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THE NET PRESENT VALUE IN DYNAMIC MODELS OF THE FIRM

Peter M. Kort

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ABSTRACT
In this paper we introduce the well-known net present value criterion in a dynamic model of the firm, namely in a model in which investment can be financed by equity and debt. It turns out that within this model the firm tries to reach its equilibrium level as soon as possible. The optimality of this level is characterized by the net present value of marginal investment being equal to zero.

1. Introduction
In business economics, the net present value approach is used as a method which evaluates the desirability of an investment proposal. The net present value of such a proposal is defined by the sum of the discounted values of the net cash receipts minus the initial investment outlay (see Levy & Sarnat (1978), p.26).

In this paper we show that within dynamic models of the firm the net present value can be used as follows: the firm must invest at its maximum if the net present value of marginal investment is positive. As soon as it equals zero marginal earnings are equal to marginal cost. Then, the firm is in its optimal situation and it starts paying out dividend.

To illustrate the net present value approach within dynamic models of the firm, we analyse a model in which investment can be financed by equity and debt. This model was originally designed by Lesourne (1973) and used as a framework for analysing the influence of taxation and activity analysis by Van Schijndel (1986, 1987) and Van Loon (1983), respectively. Van Hilten (1987) studied the contraction policies of the original model extended by profit tax and investment grants. In this paper we derive formulas for the net present value of marginal investment on growth- as well as on equilibrium-paths and we give an economic interpretation of these formulas.
2. The model with equity and debt financing

First, we formulate the model in symbols; afterwards the definition of the symbols and the interpretation of the model will be given.

\[
\text{maximize: } \int_{0}^{z} D(T) e^{-iT} dT + X(z) e^{-iZ} \quad (1)
\]

\[
\text{subject to: } K = I(T) - aK(T), K(0) = K_0 > 0 \quad (2)
\]

\[
\dot{X} = S(K) - aK(T) - rY(T) - D(T), X(0) = X_0 > 0 \quad (3)
\]

\[
K(T) = X(T) + Y(T) \quad (4)
\]

\[
0 \leq Y(T) \leq hX(T) \quad (5)
\]

\[
D(T) \geq 0 \quad (6)
\]

with:

D = dividend

I = gross investment

K = capital goods

S = earnings, \( S(K) > 0, \frac{dS}{dK} > 0, \frac{d^2S}{dK^2} < 0 \) \quad (6a)

T = time

X = equity

Y = debt

a = depreciation rate

i = discount rate of the shareholders

h = maximum debt to equity rate

r = interest rate on debt

z = planning horizon

The model describes a firm that maximizes its value to the shareholders (1). The state of the firm is fixed by the values of its capital goods (2) and equity (3). The firm finances its assets by equity and debt (4). The amount of debt available is restricted, because the firm is assumed to belong to a given risk-class (5). The firm is allowed to pay out no dividend (6) and it operates under decreasing returns to scale (6a). Later, we give an
assumption that rules out contraction policies and therefore a lower bound on investment is not necessary in the model formulation.

After substituting \( K - X \) for \( Y \) and using standard control theory application (see Feichtinger & Hartl (1986)), we derive the Lagrange function which includes the Hamiltonian and the constraints:

\[
L = D e^{-iT} + \psi_1(S(K) - (a+r)K + rX - D) + \psi_2(I - aK) + \lambda_1(K - X) + \\
\lambda_2((1+h)X - K) + \lambda_3D
\]  

(7)

then, after some rearranging, the necessary conditions are:

\[
\begin{align*}
\psi_1 &= e^{-iT} + \lambda_3 \\
\psi_2 &= 0 \\
\psi_1 \left( \frac{dS}{dK} - (a+r) \right) + \lambda_1 - \lambda_2 &= 0 \\
-\psi_1 &= r\psi_1 - \lambda_1 + (1+h)\lambda_2 \\
\lambda_1(K - X) &= 0 \\
\lambda_2((1+h)X - K) &= 0 \\
\lambda_3D &= 0 \\
\psi_1(z) &= e^{-iz}
\end{align*}
\]  

(8) \quad (9) \quad (10) \quad (11) \quad (12) \quad (13) \quad (14) \quad (15)

Applying the solution procedure of Van Loon (see Van Loon (1983), p.116) we may discern five different feasible paths, which are represented by table 1.
Path Y K K Feasible Policy  
1 hX Y K* K YX Always growth with maximum debt  
2 Y < 0 0 K* K YX Always redemption of debt  
3 0 hX K YK X Always growth without debt  
4 0 0 K* K X i < r stationary dividend without debt  
5 hX 0 K Y K* i > r stationary dividend with maximum debt

Table 1. Features of feasible paths

in which:

\[ K = K_{YX} \iff \frac{dS}{dK(K_{YX})} = a + r \]  \hspace{1cm} (16)

\[ K = K_{X} \iff \frac{dS}{dK(K_{X})} = a + i \]  \hspace{1cm} (17)

\[ K = K_{Y} \iff \frac{dS}{dK(K_{Y})} = a + \frac{h}{1+h} r + \frac{1}{1+h} i \]  \hspace{1cm} (18)

We avoid contraction by assuming:

\[ \left. \frac{dS}{dK} \right|_{K=K_0} > i + a \]  \hspace{1cm} (19)

We also assume that \( X_0 \) is that small that it is optimal to start with maximum debt (see also Feichtinger & Hartl (1986), p.378):

\[ K_0 = (1 + h) X_0 \]  \hspace{1cm} (20)

After these assumptions, Van Loon's path connecting procedure can be used to prove that two optimal policy strings are left. The first one occurs if \( i < r \) and is represented by fig. 1.
It can be proved that on path 1 through 4 the following equation holds:

$$\lambda_3(T) e^{iT} = \frac{z}{T} \int (R_X(t) + a) e^{-(i+a)(t-T)} dt + \frac{z}{T} \int (R_X(t) + a) e^{-a(t-T)} \lambda_3(t) e^{iT} dt + e^{-(i+a)(z-T)} - 1$$

in which:

- $R_X = \text{marginal return on equity, which satisfies the following relation (see also Van Hilten (1987))}$:

$$R_X = \frac{dS}{dK} - a + \left(\frac{dS}{dK} - a - r\right)Y_X$$

In the Appendix equation (21) is derived for path 1. Note that $\lambda_3$ is the Lagrange multiplier of the restriction that dividend is greater than zero. Therefore, $\lambda_3$ is equal to the discounted extra value of the Hamiltonian gained, if the lower bound of dividend is decreased by one dollar. On path 1 and path 3 the dollar would be used for investing, on path 2 for paying off debt. In this way, the left-hand side of equation (21) represents this extra value, but then discounted to $T$. 

Fig.1 Master trajectory if $i < r$
On path 1, the first term on the right-hand side is equal to the direct marginal earnings of investment which consist of the present value of additional sales over the whole period due to the new equipment (the production capacity of this equipment decreases with rate \"a\" during the rest of the planning period). The second term represents the indirect marginal earnings of investment. An extra investment of one dollar on time-point \(T\) implies an increase in capital good stock of \(e^{-a(t-T)}\) on time-point \(t > T\), which generates an extra return of \((R_X(t) + a) e^{-a(t-T)}\). This extra return will be used for investment when "\(t\)" is situated on path 1 and path 3 and for paying off debt when "\(t\)" lies on path 2. It has the same effect as a decrease of the lower bound of dividend with this value on time-point \(t\) and according to the economic definition of \(\lambda_3\), the objective discounted to \(T\) is increased by \((R_X(t) + a) e^{-a(t-T)}\lambda_3(t) e^{iT}\). The third term is equal to the present value of the remaining new equipment at the end of the planning period (the value of the new equipment decreases with depreciation rate \"a\" during the rest of the planning period), while the fourth term represents the initial investment outlay of one dollar.

From the above, we can conclude that on path 1 the right-hand side of (21) represents the net present value of marginal investment. Due to the fact that \(\lambda_3\) is greater than zero on path 1, we can conclude that this net present value is greater than zero, so marginal earnings are greater than marginal cost of investment and therefore it is optimal for the firm to invest at its maximum until the level \(K^{*}_{yx}\) is reached. Then, \(\frac{dS}{dK}\) equals \(r + a\) and it follows from (22) that debt has a negative influence on \(R_X\) if the firm continues with expansion investment. So, although the net present value of marginal investment is positive, it is optimal for the firm to stop investing at its maximum and to start paying off debt, keeping \(I\) on depreciation level (path 2). Now, due to the facts that \(\frac{dS}{dK}\) equals \(a+r\) on path 2 and debt is zero on the paths 3 and 4, it follows from (22) that we can rewrite (21) into:
\begin{equation}
\lambda_3(T) e^{iT} = \int_T^z \frac{dS(t)}{dK(t)} e^{-(i+a)(t-T)} dt + \int_T^z \frac{dS(t)}{dK(t)} e^{-a(t-T)} \lambda_3(t) e^{iT} dt + e^{-(i+a)(z-T)} \tag{23}
\end{equation}

(23) means that on path 2 the marginal value of paying off debt \((= \lambda_3(T)e^{iT})\) equals the net present value of marginal investment. When all debt is paid off, (23) continues to hold, so the net present value of marginal investment is still positive. Therefore the firm again starts investing at its maximum, but now financed by equity only (path 3). When marginal return on equity equals \(i\), which happens when \(K\) reaches \(K^*_X\), it is optimal to pay out dividend (path 4). \(\lambda_3\) then equals zero and therefore expression (23) now turns into:

\begin{equation}
\int_T^z \frac{dS(t)}{dK(t)} e^{-(i+a)(t-T)} dt + e^{-(i+a)(z-T)} = 1 \tag{24}
\end{equation}

From this expression we conclude that the net present value of the last investment unit equals zero which means that the firm is on its optimal level. On the other paths this situation cannot be reached because of the active dividend restriction.

This solution shows that the firm tries to reach its optimal situation as soon as possible. To do so, it finances its investment to some instant by debt, even if debt is more expensive than equity.

The second optimal policy string (see fig. 2) occurs if debt money is cheap compared to equity.
Fig 2. Master trajectory if \( i > r \)

In this solution, debt will be at its maximum during the whole planning period and from (22) we can derive that the following holds for \( R_X \):

\[
R_X = (1+h) \left( \frac{dS}{dK} - a \right) - hr
\]  

(25)

After substituting (25) in (21), we get that on path 1 it holds that:

\[
\lambda_3(T) e^{iT} = \int_T^Z \left( (1+h) \frac{dS}{dK}(t) - h(r+a) \right) e^{-(i+a)(t-T)} \, dt + \\
\int_T^Z \left( (1+h) \frac{dS}{dK}(t) - h(r+a) \right) e^{-a(t-T)} \lambda_3(t) e^{iT} \, dt + e^{-(i+a)(z-T)} - 1
\]  

(26)

From (26) we can conclude that the net present value of marginal investment is greater than zero. Therefore, the firm will invest at its maximum. As soon as the amount of capital goods equals \( K^*_y \), \( R_X \) becomes equal to \( i \) (to see this, substitute (18) in (25)) and it is optimal to pay out dividend. Now, expression (26) changes into:

\[
\int_T^Z \left( (1+h) \frac{dS}{dK}(t) - h(r+a) \right) e^{-(i+a)(t-T)} \, dt + e^{-(i+a)(z-T)} - 1 = 0
\]  

(27)
So, the net present value of the last dollar invested is equal to zero. Therefore, marginal earnings equal marginal cost and the firm has reached its equilibrium level. In this solution it is never optimal to pay off debt, because the marginal return on equity is always greater than \( r \). This is caused by the fact that during the whole planning period capital goods remain below \( K_{yx}^* \).

Note
1) Gradus (1987) used this criterion of optimality to solve the Lancaster open loop Stackelberg differential game model.

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Appendix. Derivation of the net present value equation on path 1 when $i < r$

On path 1, there is growth with maximum debt, so the following holds:

\[
\begin{align*}
\lambda_1 & = 0 \Rightarrow K > X \\
\lambda_2 & > 0 \Rightarrow K = (1+h)X \\
\lambda_3 & > 0 \Rightarrow D = 0
\end{align*}
\]

After substituting (28) and (29) in (10), we get:

\[
\lambda_2 = \nu_1(\frac{dS}{dK} - (a+r))
\]

Due to (11), (28) and (31) it holds that:

\[
-\nu_1 = ((1+h)\frac{dS}{dK} - h(a+r))\nu_1 - a\nu_1
\]

After using (8) and (30), we can rewrite (32) into:

\[
-\nu_1 = ((1+h)\frac{dS}{dK} - h(a+r))(e^{-iT}\lambda_3) - a\nu_1
\]

After solving this differential equation we get:
\[ \nu_1(T) = e^{iT} \int_T^{t+12} \left[ ((1+h)\frac{ds(t)}{dt} - h(a+r))e^{-(i+a)t} + \lambda_3(t) e^{-at} \right] dt + e^{iT}C \]  

(34)

When we substitute (8) and the equation for \( R_X \) in the case of maximum debt into (34), we get after multiplying with \( e^{iT} \):

\[ \lambda_3(T) e^{iT} = \int_T^{t+12} (R_X(t) + a) e^{-(i+a)(t-T)} dt + \int_T^{t+12} (R_X(t) + a)e^{-a(t-T)} \lambda_3(t)e^{iT} dt + e^{(i+a)T} \]

(35)

When we use the definition of \( R_X \) also on the paths 2, 3 and 4, it can be proved, but we will not show it here, that on the corresponding intervals (35) holds on path 2, path 3 and also on path 4 for \( \lambda_3 \) equal to zero.

Therefore, we can rewrite (35) into:

\[ \lambda_3(T) e^{iT} = \int_T^{Z} (R_X(t) + a)e^{-(i+a)(t-T)} dt + \int_T^{Z} (R_X(t) + a)e^{-a(t-T)} \lambda_3(t)e^{iT} dt + e^{-(i+a)(Z-T)} \]

(36)
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