
Large, infrequent consumption in the multi-good life cycle consumption model
- 519 Raymond Gradus, Sjak Smulders
Pollution and Endogenous Growth
- 520 Raymond Gradus en Hugo Keuzenkamp
Arbeidsongeschiktheid, subjectief ziektegevoel en collectief belang
- 521 A.G. de Kok
Basics of inventory management: Part 2
The (R,S)-model
- 522 A.G. de Kok
Basics of inventory management: Part 3
The (b,Q)-model
- 523 A.G. de Kok
Basics of inventory management: Part 4
The (s,S)-model
- 524 A.G. de Kok
Basics of inventory management: Part 5
The (R,b,Q)-model
- 525 A.G. de Kok
Basics of inventory management: Part 6
The (R,s,S)-model
- 526 Rob de Groof and Martin van Tuijl
Financial integration and fiscal policy in interdependent two-sector economies with real and nominal wage rigidity

- 527 A.G.M. van Eijs, M.J.G. van Eijs, R.M.J. Heuts
Gecoördineerde bestelsystemen
een management-georiënteerde benadering
- 528 M.J.G. van Eijs
Multi-item inventory systems with joint ordering and transportation
decisions
- 529 Stephan G. Vanneste
Maintenance optimization of a production system with buffercapacity
- 530 Michel R.R. van Bremen, Jeroen C.G. Zijlstra
Het stochastische variantie optiewaarderingsmodel
- 531 Willy Spanjers
Arbitrage and Walrasian Equilibrium in Economies with Limited Infor-
mation

IN 1992 REEDS VERSCHENEN

- 532 F.G. van den Heuvel en M.R.M. Turlings
Privatisering van arbeidsongeschiktheidsregelingen
Refereed by Prof.Dr. H. Verbon
- 533 J.C. Engwerda, L.G. van Willigenburg
LQ-control of sampled continuous-time systems
Refereed by Prof.dr. J.M. Schumacher
- 534 J.C. Engwerda, A.C.M. Ran & A.L. Rijkeboer
Necessary and sufficient conditions for the existence of a positive definite solution of the matrix equation $X + A^*X^{-1}A = Q$.
Refereed by Prof.dr. J.M. Schumacher
- 535 Jacob C. Engwerda
The indefinite LQ-problem: the finite planning horizon case
Refereed by Prof.dr. J.M. Schumacher
- 536 Gert-Jan Otten, Peter Borm, Ton Storcken, Stef Tijs
Effectivity functions and associated claim game correspondences
Refereed by Prof.dr. P.H.M. Ruys
- 537 Jack P.C. Kleijnen, Gustav A. Alink
Validation of simulation models: mine-hunting case-study
Refereed by Prof.dr.ir. C.A.T. Takkenberg
- 538 V. Feltkamp and A. van den Nouweland
Controlled Communication Networks
Refereed by Prof.dr. S.H. Tijs
- 539 A. van Schaik
Productivity, Labour Force Participation and the Solow Growth Model
Refereed by Prof.dr. Th.C.M.J. van de Klundert
- 540 J.J.G. Lemmen and S.C.W. Eijffinger
The Degree of Financial Integration in the European Community
Refereed by Prof.dr. A.B.T.M. van Schaik
- 541 J. Bell, P.K. Jagersma
Internationale Joint Ventures
Refereed by Prof.dr. H.G. Barkema
- 542 Jack P.C. Kleijnen
Verification and validation of simulation models
Refereed by Prof.dr.ir. C.A.T. Takkenberg
- 543 Gert Nieuwenhuis
Uniform Approximations of the Stationary and Palm Distributions of Marked Point Processes
Refereed by Prof.dr. B.B. van der Genugten

- 544 R. Heuts, P. Nederstigt, W. Roebroek, W. Selen
Multi-Product Cycling with Packaging in the Process Industry
Refereed by Prof.dr. F.A. van der Duyn Schouten
- 545 J.C. Engwerda
Calculation of an approximate solution of the infinite time-varying
LQ-problem
Refereed by Prof.dr. J.M. Schumacher
- 546 Raymond H.J.M. Gradus and Peter M. Kort
On time-inconsistency and pollution control: a macroeconomic approach
Refereed by Prof.dr. A.J. de Zeeuw
- 547 Drs. Dolph Cantrijn en Dr. Rezaul Kabir
De Invloed van de Invoering van Preferente Beschermingsaandelen op
Aandelenkoersen van Nederlandse Beursgenoteerde Ondernemingen
Refereed by Prof.dr. P.W. Moerland
- 548 Sylvester Eijffinger and Eric Schaling
Central bank independence: criteria and indices
Refereed by Prof.dr. J.J. Sijben
- 549 Drs. A. Schmeits
Geïntegreerde investerings- en financieringsbeslissingen; Implicaties
voor Capital Budgeting
Refereed by Prof.dr. P.W. Moerland
- 550 Peter M. Kort
Standards versus standards: the effects of different pollution
restrictions on the firm's dynamic investment policy
Refereed by Prof.dr. F.A. van der Duyn Schouten
- 551 Niels G. Noorderhaven, Bart Nooteboom and Johannes Berger
Temporal, cognitive and behavioral dimensions of transaction costs;
to an understanding of hybrid vertical inter-firm relations
Refereed by Prof.dr. S.W. Douma
- 552 Ton Storcken and Harrie de Swart
Towards an axiomatization of orderings
Refereed by Prof.dr. P.H.M. Ruys
- 553 J.H.J. Roemen
The derivation of a long term milk supply model from an optimization
model
Refereed by Prof.dr. F.A. van der Duyn Schouten
- 554 Geert J. Almekinders and Sylvester C.W. Eijffinger
Daily Bundesbank and Federal Reserve Intervention and the Conditional
Variance Tale in DM/\$-Returns
Refereed by Prof.dr. A.B.T.M. van Schaik
- 555 Dr. M. Hetebrij, Drs. B.F.L. Jonker, Prof.dr. W.H.J. de Freytas
"Tussen achterstand en voorsprong" de scholings- en personeelsvoor-
zieningsproblematiek van bedrijven in de procesindustrie
Refereed by Prof.dr. Th.M.M. Verhallen

- 556 Ton Geerts
Regularity and singularity in linear-quadratic control subject to implicit continuous-time systems
Communicated by Prof.dr. J. Schumacher
- 557 Ton Geerts
Invariant subspaces and invertibility properties for singular systems: the general case
Communicated by Prof.dr. J. Schumacher
- 558 Ton Geerts
Solvability conditions, consistency and weak consistency for linear differential-algebraic equations and time-invariant singular systems: the general case
Communicated by Prof.dr. J. Schumacher
- 559 C. Fricker and M.R. Jaïbi
Monotonicity and stability of periodic polling models
Communicated by Prof.dr.ir. O.J. Boxma
- 560 Ton Geerts
Free end-point linear-quadratic control subject to implicit continuous-time systems: necessary and sufficient conditions for solvability
Communicated by Prof.dr. J. Schumacher
- 561 Paul G.H. Mulder and Anton L. Hempenius
Expected Utility of Life Time in the Presence of a Chronic Noncommunicable Disease State
Communicated by Prof.dr. B.B. van der Genugten
- 562 Jan van der Leeuw
The covariance matrix of ARMA-errors in closed form
Communicated by Dr. H.H. Tigelaar
- 563 J.P.C. Blanc and R.D. van der Mei
Optimization of polling systems with Bernoulli schedules
Communicated by Prof.dr.ir. O.J. Boxma
- 564 B.B. van der Genugten
Density of the least squares estimator in the multivariate linear model with arbitrarily normal variables
Communicated by Prof.dr. M.H.C. Paardekooper
- 565 René van den Brink, Robert P. Gilles
Measuring Domination in Directed Graphs
Communicated by Prof.dr. P.H.M. Ruys
- 566 Harry G. Barkema
The significance of work incentives from bonuses: some new evidence
Communicated by Dr. Th.E. Nijman

- 567 Rob de Groof and Martin van Tuijl
Commercial integration and fiscal policy in interdependent, financially integrated two-sector economies with real and nominal wage rigidity.
Communicated by Prof.dr. A.L. Bovenberg
- 568 F.A. van der Duyn Schouten, M.J.G. van Eijs, R.M.J. Heuts
The value of information in a fixed order quantity inventory system
Communicated by Prof.dr. A.J.J. Talman
- 569 E.N. Kertzman
Begrotingsnormering en EMU
Communicated by Prof.dr. J.W. van der Dussen
- 570 A. van den Elzen, D. Talman
Finding a Nash-equilibrium in noncooperative N-person games by solving a sequence of linear stationary point problems
Communicated by Prof.dr. S.H. Tijs
- 571 Jack P.C. Kleijnen
Verification and validation of models
Communicated by Prof.dr. F.A. van der Duyn Schouten
- 572 Jack P.C. Kleijnen and Willem van Groenendaal
Two-stage versus sequential sample-size determination in regression analysis of simulation experiments
- 573 Pieter K. Jagersma
Het management van multinationale ondernemingen: de concernstructuur
- 574 A.L. Hempenius
Explaining Changes in External Funds. Part One: Theory
Communicated by Prof.Dr.Ir. A. Kapteyn
- 575 J.P.C. Blanc, R.D. van der Mei
Optimization of Polling Systems by Means of Gradient Methods and the Power-Series Algorithm
Communicated by Prof.dr.ir. O.J. Boxma
- 576 Herbert Hamers
A silent duel over a cake
Communicated by Prof.dr. S.H. Tijs

Bibliotheek K. U. Brabant

17 000 01108473 9