Tilburg University

Finding a Nash equilibrium in noncooperative N-person games by solving a sequence of linear stationary point problems

van den Elzen, A.H.; Talman, A.J.J.

Publication date:
1992

[Link to publication in Tilburg University Research Portal](#)

Citation for published version (APA):

van den Elzen, A. H., & Talman, A. J. J. (1992). *Finding a Nash equilibrium in noncooperative N-person games by solving a sequence of linear stationary point problems*. (Research memorandum / Tilburg University, Department of Economics; Vol. FEW 570). Unknown Publisher.

General rights

Copyright and moral rights for the publications made accessible in the public portal are retained by the authors and/or other copyright owners and it is a condition of accessing publications that users recognise and abide by the legal requirements associated with these rights.

- Users may download and print one copy of any publication from the public portal for the purpose of private study or research.
- You may not further distribute the material or use it for any profit-making activity or commercial gain
- You may freely distribute the URL identifying the publication in the public portal

Take down policy

If you believe that this document breaches copyright please contact us providing details, and we will remove access to the work immediately and investigate your claim.

Eco
CBM
R
7626
7626
1992
570

UNIVERSITY
UNIVERSITEIT
BRABANT

POSTBOX 90153
5000 LE TILBURG
THE NETHERLANDS

R40

Nash Equilibrium
Game Theory

DEPARTMENT OF ECONOMICS
RESEARCH MEMORANDUM

FINDING A NASH EQUILIBRIUM IN
NONCOOPERATIVE N-PERSON GAMES BY
SOLVING A SEQUENCE OF LINEAR
STATIONARY POINT PROBLEMS
Antoon van den Elzen, Dolf Talman

FEW 570

Communicated by Prof.dr. S.H. Tijs

K.U.B.
BIBLIOTHEEK
TILBURG

FINDING A NASH EQUILIBRIUM IN NONCOOPERATIVE N -PERSON GAMES BY SOLVING A SEQUENCE OF LINEAR STATIONARY POINT PROBLEMS*

Antoon van den Elzen

Dolf Talman

Department of Econometrics, Tilburg University,
P.O. Box 90153, 5000 LE Tilburg,
The Netherlands

Abstract

In this paper we present an algorithm for finding a Nash equilibrium in a noncooperative normal form N -person game. More generally, the algorithm can be applied for solving a nonlinear stationary point problem on a simplotope, being the Cartesian product of several simplices. The algorithm solves the problem by solving a sequence of linear stationary point problems. Each problem in the sequence is solved in a finite number of iterations. Although the overall convergence cannot be proved, the method performs rather well. Computational results suggest that this algorithm performs at least as good as simplicial algorithms do.

For the special case of a bi-matrix game ($N = 2$), the algorithm has an appealing game-theoretic interpretation. In that case, the problem is linear and the algorithm always finds a solution. Furthermore, the equilibrium found in a bi-matrix game is perfect whenever the algorithm starts from a strategy vector at which all actions are played with positive probability.

Keywords: noncooperative game, Nash equilibrium, stationary point.

---

*This research is part of the VF-program "Co-operation and Competition", which has been approved by the Netherlands Ministry of Education and Sciences.

1 Introduction

In this paper we present an algorithm for computing a Nash equilibrium (NE) in a noncooperative N -person game in normal form. First we show that a Nash equilibrium of such a game is a stationary point of a multilinear function on a simpletope, being the Cartesian product of several unit simplices. The algorithm solves the stationary point problem of a continuous function on the simpletope and to find a Nash equilibrium it roughly works as follows. It starts from an arbitrary strategy vector in the simpletope. The multilinear function is linearized by taking the first-order Taylor expansion around that starting vector and extending it over the whole strategy space. Then the algorithm finds a stationary point on the simpletope for that linear function. If that strategy vector is a sufficient approximation of a Nash equilibrium for the original game, then the algorithm stops. Else the whole procedure is repeated but then from the strategy vector just found, and so on. This procedure is motivated by Mathiesen [1985a,b], [1987] who solves a Nonlinear Complementarity Problem (NLCP) by a sequence of Linear Complementarity Problems (LCP's). In practice this method works quite satisfactory, although global convergence is not guaranteed. Contrary to our method, his procedure may even fail to solve a specific LCP (see Mathiesen [1985b]).

Because our method solves a Sequence of Linear Stationary Point Problems we often refer to it as the SLSP-algorithm. This SLSP-algorithm is a generalization of the method presented in van den Elzen and Talman [1991] for finding an NE in a bi-matrix game. In that case the relevant function is linear and an exact solution is found.

The most well-known methods for finding an NE in an N -person game are the simplicial algorithms. For an exposition and some computational results we refer to Doup and Talman [1987]. At the end of this paper we compare the results obtained with the simplicial methods and the results we obtained with the procedure discussed here. We remark that in some sense the method sketched above works similar as a simplicial algorithm. Such a method finds in each round an exact solution for a piecewise linear approximation of the original function. Thus, then the function is linearized on each simplex of some simplicial subdivision, whereas here we take a linear approximation on the whole set. The advantage of simplicial algorithms is that their global convergence is guaranteed. However, especially in the later rounds simplicial algorithms may need more iterations to reach a solution. In practice it seems best to combine both methods by applying a simplicial algorithm during the first rounds, and later on the method discussed here. Concerning other methods we refer to the articles of Rosenmüller [1971] and Wilson

[1971]. However, both articles are theoretical of nature and of hardly any use for practical implementation. Rosenmüller argues that each nondegenerated N -person game has an odd number of isolated Nash equilibria. The reasoning is similar to that in Lemke and Howson [1964] for bi-matrix games; there is a path connecting the constructed starting vector and an NE whereas the other Nash equilibria are pairwise connected by paths. However, he gives no procedure how to follow these paths. The latter is crucial because these paths are in general not linear. Wilson [1971] proves the existence of a path leading to an NE in an N -person game. This path finds in succession an equilibrium for each of certain related k -person games for k increasing from 1 to N . An equilibrium for the k -person game is then the starting point of a path leading to an NE of a $(k + 1)$ -person game and so on. However, also here merely the existence of the path is proved and no method is given to follow it.

We stress the fact that our method can be used for other problems than the problem of finding a Nash equilibrium. In fact it can be used for finding a stationary point of an arbitrary continuous function on a simplotope.

The set-up of this paper is as follows. In Section 2 we introduce some notation and show that the set of Nash equilibria coincide with the solution set to a stationary point problem on a simplotope. Furthermore, we introduce the algorithm. Next, in Section 3 we present the formal steps of the algorithm. Finally, in Section 4 we give a game-theoretic interpretation of the algorithm and present an example. Furthermore, we compare the performance of the SLSPP-algorithm with that of the simplicial algorithms.

2 Nash equilibria as solutions to a stationary point problem

We consider noncooperative games with a finite number of players each having a finite set of actions. The payoffs to a player are listed in a tensor. Each element in the tensor indicates the payoff to that player when a specific set of actions is played by the players.

Let us introduce some notation. The number of players equals N and the players are indexed by j , $j \in \{1, \dots, N\}$. The set $\{1, \dots, N\}$ is often denoted as I_N . Player j has n_j actions. Action k of player j is indicated as (j, k) . The set of actions of player j , i.e. the set $\{(j, 1), \dots, (j, n_j)\}$ is denoted by $I(j)$. The set of all actions in the game is often denoted by I , i.e. $I = \cup_{j \in I_N} I(j)$. The payoffs to player j are listed in a tensor A^j , $j \in I_N$. The payoff to player j in case each player $i \in I_N$ plays action (i, ℓ_i) is denoted by $A^j(\ell_1, \dots, \ell_{j-1}, \ell_j, \ell_{j+1}, \dots, \ell_N)$.

A strategy of player j , $j \in I_N$, is represented by a vector $x_j = (x_{j1}, \dots, x_{jn_j})^\top$ in $S^{n_j-1} := \{x_j \in \mathbb{R}_+^{n_j} \mid \sum_{k=1}^{n_j} x_{jk} = 1\}$. The number x_{jk} , $k \in \{1, \dots, n_j\}$, is then the probability with which player j plays his k -th action at strategy x_j . Since the components of x_j are all nonnegative and sum up to one, x_j is indeed a vector of probabilities. We call S^{n_j-1} the strategy space of player j . In case actions are played with probability one we speak about pure strategies. The strategy space of the game is then the simpletope S obtained by taking the Cartesian product of the strategy spaces of the players, i.e. $S = S^{n_1-1} \times \dots \times S^{n_N-1}$. An element $x = (x_1, \dots, x_N)$ in S denotes a strategy vector of the game with x_j the strategy played by player $j \in I_N$.

The marginal expected payoff function z is a function from S to $\prod_{j=1}^N \mathbb{R}^{n_j}$ defined by $z(x) = (z_1(x), \dots, z_N(x))$, with

$$z_{jk}(x) = \sum_{\ell_1=1}^{n_1} \dots \sum_{\ell_{j-1}=1}^{n_{j-1}} \sum_{\ell_{j+1}=1}^{n_{j+1}} \dots \sum_{\ell_N=1}^{n_N} A^j(\ell_1, \dots, \ell_{j-1}, k, \ell_{j+1}, \dots, \ell_N) \prod_{i \neq j} x_{i\ell_i},$$

$$(j, k) \in I(j), j \in I_N.$$

Thus $z_{jk}(x)$ is the payoff to player j in case he plays his k -th pure strategy while each other player i plays x_i . Now a Nash equilibrium is defined as a strategy vector at which no player can improve upon his situation by unilaterally deviating from his strategy. Thus, $x^* = (x_1^*, \dots, x_N^*) \in S$ is a Nash equilibrium if

$$(x_j^\top) z_j(x^*) \leq (x_j^*)^\top z_j(x^*), \quad x_j \in S^{n_j-1}, j \in I_N. \quad (2.1)$$

The number $(x_j^*)^\top z_j(x^*)$ equals the expected payoff of player $j \in I_N$ at the strategy vector x^* . Thus, at a Nash equilibrium x^* it is optimal for each player j to play x_j^* . System (2.1) shows that x^* solves the stationary point problem of the function z on S . Indeed the set of Nash equilibria corresponds to the solution set of this stationary point problem.

Because of the linearity of $x_j^\top z_j(x^*)$ in x_j we only have to check (2.1) for the vertices of S^{n_j-1} , i.e. for the pure strategies. Thus, x^* is a Nash equilibrium if and only if $z_{jk}(x^*) \leq (x_j^*)^\top z_j(x^*)$ for all $(j, k) \in I$. From this it is straightforward to derive that x^* is a Nash equilibrium if and only if

$$z_{jk}(x^*) = \max_h z_{jh}(x^*) \text{ when } x_{jk}^* > 0, (j, k) \in I. \quad (2.2)$$

In general we say that an action k of player j is optimal at strategy x when $z_{jk}(x) = \max_h z_{jh}(x)$ and that at x player j is in equilibrium when at x all nonoptimal actions of j are played with probability zero. The interpretation of (2.2) is that at a Nash equilibrium the actions being played with positive probability are all optimal and hence that all players are in equilibrium.

How does the algorithm work to find a strategy vector x^* obeying (2.2)? First we choose an arbitrary starting vector $v = (v_1, \dots, v_N)$ in S . Then we linearize z around v to obtain the function $z^v : S \rightarrow \mathbf{R}^n$, defined by

$$z^v(x) = z(v) + Dz(v)(x - v), \quad (2.3)$$

where $n = \sum_{j=1}^N n_j$, and $Dz(v)$ is the $(n \times n)$ -matrix of derivatives of z at v . More precisely, the (p, q) -th element of $Dz(v)$ equals $\delta z_{jk}(v) / \delta x_{ih}$, where $p = \sum_{\ell=1}^{j-1} n_\ell + k$ and $q = \sum_{\ell=1}^{i-1} n_\ell + h$. Observe that this element is zero if $(i, h) \in I(j)$. Note also that $z(v)$, x , and v are represented as vectors in $\mathbf{R}^n$. Now we apply a generalized version of the algorithm presented in van den Elzen and Talman [1991], to find an NE for the game with marginal expected payoff function z^v . If that strategy vector is a close enough approximation of an NE for the original game then the algorithm stops. Else the procedure is restarted from the strategy vector just found.

So, the algorithm searches for an equilibrium for the game with marginal payoff function z^v . It does so by generating from v a path of strategy vectors $x = (x_1, \dots, x_N)$ in S obeying for $(j, k) \in I$,

$$\begin{aligned} x_{jk} &= b(x, v) v_{jk} \quad \text{if } z_{jk}^v(x) < \max_h z_{jh}^v(x) \\ x_{jk} &\geq b(x, v) v_{jk} \quad \text{if } z_{jk}^v(x) = \max_h z_{jh}^v(x), \end{aligned} \quad (2.4)$$

where $0 \leq b(x, v) := \min_{(j,h)} \{x_{jh}/v_{jh} \mid v_{jh} > 0\} \leq 1$.

Observe that the strategy vector v obeys (2.4) with $b(x, v) = 1$. Also any Nash equilibrium x^* for the game with marginal payoff function z^v obeys (2.4) with $b(x^*, v) = 0$ or with $v_{jk} = 0$ for all indices (j, k) for which $z_{jk}^v(x^*) < \max_h z_{jh}^v(x^*)$. In both cases

$x_{jk}^* = b(x^*, v)v_{jk}$ is equal to zero when at x^* action (j, k) is nonoptimal. Under some nondegeneracy condition the set of points satisfying (2.4) contains a piecewise linear path, P , from v to a Nash equilibrium related to z^v . That Nash equilibrium is an approximation for a Nash equilibrium for the original game. The path P will be followed by the algorithm as described in the next section.

The notion of nondegeneracy will be made precise further on, but it is for example required that at $x = v$ it must hold that $\max_h z_{jh}^v(x), j \in I_N$, is attained for a unique index. Thus, at the starting strategy vector v each player has only one optimal action. Suppose these maxima are attained for the actions $(j, k_j), j \in I_N$. Clearly, from v , along the path $P, b(x, v)$ must decrease from 1. Thus, according to (2.4) initially vectors x in S are generated such that all the x_{jk} with $k \neq k_j, j \in I_N$, are relatively decreased ($x_{i\ell} = b(x, v)v_{i\ell}$ for $(i, \ell) \notin \{(j, k_j), j \in I_N\}$), while every $x_{jk_j}, j \in I_N$, is increased to keep x_j in S^{n-1} . This is continued till either $b(x, v)$ becomes 0 and a Nash equilibrium for z^v is reached, or a strategy vector x is reached at which $z_{jk}^v(x) = z_{jk_j}^v(x)$ for some $(j, k), k \neq k_j$. Then x_{jk} is also relatively increased. In general, the algorithm generates strategy vectors x such that all the x_{jk}/v_{jk} , related to the indices (j, k) for which $z_{jk}^v(x) < \max_h z_{jh}^v(x)$, are minimal, i.e. equal to $b(x, v)$ ($x_{jk} = 0$ if $v_{jk} = 0$). As soon as one of these components of $z^v(x)$, say $z_{j\ell}^v(x)$, becomes equal to $\max_h z_{jh}^v(x)$, then $x_{j\ell}/v_{j\ell}$ is increased from $b(x, v)$ ($x_{j\ell}$ is increased from zero if $v_{j\ell} = 0$), while keeping $z_{j\ell}^v(x)$ maximal for j . On the other hand, if a vector x is generated such that x_{jr}/v_{jr} for some (j, r) with $z_{jr}^v(x) = \max_h z_{jh}^v(x)$ becomes minimal, i.e. equal to $b(x, v)$ (x_{jr} becomes 0 if $v_{jr} = 0$), then vectors y are generated with y_{jr} equal to $b(y, v)v_{jr}$ while $z_{jr}^v(y)$ is decreased from $\max_h z_{jh}^v(y)$.

3 The procedure

The algorithm is a complementary pivoting procedure. To implement it we have to write (2.4) as a system of linear equations. Observe that for each vector x satisfying (2.4) there is at least one set $T \subset I$ with $T_j = T \cap I(j) \neq \emptyset$ for all j such that $x_{jk} \geq bv_{jk}$ and $z_{jk}^v(x) = \beta_j$ for all $(j, k) \in T$ while $z_{ih}^v(x) \leq \beta_i$ and $x_{ih} = bv_{ih}$ for all $(i, h) \notin T$, where $b = b(x, v)$ and $\beta_j = \max_h z_{jh}^v(x)$. From this observation we obtain that the procedure generates for a sequence of subsets T of I , starting from $x = v$, strategy vectors x in S , such that

$$x = bv + \sum_{(j,k) \in T} \lambda_{jk} e(j, k)$$

and

$$(3.1)$$

$$z^v(x) = - \sum_{(i,h) \notin T} \mu_{ih} e(i,h) + \sum_{j=1}^N \beta_j \bar{e}_j,$$

with $\sum_{(j,k) \in T} \lambda_{jk} = 1 - b$ for $j \in I_N$, $\lambda_{jk} \geq 0$ for $(j,k) \in T$, $\mu_{ih} \geq 0$ for $(i,h) \notin T$, $b \in [0, 1]$. The vector $e(j,k)$ denotes the $(\sum_{i=1}^{j-1} n_i + k)$ -th unit vector in $\mathbb{R}^n$ and $\bar{e}_j$ denotes the vector in $\mathbb{R}^n$ with ones on the places $\sum_{\ell=1}^{j-1} n_\ell + k$, $k \in \{1, \dots, n_j\}$, and zeros elsewhere.

So, we obtain the linear system (3.1) from (2.4) by introducing slacks for each inequality. Next, we substitute the first system of equations in (3.1) into the second system and together with (2.3) we obtain the system of linear equations

$$bDz(v)v + \sum_{(j,k) \in T} \lambda_{jk} D^{jk} z(v) + \sum_{(i,h) \notin T} \mu_{ih} e(i,h) - \sum_{j \in I_N} \beta_j \bar{e}_j = Dz(v)v - z(v) \quad (3.2)$$

$$\text{with } b + \sum_{(j,k) \in T} \lambda_{jk} = 1, \text{ for } j \in I_N.$$

Here $D^{jk} z(v)$ denotes the $(\sum_{i=1}^{j-1} n_i + k)$ -th column of the matrix $Dz(v)$.

We obtain the final system by substituting $b' = 1 - b$ and for $j \in I_N$, λ_{jk} , by $b' - \sum_{k \neq k_j} \lambda_{jk}$ where k_j is the index for which $z_{jk_j}(v) = \max_{\ell} z_{j\ell}(v)$. In this way, we obtain a system of n equations with $n + 1$ variables. More precise, we get the system of equations

$$\begin{aligned} b' \left(\sum_{j \in I_N} D^{jk_j} z(v) - Dz(v)v \right) + \sum_{\substack{(j,k) \in T \\ k \neq k_j}} \lambda_{jk} (D^{jk} z(v) - D^{jk_j} z(v)) \\ + \sum_{(i,h) \notin T} \mu_{ih} e(i,h) - \sum_{j \in I_N} \beta_j \bar{e}_j = -z(v), \end{aligned} \quad (3.3)$$

with additional restrictions $\lambda_{jk} \geq 0$ for $(j,k) \in T_j \setminus \{(j,k_j)\}$, $\sum_{k \neq k_j} \lambda_{jk} \leq b'$ for $j \in I_N$, $0 \leq b' \leq 1$, and $\mu_{ih} \geq 0$ for $(i,h) \notin T$. Observe that (3.3) is equivalent to system (3.7) in van den Elzen and Talman [1991] in case $N = 2$ because in that case

$$Dz(v) = \begin{bmatrix} 0 & A \\ B^T & 0 \end{bmatrix}.$$

A solution to (3.3) is denoted as $(b', \lambda, \mu, \beta)$ with $\lambda = (\lambda_{jk})_{(j,k) \in T}$, $\mu = (\mu_{ih})_{(i,h) \notin T}$, and $\beta = (\beta_1, \dots, \beta_N)$. The algorithm is a complementary pivoting procedure operating in

system (3.3), with the variables in λ and μ being complementary. So, if λ_{jk} for some $(j, k) \in T$ becomes zero then μ_{jk} is increased from zero and vice versa. To guarantee convergence we need that not more than one variable becomes zero at the same time.

Assumption 3.1 (Nondegeneracy Assumption). At each solution $(b', \lambda, \mu, \beta)$ of (3.3) at most one of the constraints $0 \leq b' \leq 1, \lambda_{jk} \geq 0$ for $(j, k) \in T^1 = T^1 \setminus \{(j, k_j), j \in I_N\}$, $b' \geq \sum_{(j,k) \in T_j^1} \lambda_{jk}, \mu_{ih} \geq 0$ for $(i, h) \notin T$, is binding, unless $b' = 1$ or $v_{ih} = 0$ for all $(i, h) \notin T$.

The assumption above implies that at v each player should have only one optimal action. If for example $z_{j\ell}(v) = z_{jk_j}(v) = \beta_j$ for some index ℓ then two constraints would hold at the start ($b' = 0$ and $\mu_{j\ell} = 0$).

The formal steps of the procedure are the following.

Step 0 [Initialization]

Choose an arbitrary vector v in S . If v is a Nash equilibrium then the algorithm stops. Else choose a measure of inaccuracy δ , a maximum number of rounds $\bar{t}$, and set t equal to 1.

Step 1

If $t > \bar{t}$ then the algorithm stops and no approximate Nash equilibrium is found within $\bar{t}$ rounds. Else construct the first-order Taylor expansion z^v of z at v as in (2.3). Calculate for $j \in I_N$ the unique index (j, k_j) for which $z_{jk_j}(v) = \max_h z_{jh}(v)$. Furthermore, set $T^1 = \emptyset, b' = 0, \beta_j = z_{jk_j}(v)$ for $j \in I_N, \mu_{ih} = \beta_i - z_{ih}(v)$ for $h \neq k_i$ and $i \in I_N$. Increase b' from 0 in (3.3) and go to Step 2.

Step 2

- a. If b' becomes 1 then let the solution of (3.3) be $(1, \lambda^*, \mu^*, \beta^*)$. The vector x^* , with $x^* = \sum_{(j,k) \in T} \lambda_{jk}^* e(j, k)$ is a solution to the SPP of z^v on S (cf. (2.2)). If $\max_{(j,k)} (z_{jk}(x^*) - (x_j^*)^\top z_j(x^*)) < \delta$ then x^* is an approximate Nash equilibrium and the algorithm stops after round t . Else t becomes $t + 1$ and return to Step 1 with v equal to x^* .
- b. If λ_{jk} becomes 0 for some $(j, k) \in T^1$ then T^1 becomes $T^1 \setminus \{(j, k)\}$ and go to Step 3a.
- c. If $\sum_{(j,k) \in T_j^1} \lambda_{jk}$ becomes equal to b' for some $j \in I_N$ then λ_{jk_j} becomes 0. Go to

Step 3b.

- d. If μ_{ih} becomes zero for some $(i, h) \notin T$ then go to Step 4.

Step 3

- a. Increase the complementary variable μ_{jk} from zero by pivoting into system (3.3) the column $e(j, k)$. Return to Step 2.
- b. Substitute the largest λ_{jk} , say $\lambda_{j\ell}$, by $b' - \sum_{\substack{(j,h) \in T_j \\ h \neq k_j, \ell}} \lambda_{jh}$. Increase μ_{jk} from zero by pivoting the related column into system (3.3). T^1 becomes $T^1 \setminus \{(j, \ell)\}$, k_j becomes ℓ , and return to Step 2.

Step 4

- a. If $v_{jk} = 0$ for all $(j, k) \notin T \cup \{(i, h)\}$ then let the solution be $(b^*, \lambda^*, \mu^*, \beta^*)$. The vector $x^* = (1 - b^*)v + \sum_{(j,k) \in T} \lambda_{jk}^* e(j, k)$ is a solution to the SPP of z^v on S (cf. (2.2)). If $\max_{(j,k)} (z_{jk}(x^*) - (x_j^*)^\top z_j(x^*)) < \delta$ then x^* is an approximate Nash equilibrium and the algorithm stops after round t . Else t becomes $t + 1$ and return to Step 1 with v equal to x^* .
- b. Otherwise increase the complementary variable λ_{ih} from zero by pivoting its related column into system (3.3). T^1 becomes $T^1 \cup \{(i, h)\}$ and return to Step 2.

4 Game theoretic interpretation and computation

The procedure presented in Section 3 has an intuitive game-theoretic interpretation for the game with marginal expected payoff function z^v . In case $N = 2$ the function z^v corresponds with z and the interpretation is even more appealing.

The procedure starts by increasing the probabilities with which at v the optimal actions are played whereas the probabilities related to the other actions are decreased relatively equally (b decreases from 1 and λ_{jk} increases from 0 for $j \in I_N$). With relatively we mean relative to the initial probabilities. In general the algorithm generates strategy vectors at which the relative probabilities related to the nonoptimal actions are all equal and minimal whereas they are zero if the probability is initially zero. As soon as a strategy vector is generated at which an action that was nonoptimal becomes optimal ($\mu_{i\ell}$ becomes zero for some $(i, \ell) \notin T$) then that action is kept optimal whereas the related relative probability is allowed to increase from the minimum ($\lambda_{i\ell}$ is increased). On the other hand, if a vector is generated at which the probability related to an optimal

action becomes relatively minimal (λ_{jk} becomes zero for some $(j, k) \in T$) or zero if v_{jk} is zero, then the probability is kept relatively minimal (zero if v_{jk} is zero) and the action is made nonoptimal (μ_{jk} is increased from zero).

We illustrate the working of the algorithm by an example.

Example 4.1. We consider a noncooperative 3-person game, in which each player has two actions. The payoffs are listed below.

	(1,1)			(1,2)	
	(2,1)	(2,2)		(2,1)	(2,2)
(3,1)	(1,3,-2)	(2,1,4)	(3,1)	(1,-3,2)	(1,1,-3)
(3,2)	(-1,-2,3)	(4,5,-6)	(3,2)	(1,-3,4)	(-5,1,-2)

The left matrix entails the payoffs in case player 1 plays his first action, the right matrix corresponds to strategy vectors at which player 1 plays his second action. Each entry in the matrix consists of three elements corresponding to the payoffs for each player. For example, if player 1 plays (1,2), player 2 plays (2,2) and 3 plays (3,1) then player 1 and 2 get a payoff of 1 whereas player 3 gets -3.

Let us apply our algorithm starting from $v = ((1/2, 1/2), (1/2, 1/2), (1/2, 1/2))$. Linearizing the marginal payoff function z at v gives

$$z^v(x) = \begin{bmatrix} 1.5 \\ -0.5 \\ -1.25 \\ 2 \\ 0.25 \\ -0.25 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 3 & 1.5 & 1.5 \\ 0 & 0 & 1 & -2 & 1 & -2 \\ 0.5 & -3 & 0 & 0 & 0 & -2.5 \\ 3 & 1 & 0 & 0 & 1 & 3 \\ 1 & -0.5 & 0 & 0.5 & 0 & 0 \\ -1.5 & 1 & 3.5 & -4 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_{11} - 0.5 \\ x_{12} - 0.5 \\ x_{21} - 0.5 \\ x_{22} - 0.5 \\ x_{31} - 0.5 \\ x_{32} - 0.5 \end{bmatrix}.$$

At $x = v$, $z^v(x)$ equals $z(v)$ and the optimal action for player 1 is (1,1), for player 2 action (2,2), for player 3 action (3,1). Thus, from the start the related probabilities are increased by increasing b' from zero (Step 1). After one step a stationary point

for z^v is found because b' becomes 1 (Step 2a). It is easily checked that the vector $\bar{x} = ((1,0), (0,1), (1,0))$ is indeed a Nash equilibrium related to the marginal payoff function z^v . However, $z(\bar{x})$ equals $(2, 1, 3, 1, 4, -6)^\top$, i.e. $\bar{x}$ is not a Nash equilibrium for the original game because player 2 is not in equilibrium at $\bar{x}$. In the second round we linearize z at $v = x$ and obtain

$$z^v(x) = \begin{bmatrix} 2 \\ 1 \\ 3 \\ 1 \\ 4 \\ -6 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 1 & 2 & 2 & 4 \\ 0 & 0 & 1 & 1 & 1 & -5 \\ 3 & -3 & 0 & 0 & 3 & -2 \\ 1 & 1 & 0 & 0 & 1 & 5 \\ 4 & -3 & -2 & 4 & 0 & 0 \\ -6 & -2 & 3 & -6 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_{11} - 1 \\ x_{12} \\ x_{21} \\ x_{22} - 1 \\ x_{31} - 1 \\ x_{32} \end{bmatrix}.$$

At the start we have that $\beta_1 = z_{11}(v) = 2$, $\beta_2 = z_{21}(v) = 3$, and $\beta_3 = z_{31}(v) = 4$. Furthermore $\mu_{12} = 1$, $\mu_{22} = 2$, and $\mu_{32} = 10$. Then b' is increased from 0 (Step 1). If b' becomes $2/3$ then μ_{32} becomes 0 (Step 2d). For the solution to (3.3) it further holds that $\beta = (\beta_1, \beta_2, \beta_3) = (4/3, 3, 0)$, $\mu_{12} = 1/3$, $\mu_{22} = 2$. Thus, now λ_{32} is increased (Step 4) till $2/9$ when μ_{22} becomes zero (Step 2d). For the solution to (3.3) we have $\beta = (16/9, 17/9, 0)$, $\mu_{12} = 19/9$, $\lambda_{31} = 4/9$, $b' = 2/3$ with corresponding vector $x^* = ((1,0), (2/3, 1/3), (7/9, 2/9))$. Since $v_{jk} = 0$ for all (j, k) such that $\mu_{jk} > 0$, x^* is an equilibrium for z^v (Step 4a). It is easily verified that x^* is also a Nash equilibrium for the original game.

Finally we compare the speed of our algorithm with that of simplicial algorithms. We applied our algorithm to the three games given in Doup and Talman [1987] and compared the results with those given in Doup [1988, ch. 11]. Three different simplicial algorithms are applied to the games. In Table 4.1 we compare our results with the best results obtained by the simplicial algorithms. It turns out that our algorithm is at least as good.

Game	Simplicial algorithms		SLSPP algorithm	
	LP	ν	LP	ν
1	35	4	28	4
2	14	1	6	2
3	14	3	14	3

Table 4.1. Computational results obtained for three noncooperative more-person games. LP denotes the number of linear programming pivoting steps whereas ν denotes the number of rounds.

References

- Doup, T.M. [1988], *Simplicial Algorithms on the Simplotope*, Lecture Notes in Economics and Mathematical Systems 318, Springer-Verlag, Berlin.
- Doup, T.M. and A.J.J. Talman [1987], "A new variable dimension simplicial algorithm to find equilibria on the product space of unit simplices", *Mathematical Programming* 37, 319-355.
- Elzen, A.H. van den, and A.J.J. Talman [1991], "A procedure for finding Nash equilibria in bi-matrix games", *ZOR-Methods and Models of Operations Research* 35, 27-43.
- Lemke, C.E. and J.T. Howson [1964], "Equilibrium points of bi-matrix games", *SIAM Journal of Applied Mathematics* 12, 413-423.
- Mathiesen, L. [1985a], "Computational experience in solving equilibrium models by a sequence of linear complementarity problems", *Operations Research* 33, 1225-1250.
- Mathiesen, L. [1985b], "Computation of economic equilibria by a sequence of linear complementarity problems", *Mathematical Programming Study* 23, 144-162.
- Mathiesen, L. [1987], "An algorithm based on a sequence of linear complementarity problems applied to a Walrasian equilibrium model: an example", *Mathematical Programming* 37, 1-18.
- Rosenmüller, J. [1971], "On a generalization of the Lemke-Howson algorithm to noncooperative N-person games", *SIAM Journal of Applied Mathematics* 21, 73-79.
- Wilson, R. [1971], "Computing equilibria of N-person games", *SIAM Journal of Applied Mathematics* 21, 80-87.

IN 1991 REEDS VERSCHENEN

- 466 Prof.Dr. Th.C.M.J. van de Klundert - Prof.Dr. A.B.T.M. van Schaik
Economische groei in Nederland in een internationaal perspectief
- 467 Dr. Sylvester C.W. Eijffinger
The convergence of monetary policy - Germany and France as an example
- 468 E. Nijssen
Strategisch gedrag, planning en prestatie. Een inductieve studie binnen de computerbranche
- 469 Anne van den Nouweland, Peter Borm, Guillermo Owen and Stef Tijs
Cost allocation and communication
- 470 Drs. J. Grazell en Drs. C.H. Veld
Motieven voor de uitgifte van converteerbare obligatieleningen en warrant-obligatieleningen: een agency-theoretische benadering
- 471 P.C. van Batenburg, J. Kriens, W.M. Lammerts van Bueren and R.H. Veenstra
Audit Assurance Model and Bayesian Discovery Sampling
- 472 Marcel Kerkhofs
Identification and Estimation of Household Production Models
- 473 Robert P. Gilles, Guillermo Owen, René van den Brink
Games with Permission Structures: The Conjunctive Approach
- 474 Jack P.C. Kleijnen
Sensitivity Analysis of Simulation Experiments: Tutorial on Regression Analysis and Statistical Design
- 475 C.P.M. van Hoesel
An $O(n \log n)$ algorithm for the two-machine flow shop problem with controllable machine speeds
- 476 Stephan G. Vanneste
A Markov Model for Opportunity Maintenance
- 477 F.A. van der Duyn Schouten, M.J.G. van Eijs, R.M.J. Heuts
Coordinated replenishment systems with discount opportunities
- 478 A. van den Nouweland, J. Potters, S. Tijs and J. Zarzuelo
Cores and related solution concepts for multi-choice games
- 479 Drs. C.H. Veld
Warrant pricing: a review of theoretical and empirical research
- 480 E. Nijssen
De Miles and Snow-typologie: Een exploratieve studie in de meubelbranche
- 481 Harry G. Barkema
Are managers indeed motivated by their bonuses?

- 482 Jacob C. Engwerda, André C.M. Ran, Arie L. Rijkeboer
Necessary and sufficient conditions for the existence of a positive definite solution of the matrix equation $X + A'X^{-1}A = I$
- 483 Peter M. Kort
A dynamic model of the firm with uncertain earnings and adjustment costs
- 484 Raymond H.J.M. Gradus, Peter M. Kort
Optimal taxation on profit and pollution within a macroeconomic framework
- 485 René van den Brink, Robert P. Gilles
Axiomatizations of the Conjunctive Permission Value for Games with Permission Structures
- 486 A.E. Brouwer & W.H. Haemers
The Gewirtz graph - an exercise in the theory of graph spectra
- 487 Pim Adang, Bertrand Melenberg
Intratemporal uncertainty in the multi-good life cycle consumption model: motivation and application
- 488 J.H.J. Roemen
The long term elasticity of the milk supply with respect to the milk price in the Netherlands in the period 1969-1984
- 489 Herbert Hamers
The Shapley-Entrance Game
- 490 Rezaul Kabir and Theo Vermaelen
Insider trading restrictions and the stock market
- 491 Piet A. Verheyen
The economic explanation of the jump of the co-state variable
- 492 Drs. F.L.J.W. Manders en Dr. J.A.C. de Haan
De organisatorische aspecten bij systeemontwikkeling een beschouwing op besturing en verandering
- 493 Paul C. van Batenburg and J. Kriens
Applications of statistical methods and techniques to auditing and accounting
- 494 Ruud T. Frambach
The diffusion of innovations: the influence of supply-side factors
- 495 J.H.J. Roemen
A decision rule for the (des)investments in the dairy cow stock
- 496 Hans Kremers and Dolf Talman
An SLSP algorithm to compute an equilibrium in an economy with linear production technologies

- 497 L.W.G. Strijbosch and R.M.J. Heuts
Investigating several alternatives for estimating the compound lead time demand in an (s,Q) inventory model
- 498 Bert Bettonvil and Jack P.C. Kleijnen
Identifying the important factors in simulation models with many factors
- 499 Drs. H.C.A. Roest, Drs. F.L. Tijssen
Beheersing van het kwaliteitsperceptieproces bij diensten door middel van keurmerken
- 500 B.B. van der Genugten
Density of the F-statistic in the linear model with arbitrarily normal distributed errors
- 501 Harry Barkema and Sytse Douma
The direction, mode and location of corporate expansions
- 502 Gert Nieuwenhuis
Bridging the gap between a stationary point process and its Palm distribution
- 503 Chris Veld
Motives for the use of equity-warrants by Dutch companies
- 504 Pieter K. Jagersma
Een etiologie van horizontale internationale ondernemingsexpansie
- 505 B. Kaper
On M-functions and their application to input-output models
- 506 A.B.T.M. van Schaik
Productiviteit en Arbeidsparticipatie
- 507 Peter Borm, Anne van den Nouweland and Stef Tijs
Cooperation and communication restrictions: a survey
- 508 Willy Spanjers, Robert P. Gilles, Pieter H.M. Ruys
Hierarchical trade and downstream information
- 509 Martijn P. Tummers
The Effect of Systematic Misperception of Income on the Subjective Poverty Line
- 510 A.G. de Kok
Basics of Inventory Management: Part 1
Renewal theoretic background
- 511 J.P.C. Blanc, F.A. van der Duyn Schouten, B. Pourbabai
Optimizing flow rates in a queueing network with side constraints
- 512 R. Peeters
On Coloring j-Unit Sphere Graphs

- 513 Drs. J. Dagevos, Drs. L. Oerlemans, Dr. F. Boekema
Regional economic policy, economic technological innovation and networks
- 514 Erwin van der Krabben
Het functioneren van stedelijke onroerend-goed-markten in Nederland - een theoretisch kader
- 515 Drs. E. Schaling
European central bank independence and inflation persistence
- 516 Peter M. Kort
Optimal abatement policies within a stochastic dynamic model of the firm
- 517 Pim Adang
Expenditure versus consumption in the multi-good life cycle consumption model
- 518 Pim Adang
Large, infrequent consumption in the multi-good life cycle consumption model
- 519 Raymond Gradus, Sjak Smulders
Pollution and Endogenous Growth
- 520 Raymond Gradus en Hugo Keuzenkamp
Arbeidsongeschiktheid, subjectief ziektegevoel en collectief belang
- 521 A.G. de Kok
Basics of inventory management: Part 2
The (R,S)-model
- 522 A.G. de Kok
Basics of inventory management: Part 3
The (b,Q)-model
- 523 A.G. de Kok
Basics of inventory management: Part 4
The (s,S)-model
- 524 A.G. de Kok
Basics of inventory management: Part 5
The (R,b,Q)-model
- 525 A.G. de Kok
Basics of inventory management: Part 6
The (R,s,S)-model
- 526 Rob de Groof and Martin van Tuijl
Financial integration and fiscal policy in interdependent two-sector economies with real and nominal wage rigidity

- 527 A.G.M. van Eijs, M.J.G. van Eijs, R.M.J. Heuts
Gecoördineerde bestelsystemen
een management-georiënteerde benadering
- 528 M.J.G. van Eijs
Multi-item inventory systems with joint ordering and transportation
decisions
- 529 Stephan G. Vanneste
Maintenance optimization of a production system with buffercapacity
- 530 Michel R.R. van Bremen, Jeroen C.G. Zijlstra
Het stochastische variantie optiewaarderingsmodel
- 531 Willy Spanjers
Arbitrage and Walrasian Equilibrium in Economies with Limited Infor-
mation

IN 1992 REEDS VERSCHENEN

- 532 F.G. van den Heuvel en M.R.M. Turlings
Privatisering van arbeidsongeschiktheidsregelingen
Refereed by Prof.Dr. H. Verbon
- 533 J.C. Engwerda, L.G. van Willigenburg
LQ-control of sampled continuous-time systems
Refereed by Prof.dr. J.M. Schumacher
- 534 J.C. Engwerda, A.C.M. Ran & A.L. Rijkeboer
Necessary and sufficient conditions for the existence of a positive definite solution of the matrix equation $X + A^*X^{-1}A = Q$.
Refereed by Prof.dr. J.M. Schumacher
- 535 Jacob C. Engwerda
The indefinite LQ-problem: the finite planning horizon case
Refereed by Prof.dr. J.M. Schumacher
- 536 Gert-Jan Otten, Peter Borm, Ton Storcken, Stef Tijs
Effectivity functions and associated claim game correspondences
Refereed by Prof.dr. P.H.M. Ruys
- 537 Jack P.C. Kleijnen, Gustav A. Alink
Validation of simulation models: mine-hunting case-study
Refereed by Prof.dr.ir. C.A.T. Takkenberg
- 538 V. Feltkamp and A. van den Nouweland
Controlled Communication Networks
Refereed by Prof.dr. S.H. Tijs
- 539 A. van Schaik
Productivity, Labour Force Participation and the Solow Growth Model
Refereed by Prof.dr. Th.C.M.J. van de Klundert
- 540 J.J.G. Lemmen and S.C.W. Eijffinger
The Degree of Financial Integration in the European Community
Refereed by Prof.dr. A.B.T.M. van Schaik
- 541 J. Bell, P.K. Jagersma
Internationale Joint Ventures
Refereed by Prof.dr. H.G. Barkema
- 542 Jack P.C. Kleijnen
Verification and validation of simulation models
Refereed by Prof.dr.ir. C.A.T. Takkenberg
- 543 Gert Nieuwenhuis
Uniform Approximations of the Stationary and Palm Distributions of Marked Point Processes
Refereed by Prof.dr. B.B. van der Genugten

- 544 R. Heuts, P. Nederstigt, W. Roebroek, W. Selen
Multi-Product Cycling with Packaging in the Process Industry
Refereed by Prof.dr. F.A. van der Duyn Schouten
- 545 J.C. Engwerda
Calculation of an approximate solution of the infinite time-varying
LQ-problem
Refereed by Prof.dr. J.M. Schumacher
- 546 Raymond H.J.M. Gradus and Peter M. Kort
On time-inconsistency and pollution control: a macroeconomic approach
Refereed by Prof.dr. A.J. de Zeeuw
- 547 Drs. Dolph Cantrijn en Dr. Rezaul Kabir
De Invloed van de Invoering van Preferente Beschermingsaandelen op
Aandelenkoersen van Nederlandse Beursgenoteerde Ondernemingen
Refereed by Prof.dr. P.W. Moerland
- 548 Sylvester Eijffinger and Eric Schaling
Central bank independence: criteria and indices
Refereed by Prof.dr. J.J. Sijben
- 549 Drs. A. Schmeits
Geïntegreerde investerings- en financieringsbeslissingen; Implicaties
voor Capital Budgeting
Refereed by Prof.dr. P.W. Moerland
- 550 Peter M. Kort
Standards versus standards: the effects of different pollution
restrictions on the firm's dynamic investment policy
Refereed by Prof.dr. F.A. van der Duyn Schouten
- 551 Niels G. Noorderhaven, Bart Nooteboom and Johannes Berger
Temporal, cognitive and behavioral dimensions of transaction costs;
to an understanding of hybrid vertical inter-firm relations
Refereed by Prof.dr. S.W. Douma
- 552 Ton Storcken and Harrie de Swart
Towards an axiomatization of orderings
Refereed by Prof.dr. P.H.M. Ruys
- 553 J.H.J. Roemen
The derivation of a long term milk supply model from an optimization
model
Refereed by Prof.dr. F.A. van der Duyn Schouten
- 554 Geert J. Almekinders and Sylvester C.W. Eijffinger
Daily Bundesbank and Federal Reserve Intervention and the Conditional
Variance Tale in DM/\$-Returns
Refereed by Prof.dr. A.B.T.M. van Schaik
- 555 Dr. M. Hetebrij, Drs. B.F.L. Jonker, Prof.dr. W.H.J. de Freytas
"Tussen achterstand en voorsprong" de scholings- en personeelsvoor-
zieningsproblematiek van bedrijven in de procesindustrie
Refereed by Prof.dr. Th.M.M. Verhallen

- 556 Ton Geerts
Regularity and singularity in linear-quadratic control subject to implicit continuous-time systems
Communicated by Prof.dr. J. Schumacher
- 557 Ton Geerts
Invariant subspaces and invertibility properties for singular systems: the general case
Communicated by Prof.dr. J. Schumacher
- 558 Ton Geerts
Solvability conditions, consistency and weak consistency for linear differential-algebraic equations and time-invariant singular systems: the general case
Communicated by Prof.dr. J. Schumacher
- 559 C. Fricker and M.R. Jaïbi
Monotonicity and stability of periodic polling models
Communicated by Prof.dr.ir. O.J. Boxma
- 560 Ton Geerts
Free end-point linear-quadratic control subject to implicit continuous-time systems: necessary and sufficient conditions for solvability
Communicated by Prof.dr. J. Schumacher
- 561 Paul G.H. Mulder and Anton L. Hempenius
Expected Utility of Life Time in the Presence of a Chronic Noncommunicable Disease State
Communicated by Prof.dr. B.B. van der Genugten
- 562 Jan van der Leeuw
The covariance matrix of ARMA-errors in closed form
Communicated by Dr. H.H. Tigelaar
- 563 J.P.C. Blanc and R.D. van der Mei
Optimization of polling systems with Bernoulli schedules
Communicated by Prof.dr.ir. O.J. Boxma
- 564 B.B. van der Genugten
Density of the least squares estimator in the multivariate linear model with arbitrarily normal variables
Communicated by Prof.dr. M.H.C. Paardekooper
- 565 René van den Brink, Robert P. Gilles
Measuring Domination in Directed Graphs
Communicated by Prof.dr. P.H.M. Ruys
- 566 Harry G. Barkema
The significance of work incentives from bonuses: some new evidence
Communicated by Dr. Th.E. Nijman

- 567 Rob de Groof and Martin van Tuijl
Commercial integration and fiscal policy in interdependent, financially integrated two-sector economies with real and nominal wage rigidity.
Communicated by Prof.dr. A.L. Bovenberg
- 568 F.A. van der Duyn Schouten, M.J.G. van Eijs, R.M.J. Heuts
The value of information in a fixed order quantity inventory system
Communicated by Prof.dr. A.J.J. Talman
- 569 E.N. Kertzman
Begrotingsnormering en EMU
Communicated by Prof.dr. J.W. van der Dussen

Bibliotheek K. U. Brabant

17 000 01108477 0