2-Designs having an intersection number $k \leq n$

Haemers, Willem; Beker, H.

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2-Designs Having an Intersection Number $k - n$

HENRY BEKER

Racal-Datacom Ltd., Milford Industrial Estate, Tollgate Road, Salisbury, Wilshire SP1 2JG, Great Britain

AND

WILLEM HAEMERS

Technical University, Eindhoven, Department of Mathematics, P.O. Box 513, Eindhoven, The Netherlands

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In this paper we examine 2-designs having an intersection number $k - n$. This intersection number gives rise to an equivalence relation on the blocks of the design. Conditions on the sizes of these equivalence classes and some properties of any further intersection numbers are obtained. If such a design has at most three intersection numbers then it gives rise to a strongly regular graph. This leads to a result on the embedding of quasi-residual designs. As an example a quasi-residual 2-(56, 12, 3) design is constructed and embedded in a symmetric 2-(71, 15, 3) design.

INTRODUCTION

Many interesting types of 2-designs do have an intersection number $k - n$, such as symmetric and affine designs, and those which admit a strong tactical decomposition, see [1]. Majumdar [12] showed that this intersection number $k - n$ gives rise to an equivalence relation on the blocks. The aim of this paper is to study 2-designs with an intersection number $k - n$.

We shall see that for a given value of $k - n \notin \{0, \lambda\}$ there are only finitely many such designs. Bounds are obtained for the size of an (equivalence) class and for the remaining intersection numbers; also the case of tightness is treated.

In section 4 we prove that all classes have equal size $m$, say, if the design has just three intersection numbers. In some special cases the converse is true, such as when $\lambda v/b$ and $n/m + k - n$ are consecutive integers. Goethals and Seidel [8] proved that any 2-design with just 2 intersection numbers...
"carries" a strongly regular graph. A similar situation exists in the case when the design has three intersection numbers, provided one of them equals \( k - n \).

For the case when \( k - n = 0 \), and the design is quasi-residual we give conditions under which the design is embeddable in a symmetric 2-design. As an example we construct a 2-(56, 12, 3) design with intersection numbers 3, 2 and 0 (=\( k - n \)), and with non-trivial class graph. This design turns out to be embeddable in a symmetric 2-(71, 15, 3) design.

1. Preliminaries

For general references about designs we refer to [5], [7] and [10]. Suppose \( \mathcal{D} \) is a 2-(\( v, k, \lambda \)) design of order \( n \) with \( b \) blocks and \( r \) blocks through a point, so \( bk = vr \), \( \lambda (v - 1) = r(k - 1) \), \( n = r - \lambda \), \( b \geq v \). The intersection numbers of \( \mathcal{D} \) are the possible values of \( |x \cap y| \), where \( x \) and \( y \) are two distinct blocks of \( \mathcal{D} \). We do not allow \( \mathcal{D} \) to have repeated blocks, so \( k \) is never an intersection number. A 2-design with one intersection number is a symmetric design (\( b = v \)). A 2-design with two intersection numbers is called quasi-symmetric. The following property of the intersection numbers will be frequently used, see [13].

**Result 1.1.** Let \( y \) be a block of \( \mathcal{D} \), let \( m_i \) be the number of other blocks in \( \mathcal{D} \) intersecting \( y \) in \( i \) points, \( 0 < i < k \). Then

\[
\sum_{i=0}^{k-1} m_i = b - 1, \quad \sum_{i=0}^{k-1} im_i = k(r - 1),
\]

\[
\sum_{i=0}^{k-1} i^2 m_i = k(r - 1) + k(k - 1)(\lambda - 1).
\]

A resolution of \( \mathcal{D} \) is a partition of its blocks into classes \( B_1, \ldots, B_e \) such that any point occurs a constant number of times in each class. Note that the constant equals \( k \mid B_i \mid /v \). A resolution is called strong if the number of points in the intersection of two distinct blocks depends only on whether these blocks are from different classes or from the same class. So if \( \mathcal{D} \) is strongly resolvable, which means \( \mathcal{D} \) admits a strong resolution, \( \mathcal{D} \) is quasi-symmetric or symmetric. The following is proved in [11].

**Result 1.2.** (i) \( \mathcal{D} \) is strongly resolvable if and only if \( \mathcal{D} \) has a resolution with \( b - v + 1 \) classes.

(ii) If \( \mathcal{D} \) is strongly resolvable then two distinct blocks of the same class have \( k - n \) points in common.

Now let \( \mathcal{D} \) be a symmetric 2-(\( v, k, \lambda \)) design (i.e., \( b = v = 1 + k(k - 1)/\lambda \)) with \( 2k < v \). Let \( x \) be a block of \( \mathcal{D} \) and let \( \mathcal{D}_x \) be the incidence structure
which is obtained from $\mathcal{D}$ by deleting the points of $x$ from the points of $\mathcal{D}$ and the blocks of $\mathcal{D}$ (disregarding the empty block). Then $\mathcal{D}_x$ is a 2-$(v - k, k - \lambda, \lambda)$ design of order $k - \lambda$, having $v - 1$ blocks and $k$ blocks through a point. We call $\mathcal{D}_x$ the residual design of $\mathcal{D}$ with respect to $x$. A design having the parameters of $\mathcal{D}_x$ is called a quasi-residual design. It is easily seen that a 2-$(v, k, \lambda)$ design of order $n$ is quasi-residual if and only if $k = n$.

We shall also need some definitions and results on strongly regular graphs for which we refer to [5], [8] or [15]. Throughout this paper $I$, $J$ and $j$ will denote respectively, the identity matrix, the all-one matrix, and the all-one vector of appropriate size.

**Result 1.3.** Given a graph $\Gamma$ with adjacency matrix $A$, $\Gamma$ is regular if and only if $j$ is an eigenvector of $A$; the corresponding eigenvalue is the valency. $\Gamma$ is strongly regular if and only if $\Gamma$ is regular and $A|j^4$ has just two eigenvalues.

The eigenvalues of a graph are the eigenvalues of its adjacency matrix.

**Result 1.4.** Let $\Gamma$ be a strongly regular graph on $p$ vertices with eigenvalues $\theta_1$, $\theta_2$ and the valency $\theta_0$. The complement of $\Gamma$ has eigenvalues $-1 - \theta_1$, $-1 - \theta_2$ and $p - 1 - \theta_0$ and the same multiplicities.

The disjoint union of $d$ complete graphs on $p$ vertices is a strongly regular graph, which we shall denote by $\Gamma(d, p)$. The complement of $\Gamma(d, p)$ is the complete $d$-partite graph on $dp$ vertices.

**Result 1.5.** A strongly regular graph with eigenvalues $\theta_1$, $\theta_2$ ($\theta_1 > \theta_2$) and the valency $\theta_0$, is a $\Gamma(d, p)$ if and only if $\theta_0 = \theta_1$.

## 2. The Intersection Number $k - n$

We quote the following result due to Majumdar [12].

**Result 2.1.** Given any two blocks $x$ and $y$ of a 2-design, the number of points in their intersection is at least $k - n$. Two distinct blocks, $x$ and $y$ intersect in exactly $k - n$ points if and only if $|x \cap z| = |y \cap z|$ for all blocks $z$ different from $x$ and $y$.

Hence if a 2-design, $\mathcal{D}$, has $k - n$ as an intersection number, and $x$ and $y$ are two blocks of $\mathcal{D}$, we may define $x \sim y$ if and only if $x = y$ or $|x \cap y| = k - n$. Then $\sim$ is an equivalence relation on the blocks of $\mathcal{D}$. The partition of the blocks into equivalence classes will be called the maximal decomposition of $\mathcal{D}$. A refinement of the maximal decomposition is called a decomposition of the design. A decomposition of a 2-design has the property that the number of points in the intersection of two blocks depends only on their decomposition classes. A decomposition is said to be regular if each class
THE INTERSECTION NUMBER \( k - n \)

contains the same number of blocks. We shall show in subsequent sections that if a 2-design has just two or three intersection numbers then the maximal decomposition is regular.

Before taking a closer look at decompositions we first examine what type of designs do admit an intersection number \( k - n \).

**Theorem 2.2.** If a 2-design admits an intersection number \( k - n \) then

(i) \( k - n \geq 0 \)

(ii) \( v - k - n \geq 0 \)

(iii) \( v - 1 \geq b - r \)

(iv) \( v - 1 \geq r \)

(v) \( k > \frac{1}{2}r \).

**Proof.** (i) is trivial. (ii): Take two blocks having \( k - n \) points in common, then the number of points in neither of these blocks equals \( v - k - n \).

(iii): Multiply (i) by \( (v - 1)/k \).

(iv): Multiply (ii) by \( (v - 1)/(v - k) \).

(v): Add (iii) and (iv).

From \( b \geq v \) it follows \( k - n \leq \lambda \) hence \( 0 \leq k - n \leq \lambda \). If \( k - n = \lambda \) then \( b = v \) and the 2-design is symmetric. We know that in a symmetric 2-design any two blocks have \( \lambda \) points in common. If \( k - n = 0 \) then the 2-design is quasi-residual; not all quasi-residual designs have an intersection number \( k - n = 0 \), but for instance all affine planes do. Since there are infinitely many affine planes and symmetric 2-designs, there exist infinitely many 2-designs with an intersection number \( k - n \) whenever \( k - n = 0 \) or \( k - n = \lambda \). However, following a suggestion by J. H. van Lint, we can prove that for other values of \( k - n \) this is not true.

**Theorem 2.3.** For a given value of \( k - n \notin \{0, \lambda \} \) there exist only finitely many 2-designs admitting the intersection number \( k - n \).

**Proof.** Take \( 0 < k - n < \lambda \). From \( bk = vr \) and \( \lambda(v - 1) = r(k - 1) \) it follows \( b = r(rk - n)/k\lambda \), hence \( k \mid rn \), thus \( k \mid (r - k)(k - n) \). So we have

\[
k \leq (r - k)(k - n) = (\lambda - (k - n))(k - n) \leq \frac{1}{2}\lambda^2.
\]

So given \( \lambda \) we have finitely many possibilities for \( k \), hence for \( n \), and hence for all parameters of the 2-design. This implies that for a given \( \lambda \) the number of such 2-designs is finite. Now put \( \alpha := (r - k)(k - n)/k \). With 2.2.v we have \( r - k < k \) hence \( 0 < \alpha < k - n \). Now using \( \lambda(v - 1) = r(k - 1) \) we have

\[
v - 1 = ((\lambda - k + n)(k - n) - \alpha) \\
\times (\alpha\lambda - \alpha(k - n) + (\lambda - k + n)(k - n))/\alpha^2\lambda.
\]
Thus
\[ \lambda \mid ((k - n)^2 + \alpha)(k - n)(k - n + \alpha). \]

Now with \(0 < \alpha < k - n\) we have \(\lambda < 2(k - n)^3(k - n + 1)\). So for a given value of \(k - n\), only finitely many values of \(\lambda\) apply. With the former result it follows that then the number of such 2-designs is finite. □

From the proof of 2.3 we have \(0 < \alpha < k - n\). This implies that \(k - n = 1\) cannot occur. Also we have seen

**Corollary 2.4.** For a given value of \(\lambda\) there exist only finitely many 2-designs with an intersection number \(k - n \notin \{0, \lambda\}\).

We also point out that the total number of 2-designs admitting an intersection number \(k - n\), satisfying \(0 < k - n < \lambda\), is not finite. Indeed the complements of affine planes do the job.

**Remark.** One could ask whether a \(t\)-design with \(t > 2\) can have an intersection number \(k - n\). The answer is affirmative if and only if it is a Hadamard 3-design. This follows easily from 2.2 and some properties of Hadamard 3-designs, cf. [5].

Designs admitting a nontrivial strong tactical decomposition, see [1], [2], [3] and [14] are examples of 2-designs with an intersection number \(k - n\). This includes all strongly resolvable designs. In the next section we will obtain conditions for a 2-design, having an intersection number \(k - n\), to be strongly resolvable.

### 3. Inequalities

**Theorem 3.1.** If \(B_1, \ldots, B_e\) are the classes of a decomposition of a 2-design \(\mathcal{D}\) then
\[ |B_j| \leq \frac{b}{b - v + 1}, \]
with equality for all \(j\) if and only if \(B_1, \ldots, B_e\) are the classes of a resolution of \(\mathcal{D}\).

**Proof.** Let \(A_j\) be the matrix of size \(v \times |B_j|\) giving the incidence between the points of \(\mathcal{D}\) and the blocks of \(B_j\). Then
\[ A_j^T A_j = (k - n)J + nI, \quad A_j^T J = kJ. \]

Hence
\[ 0 \leq \mathbf{j}^T \left( A_j - \frac{k}{v} J \right)^T \left( A_j - \frac{k}{v} J \right) \mathbf{j} \]
\[ = \mathbf{j}^T (nI + (k - n - k^2/v) J) \mathbf{j} \]
\[ = |B_j| (n + |B_j| (k - n - k^2/v)). \]
With \( bk = vr \) and \( \lambda(v - 1) = r(k - 1) \) this implies

\[
n - n | B_j | (b - v + 1)/b \geq 0,
\]

which yields the desired inequality.

Clearly equality holds if and only if

\[
\left( A_j - \frac{k}{v} J \right) j = 0,
\]

which is equivalent to \( A_j j = k \mid B_j \mid /v \). This means that every point of \( D \) is in \( k \mid B_j \mid /v \) blocks of \( B_j \). Hence we have equality for all \( j \) if and only if \( B_1, \ldots, B_c \) are the classes of a resolution of \( D \).

**COROLLARY 3.2.** If a decomposition of a 2-design is regular then

\[
v + c \geq b + 1
\]

with equality if and only if the decomposition is a resolution and hence a strong resolution.

**Proof.** Since the decomposition is regular we have \( |B_j| = b/c \) for all \( j \). Hence 3.1 gives the result.

From now on \( m \) will denote the number of blocks in a class of a regular decomposition. The number of classes is always denoted by \( c \).

**THEOREM 3.3.** If \( D \) is a nonsymmetric 2-design with a regular decomposition then

\[
k - n < \lambda v/b < n/m + k - n.
\]

**Proof.** With \( bk = vr \) we have \( \lambda v/b - (k - n) = (r - \lambda)(b - v)/b \). Hence by \( b > v \) we have \( k - n < \lambda v/b \).

Theorem 3.1 implies \( m \leq b/(b - v + 1) \), hence

\[
n/m + k - n \geq n(b - v + 1)/b + k - n
\]

\[= (-nv + n + bk)/b = (-nv + n + vr)/b\]

\[= \lambda v/b + n/b > \lambda v/b.\]

It will return out that the numbers \( \lambda v/b \) and \( n/m + k - n \) play a special role with respect to regular decompositions. The next theorem shows that \( n/m + k - n \) is an upper bound for the intersection numbers.
THEOREM 3.4. Let $\mathcal{D}$ be a 2-design with a regular decomposition and let $\rho_{ij}$ be the number of points in the intersection of two blocks from different classes $B_i$ and $B_j$. Then

(i) $\rho_{ij} \geq 2k^2/v - n/m - k + n$

with equality if and only if all points of $\mathcal{D}$ are in the same number of blocks from $B_i \cup B_j$, and

(ii) $\rho_{ij} \leq n/m + k - n$

with equality if and only if each point occurs the same number of times in blocks from $B_i$ as in blocks from $B_j$.

Proof. As in the proof of Theorem 3.1 let $A_i$ denote the matrix describing the incidence between the points of $\mathcal{D}$ and the blocks of $B_i$. Put

$$N = [A_i - k/v J | A_j - k/v J]$$

then

$$N^TN = \begin{bmatrix}
iI + (k - n - k^2/v) J & (\rho_{ij} - k^2/v) J \\
(\rho_{ij} - k^2/v) J & nI + (k - n - k^2/v) J
\end{bmatrix}$$

So we have

$$0 \leq j^TN^TNj = 2m^2(k - n - 2k^2/v + \rho_{ij}) + 2mn,$$

hence

$$\rho_{ij} \geq 2k^2/v - k + n - n/m.$$

Clearly equality holds if and only if $Nj = 0$, which is equivalent to $[A_i A_j]j = 2mk/v j$. This proves (i). Now define

$$\tilde{j} := (j^T - j^T)^T,$$

with $j$ and $-j$ both of length $m$.

Then $0 \leq \tilde{j}^TN^TN\tilde{j} = 2m^2(k - n - \rho_{ij}) + 2mn$, hence $\rho_{ij} \leq n/m + k - n$. Equality holds if and only if $N\tilde{j} = 0$, which means $A_i\tilde{j} = A_j\tilde{j}$. This proves (ii).

Majumdar [12] proved that any intersection number $\rho$ of a 2-design satisfies

$$k - n \leq \rho \leq 2\lambda v/b - k + n.$$
similar thing. Define two classes $B_i$ and $B_j$ to be related if and only if $i = j$ or $\rho_{ij} = n|m + k - n$, then from 3.4.ii it follows immediately that we have an equivalence relation.

In the case when $\mathcal{D}$ is strongly resolvable it can be seen that both bounds of Theorem 3.4 are tight.

4. REGULAR DECOMPOSITIONS

We now show that every 2-design with exactly three intersection numbers has a regular maximal decomposition.

**Lemma 4.1.** Let $\mathcal{D}$ be a 2-design with three intersection numbers $\rho_1$, $\rho_2$, and $\rho_3$. Then the number of blocks that intersect a given block in $\rho_i$ points $(i = 1, 2, 3)$ is a constant independent of the block chosen.

**Proof.** Let $y$ be any block of $\mathcal{D}$ and let $y$ intersect $x_1$ blocks in $\rho_1$ points and $x_2$ blocks in $\rho_2$ points. Then by result 1.1 we have

\[
k + x_1\rho_1 + x_2\rho_2 + (b - 1 - x_1 - x_2)\rho_3 = rk
\]

\[
k^2 + x_1\rho_1^2 + x_2\rho_2^2 + (b - 1 - x_1 - x_2)\rho_3^2 = \lambda k^2 + nk.
\]

Eliminating one of the $x_i$ $(i = 1, 2)$ from the two equations above we find

\[
x_i = \frac{\lambda k^2 + nk - k^2 + (b - 1)\rho_j\rho_k - k(r - 1)(\rho_j + \rho_k)}{(\rho_i - \rho_j)(\rho_i - \rho_k)} \quad (*)
\]

where $i, j, k$ are all distinct values 1, 2 or 3. Hence $x_i$ is independent of the chosen block. 

**Corollary 4.2.** Let $\mathcal{D}$ be a 2-design with exactly three intersection numbers $k - n$, $\rho_1$ and $\rho_2$. The maximal decomposition is regular, and the size $m$ of the classes equals

\[
\frac{\lambda k^2 - kn + n^2 + b\rho_1\rho_2 - \lambda v(\rho_1 + \rho_2)}{(k - n - \rho_1)(k - n - \rho_2)}
\]

**Proof.** Take $\rho_i = k - n$ in (*). The size of the class containing a given block $y$ equals $x_i + 1$.

It is clear that this method of proof cannot be used to prove regularity of the maximal decomposition of a 2-design with four or more intersection numbers one of which is $k - n$. In fact, the design of Bhattacharya provides a counterexample; cf. [12].
COROLLARY 4.3. Let \( D \) be a 2-design with exactly three intersection numbers: \( k - n, \rho_1 \) and \( \rho_2 \). Then \( \rho_1 = n/m + k - n \) if and only if \( \rho_2 = \lambda v/b \).

Proof. This follows from Corollary 4.2 by straightforward verification.

Although we have no counterexample it is not likely that a 2-design with a regular maximal decomposition needs to have three or less intersection numbers. However in some special cases this is true. One of these is a result of Singhi and Shrikhande [16].

Result 4.4. If a quasi-residual 2-design has a regular decomposition with \( \lambda - 1 \) blocks in each class, then the only possible other intersection numbers are \( \lambda \) and \( \lambda - 1 \).

Another case is the corollary of the next theorem.

THEOREM 4.5. Let \( D \) be a design with a block \( x \), which intersects \( m_x - 1 \) blocks in \( k - n \) points. Suppose \( \lambda v/b \) and \( n/m_x + k - n \) are consecutive integers. Then for any other block \( y \) of \( D \) we have \( |x \cap y| = \lambda v/b \) or \( |x \cap y| = n/m_x + k - n \).

Proof. Let \( x_i \) be the number of blocks that intersect \( x \) in \( i \) points. Then \( x_{k-n} = m_x - 1 \) and \( x_i = 0 \) if \( 0 < i < k - n \).

Using result 1.1 we have

\[
\sum_{i=k-n+1}^{k-1} (i - \lambda v/b)(i - (n/m_x + k - n)) x_i
\]

\[
= \lambda k^2 + nk - k^2 - (m_x - 1)(k - n)^2 - (v/b + n/m_x + k - n)
\]

\[
\times (rk - m_x k + m_x n - n) + \lambda v/b(n/m_x + k - n)(b - m_x)
\]

\[
= nrk - n^2 - \lambda v n - rk^2 + nk + \lambda v k - (nrk - n^2 - \lambda v n)/m_x
\]

\[
= 0 \quad \text{on applying } \lambda(v - 1) = r(k - 1).
\]

Thus if \( \lambda v/b \) and \( n/m_x + k - n \) are consecutive integers all terms of (*) are nonnegative and so we must have \( x_i = 0 \) for all \( i \neq k - n, \lambda v/b \) or \( n/m_x + k - n \).

In the case when all \( m_x \) are equal Theorem 4.5 implies:

COROLLARY 4.6. If \( D \) is a design with a regular maximal decomposition and \( \lambda v/b \) and \( n/m + k - n \) are consecutive integers, then \( D \) has at most three intersection numbers.

The family of 2-designs admitting a strong tactical decomposition which was constructed in [3] yields examples of both 4.4 and 4.6.
5. Strongly Regular Graphs

Let \( \mathcal{D} \) be any quasi-symmetric 2-design with intersection numbers \( \rho_1 \) and \( \rho_2 \) (take \( \rho_1 > \rho_2 \)). Define the block graph of \( \mathcal{D} \) to be the graph whose vertices are the blocks of \( \mathcal{D} \), two vertices being adjacent if and only if the corresponding blocks have \( \rho_1 \) points in common. Goethals and Seidel proved the following result (cf. [5] or [8]).

Result 5.1. The block graph of a quasi-symmetric 2-design is strongly regular.

We shall show later on that a similar method may be used to obtain a strongly regular graph from a design with three intersection numbers, provided one of them is \( k - n \). First we consider the case of two intersection numbers one of which is \( k - n \).

Lemma 5.2. Let \( \mathcal{D} \) be a quasi-symmetric 2-design. Then

(i) the number of blocks that intersect a given block in \( \rho_1 \) points equals

\[
\frac{k(r - 1) - \rho_2(b - 1)}{\rho_1 - \rho_2},
\]

(ii) \( \rho_1 = \frac{k^2}{v} \) if and only if \( \rho_2 = k - n \).

Proof. Let \( y \) be any block of \( \mathcal{D} \) and let \( x_1 \) denote the number of blocks that intersect \( y \) in \( \rho_1 \) points (we know this number is a constant by result 5.1). By result 1.1

\[
x_1 \rho_1 + (b - x_1 - 1) \rho_2 = k(r - 1),
\]

i.e.,

\[
x_1 = \frac{k(r - 1) - \rho_2(b - 1)}{\rho_1 - \rho_2}.
\]

(1)

Also by result 1.1 we have

\[
k^2 + x_1 \rho_1^2 + (b - x_1 - 1) \rho_2^2 = \lambda k^2 + nk.
\]

(2)

Hence eliminating \( x_1 \) in (1) and (2) gives

\[
\rho_1(kr - k - bp_2 + \rho_2) - \lambda k^2 + nk + k\rho_2 - rk\rho_2 - k^2,
\]

from which (ii) follows by straightforward verification. \( \blacksquare \)

Theorem 5.3. For a 2-design \( \mathcal{D} \) the following are equivalent

(i) \( \mathcal{D} \) is quasi-symmetric with an intersection number \( k - n \),

(ii) \( \mathcal{D} \) is strongly resolvable,

(iii) \( \mathcal{D} \) is quasi-symmetric and its block graph is a complete \( c \)-partite graph.
(i) \rightarrow (ii): According to Lemma 5.2, \( \mathcal{D} \) has a maximal decomposition in which every class has size

\[
\frac{k(r - 1) - (b - 1) k^2/v}{k - n - k^2/v} + 1 = b/(b - v + 1)
\]
on applying \( bk = vr \). Thus by Theorem 3.1 the maximal decomposition is a resolution which must be strong by Corollary 3.2.

(ii) \rightarrow (iii): By definition \( \mathcal{D} \) is quasi-symmetric and two vertices of its block graph are adjacent if and only if the corresponding blocks belong to different classes, so it is a complete \( c \)-partite graph.

(iii) \rightarrow (i): Take two nonadjacent vertices of the complete \( c \)-partite graph. We know that every other vertex is either adjacent or nonadjacent to both. This means that for the corresponding two blocks of \( \mathcal{D} \) we have the property that every further block intersects both blocks in the same number of points. With result 2.1 it follows that the two blocks intersect in \( k - n \) points. \( \square \)

From now on \( \mathcal{D} \) is a 2-design with possibly an intersection number \( k - n \) and just two further intersection numbers \( \rho_1 \) and \( \rho_2 \) (\( \rho_1 > \rho_2 \)). Let the class graph of \( \mathcal{D} \) be the graph whose vertices are the classes of the maximal decomposition, where two vertices are adjacent if and only if two blocks, one from each of the corresponding classes, meet in \( \rho_1 \) points. So the number of vertices of the class graph equals the number \( c \) of classes.

**Theorem 5.4.** The class graph of \( \mathcal{D} \) is strongly regular with eigenvalues

\[
\theta_0 = \frac{\lambda vc - b(k - n - \rho_2 + \rho_2 c)}{b(\rho_1 - \rho_2)}
\]

\[
\theta_1 = \frac{\rho_2 - k + n}{\rho_1 - \rho_2}
\]

\[
\theta_2 = \frac{b(\rho_2 - k + n - cn)}{b(\rho_1 - \rho_2)}
\]

and multiplicities 1, \( c - b + v - 1 \) and \( b - v \), respectively.

**Proof.** Let \( N \) be the incidence matrix of \( \mathcal{D} \). Then \( N \) satisfies \( NNT = \lambda J + nI \). This implies that \( NNT \) has eigenvalues \( n \) with multiplicity \( v - 1 \) and \( rk \) with multiplicity one. Hence \( N^TN - nI \) has eigenvalues \( -n, 0 \) and \( \lambda v \) where \( \lambda v \) has multiplicity one. By Result 2.1 and Corollary 4.2 we can write

\[
N^TN - nI = D^TB
\]
where $B$ is a symmetric $c \times c$ matrix with $k - n$ on the diagonal and $\rho_1$ and $\rho_2$ elsewhere, and $D$ is the $c \times b$ matrix:

$$
D = \begin{bmatrix}
    m & m & m \\
    1 & \cdots & 1 \\
    1 & \cdots & 1 \\
    \vdots & \ddots & \vdots \\
    1 & \cdots & 1 \\
\end{bmatrix}.
$$

Suppose $\alpha$ is an eigenvalue of $B$ with eigenvector $a$. Then $\alpha a = Ba = \frac{1}{m} BDD^T a$, since $DD^T = mI$. Thus $D^T BDD^T a = \alpha m D^T a$. So any eigenvalue of $B$ multiplied by $m$ is an eigenvalue of $N^T N - nI$.

Hence the only possible eigenvalues of $B$ are $(\lambda - r)/m$, 0 and $\lambda v/m$, where $\lambda v/m$ is a simple eigenvalue belonging to the all-one eigenvector $j$. Define

$$
A = \frac{1}{\rho_1 - \rho_2} (B - \rho_2 J - (k - n - \rho_2) I).
$$

Then $A$ is the adjacency matrix of the class graph of $\mathcal{D}$. Clearly

$$
\theta_0 = \frac{\lambda v c - b (k - n - \rho_2 + \rho_2 c)}{b (\rho_1 - \rho_2)}
$$

is an eigenvalue of $A$ belonging to the eigenvector $j$. The only two possible eigenvalues of $A | j^k$ are

$$
\theta_1 = \frac{\rho_2 - k + n}{\rho_1 - \rho_2} \quad \text{and} \quad \theta_2 = \frac{b (\rho_2 - k + n) - cn}{b (\rho_1 - \rho_2)}.
$$

Now let $f_1$ and $f_2$ be the multiplicities of $\theta_1$ and $\theta_2$ respectively. Then we have $f_1 + f_2 + 1 = c$ and $\theta_1 f_1 + \theta_2 f_2 + \theta_0 = \text{trace}(A) = 0$, hence $f_1 = (\theta_2 - c \theta_2 - \theta_0)/(\theta_1 - \theta_2) = c - b - 1 + (bk - \lambda v)/n = v + c - b - 1$ on applying $bk = vr$. Thus $f_2 = c - 1 - (v + c - b - 1) = b - v$. By Theorem 5.3 $\mathcal{D}$ is not strongly resolvable and hence Corollary 3.2 gives $v + c - b - 1 > 0$. $\mathcal{D}$ is clearly not symmetric, so $b - v > 0$, and both eigenvalues $\theta_1$ and $\theta_2$ do occur. Now result 1.3 implies that our graph is strongly regular.

**Corollary 5.5.** If two blocks $x$ and $y$ of $\mathcal{D}$ intersect in $\rho_i$ points then the number of blocks intersecting $x$ in $\rho_j$ points and $y$ in $\rho_k$ points is a constant dependent on $i$, $j$ and $k$ but independent of $x$ and $y$, for all $i, j, k \in \{1, 2\}$.

**Proof.** This follows immediately from Theorem 5.4 and the definition of strongly regular graphs. The values of these constants can be readily calculated from the eigenvalues of the graph.
If all classes of the maximal decomposition have size one, so \( c = b \), then \( \mathcal{D} \) is quasi-symmetric and Theorem 5.4 reduces to Result 5.1.

**Remark.** It does not often happen that a 2-design with more than 2 intersection numbers has the property that the block incidence structure gives an association scheme (or equivalently the dual of the design is a PBIBD). However from Corollary 5.5 it follows that \( \mathcal{D} \) has this property.

The class graph of a 2-design constructed in [1], [3] or [14] is always a \( \Gamma(d, p) \). This is easily checked with the help of the following lemma.

**Lemma 5.6.** The class graph of \( \mathcal{D} \) is a \( \Gamma(d, p) \) if and only if \( \rho_2 = \lambda v/b \).

**Proof.** By Result 1.5 the graph is a \( \Gamma(d, p) \) if and only if \( \theta_1 = \theta_0 \), i.e., if and only if \( \rho_2 = \lambda v/b \). \( \blacksquare \)

We remark that the "if" part of Lemma 5.6 is also a consequence of 3.4.ii and 4.3. Lemma 5.6 implies that the class graph of a 2-design satisfying Corollary 4.6 is a \( \Gamma(d, p) \). In Section 7 we construct a design \( \mathcal{D} \) whose class graph is not a \( \Gamma(d, p) \).

### 6. Quasi-Residual Designs

Throughout this section let \( \mathcal{D} \) be a quasi-residual 2-\( (k - 1)(k - \lambda)/\lambda, k - \lambda, \lambda \) design with intersection numbers \( \rho_1, \rho_2 \) (\( \rho_1 > \rho_2 \)) and possibly \( k - n = 0 \). Then \( \mathcal{D} \) has \( k(k - 1)/\lambda \) blocks and \( k \) blocks through a point, and the symmetric design into which it possibly embeds is a 2-(\( v, k, \lambda \)) design where \( \lambda(v - 1) = k(k - 1) \).

**Lemma 6.1.** Let \( \mathcal{D} \) be a quasi-symmetric 2-(\( k, \lambda, (\lambda - 1)/m \)) design (\( m \) is the size of the classes of \( \mathcal{D} \)). Then the block graph of \( \mathcal{D} \) has the same eigenvalues as the complement of the class graph of \( \mathcal{D} \) if and only if the intersection numbers of \( \mathcal{D} \) are \( \lambda - \rho_1 \) and \( \lambda - \rho_2 \).

**Proof.** With Theorem 5.4 and Result 1.4 it follows that the eigenvalues of the complement of the class graph of \( \mathcal{D} \) are \( (\rho_1 - 1)(k - 1)(k - \lambda)/\lambda, (k - \lambda)/\lambda(m - \rho_1)/\rho_1 - \rho_2) \) and \( -\rho_1/(\rho_1 - \rho_2) \) with multiplicities 1, \( k - 1 \) and \( c - k \) respectively. Let \( \tilde{\rho}_1 \) and \( \tilde{\rho}_2 \) be the intersection numbers of \( \mathcal{D} \). Again using Theorem 5.4 we find that the block graph of \( \mathcal{D} \) has eigenvalues \( ((\lambda - \tilde{\rho}_2)(k - 1)(k - \lambda)/\lambda(m - \tilde{\rho}_1))/\tilde{\rho}_1 - \tilde{\rho}_2, ((k - \lambda)/\lambda(m + \tilde{\rho}_2 - \lambda))/\tilde{\rho}_1 - \tilde{\rho}_3 \) and \( (\tilde{\rho}_2 - \lambda)/\tilde{\rho}_1 - \tilde{\rho}_3 \)) with multiplicities 1, \( k - 1 \) and \( c - k \) respectively. We observe that the eigenvalues of both graphs are the same if and only if \( \tilde{\rho}_1 = \lambda - \rho_2 \) and \( \tilde{\rho}_2 = \lambda - \rho_1 \). \( \blacksquare \)
THEOREM 6.2. $\mathcal{D}$ is embeddable in a symmetric 2-design $\mathcal{I}$ if and only if there exists a quasi-symmetric 2-$(k, \lambda, (\lambda - 1)/m)$ design $\mathcal{D}$, whose block graph is isomorphic to the complement of the class graph of $\mathcal{D}$.

Proof. Suppose $\mathcal{D}$ exists. Let $\Phi$ be the given isomorphism, mapping the classes of $\mathcal{D}$ onto the blocks of $\mathcal{I}$. We extend $\mathcal{D}$ to $\mathcal{I}$ as follows: The points of $\mathcal{D}$ together with the points of $\mathcal{I}$ are the points of $\mathcal{I}$. If $x$ is a block of $\mathcal{D}$ belonging to the class $B(x)$, say, then $x \cup \Phi(B(x))$ is a block of $\mathcal{I}$; the pointset of $\mathcal{D}$ is a block of $\mathcal{I}$.

It immediately follows that $\mathcal{I}$ has $b = v = k(k - 1)/\lambda + 1$ and $k$ blocks through every point. We now prove that any two blocks of $\mathcal{I}$ intersect in $\lambda$ points. Let $x \cup \Phi(B(x))$ and $y \cup \Phi(B(y))$ be two distinct blocks of $\mathcal{I}$ then there are two cases:

(i) $|x \cap y| = 0$, then $B(x) = B(y)$ thus $\Phi(B(x)) = \Phi(B(y))$, so $|(x \cup \Phi(B(x))) \cap (y \cup \Phi(B(y)))| = \lambda$, i.e., the number of points in a block of $\mathcal{D}$.

(ii) $|x \cap y| = \rho_i$ ($i = 1, 2$), then $\Phi(B(x)) \cap \Phi(B(y)) = \lambda - \rho_i$ by Lemma 6.1, and $|(x \cup \Phi(B(x))) \cap (y \cup \Phi(B(y)))| = \lambda$.

Clearly the block made up of the points of $\mathcal{D}$ intersects every other block of $\mathcal{I}$ in $\lambda$ points. Thus any two blocks have $\lambda$ points in common. Therefore the dual of $\mathcal{I}$, and hence $\mathcal{I}$ itself, is a symmetric 2-$(v, k, \lambda)$ design. Conversely, let $\mathcal{I}$ be a symmetric design and suppose $\mathcal{D}$ is the residual design of $\mathcal{I}$ with respect to a block $z$, say. Let $\mathcal{D}$ be the incidence structure whose point set is $z$ and whose blocks are the intersections of $z$ with the other blocks of $\mathcal{I}$. From the block incidence structure of $\mathcal{D}$ it now follows that $\mathcal{D}$ consists of $m$ copies of a quasi-symmetric 2-design, whose block graph is isomorphic to the complement of the class graph of $\mathcal{D}$. 

Theorem 6.2 is easy to apply if the isomorphism is forced by the eigenvalues. This gives rise to the following corollaries.

COROLLARY 6.3. A quasi-residual 2-$(((k - 1)(k - \lambda))/\lambda, k, \lambda, \lambda)$ design $\mathcal{D}$ with three intersection numbers $0, \lambda(k - \lambda)/k$ and $(k - \lambda)/m$ is embeddable in a symmetric 2-$(v, k, \lambda)$ design if and only if there exists a strongly resolvable 2-$(k, \lambda, (\lambda - 1)/m)$ design $\mathcal{D}$.

Proof. Theorem 5.3 implies that $\mathcal{D}$ is quasi-symmetric and that its block graph is the complete $(c - k + 1)$-partite graph, determined by its eigenvalues. By Lemma 6.1 this graph has the same eigenvalues and hence is isomorphic to the complement of the class graph of $\mathcal{D}$. Now apply Theorem 6.2.

COROLLARY 6.4. A quasi-residual 2-$(\frac{1}{2}(\lambda + 2)(\lambda - 1)^2, \frac{1}{2}\lambda(\lambda - 1), \lambda)$ design $\mathcal{D}$ with intersection numbers $\lambda - 1$, $\lambda - 2$ and possibly 0 is embeddable in a
symmetric 2-(\frac{1}{3}(\lambda + 2)(\lambda^2 - 1) + 1, \frac{1}{3}\lambda(\lambda + 1), \lambda) design, if and only if there exists a biplane of order \lambda.

Proof. A biplane of order \lambda is a symmetric 2-(\frac{1}{3}(\lambda + 2)(\lambda + 1) + 1, \lambda + 2, 2) design. According to Hall and Connor [10], a biplane of order \lambda exists, if and only if a 2-(\frac{1}{3}\lambda(\lambda + 1), \lambda, 2) design \mathcal{D} exists. \mathcal{D} is quasi-symmetric by Result 4.4, and its block graph has the eigenvalues of the complement of the triangular graph of order \lambda + 2, which for \lambda \neq 6 is characterized by its eigenvalues, see [15]. From Lemma 4.2 it follows \( m = \frac{1}{3}(\lambda - 1) \), so here \( \lambda = 6 \) cannot occur. Lemma 6.1 implies that the two involved graphs have the same eigenvalues, and thus are isomorphic. Now Theorem 6.2 gives the result.

Examples for Corollary 6.3 can be found in [3] and [14]. Unfortunately we do not have any examples for Corollary 6.4 except when \( \lambda = 3 \), but then \( m = 1 \) and the result is trivial. The next section is devoted to an example of a design, which satisfies Theorem 6.2.

7. Example

In this section we construct a 2-(56, 12, 3) design \( \mathcal{D} \) with three intersection numbers 0 (==k - n), 2 and 3. \( \mathcal{D} \) turns out to be embeddable in a symmetric 2-(71, 15, 3) design \( \mathcal{S} \). Both designs were as far as we know not known to exist before.

The points of \( \mathcal{D} \) are represented by the fiftysix ordered pairs of different elements of \( GF(8) \). The blocks of \( \mathcal{D} \) are represented by the seventy 4-subsets of \( GF(8) \). Let \( \alpha \), satisfying \( \alpha^3 = \alpha + 1 \) be a primitive element of \( GF(8) \). We represent the element \( \alpha^i \) by \( i \). We write 0 for the zero of the field. Now we define the blocks of \( \mathcal{D} \) through the point \((0, 0)\) to be

\[
\begin{align*}
\{0, 0, 1, 2\}, & \quad \{0, 0, 2, 4\}, \quad \{0, 0, 4, 1\}, \\
\{0, 0, 2, 3\}, & \quad \{0, 0, 4, 6\}, \quad \{0, 0, 1, 5\}, \\
\{0, 0, 1, 3\}, & \quad \{0, 0, 2, 6\}, \quad \{0, 0, 4, 5\}, \\
\{1, 2, 3, 4\}, & \quad \{2, 4, 6, 1\}, \quad \{4, 1, 5, 2\}, \\
\{2, 3, 4, 5\}, & \quad \{4, 6, 1, 3\}, \quad \{1, 5, 2, 6\}.
\end{align*}
\]

Let \( H \) be the group of order fiftysix represented by \( x \to ax + b, a, b \in GF(8), a \neq 0 \). Then \( H \) acts 2-transitively on the elements of \( GF(8) \), and hence transitively on the points of \( \mathcal{D} \). Now by letting \( H \) act on the points and blocks of our design, \( \mathcal{D} \) is defined. Before we show that \( \mathcal{D} \) is a 2-design we state the following lemma.
LEMMA 7.1. Let \((a, b)\) be a point of \(\mathcal{D}\). Of the fifteen 4-subsets of \(GF(8)\) that contain the pair \(\{a, b\}\) either the 4-subset itself or the complementary 4-subset is a block incident with \((a, b)\).

Proof. If \((a, b) = (\emptyset, 0)\) it is easily verified. If \((a, b) \neq (\emptyset, 0)\) it follows from the transitivity of \(H\). \(\square\)

THEOREM 7.2. \(\mathcal{D}\) is a 2-(56, 12, 3) design with intersection numbers 0, 2 and 3.

Proof. First observe that the map \(x \rightarrow x^2\), in \(GF(8)\) stabilizes the point \((\emptyset, 0)\) and the set of blocks through this point. This implies that the group \(H^*\) defined by \(x \rightarrow ax^2 + b, a, b \in GF(8), i \in \mathbb{Z}, a \neq \emptyset\) is an automorphism group of \(\mathcal{D}\).

We now show that each block of \(\mathcal{D}\) contains twelve points. It is easily checked that \(H^*\) has two orbits on the blocks of \(\mathcal{D}\), of sizes fifty-six and fourteen. Hence there are two blocks corresponding to complementary 4-subsets in one orbit. However Lemma 7.1 implies that the two blocks are disjunct and together contain all the twenty-four points, corresponding to the pairs contained in one of the two 4-subsets. Thus those two blocks, whence all blocks in that orbit, contain twelve points. But the sum of all block sizes equals \(15 \cdot 36 = 70 \cdot 12\), thus all blocks must have size twelve.

Now we check that any pair of points is incident with three blocks. Obviously \(H^*\) acts transitively on the points of \(\mathcal{D}\), and it is easily seen that \(H^*\) has eleven orbits on the unordered pairs of the points of \(\mathcal{D}\) for which the following pairs of points are representatives:

\[
\begin{align*}
\{(\emptyset, 0), (0, \emptyset)\}, & \quad \{(\emptyset, 0), (0, 1)\}, \quad \{(\emptyset, 0), (1, \emptyset)\}, \\
\{(\emptyset, 0), (3, \emptyset)\}, & \quad \{(\emptyset, 0), (1, 3)\}, \quad \{(\emptyset, 0), (3, 1)\}, \\
\{(0, 0), (5, 0)\}, & \quad \{(0, 0), (4, 2)\}, \quad \{(0, 0), (2, 4)\}, \\
\{(0, 0), (5, 3)\}, & \quad \{(0, 0), (3, 5)\}.
\end{align*}
\]

By verification it follows that each of these pairs of points of \(\mathcal{D}\) has three blocks incident with both. So \(\mathcal{D}\) is a 2-(56, 12, 3) design. Lemma 7.1 implies that \(\mathcal{D}\) admits a regular decomposition with \(m = 2\). From Result 4.4 it follows that 2 and 3 are the only other intersection numbers. \(\square\)

Let \(G\) be the complement of the class graph of \(\mathcal{D}\) then Theorem 5.4 gives the eigenvalues of \(G\): 18, 3 and \(-3\) with multiplicities 1, 14 and 20 respectively. If we want to apply Theorem 6.2 we need a 2-(15, 3, 1) design \(\mathcal{G}\) (which obviously is quasi-symmetric because \(\lambda - 1\)) whose block graph is isomorphic to \(G\). There are many designs with the parameters of \(\mathcal{G}\), and there are even more strongly regular graphs with the eigenvalues of \(G\), see [4]. So
we have to examine $G$ more deeply. For this purpose we need the following result, see [4] or [6].

**Result 7.3.** The thirty-five lines of $PG(3, 2)$ can be represented by the thirty-five partitions of an 8-set into 4-subsets, such that two lines of $PG(3, 2)$ intersect if the intersections of the sets of the two partitions form a partition into four 2-subsets, and lines are skew if those intersections form a partition into two 3-subsets and two 1-subsets.

**Remark.** It is well-known, see [6], that the automorphism group of the lines of $PG(3, 2)$ is isomorphic to $S_8$. This in fact is equivalent to the above result.

**Lemma 7.4.** $G$ is isomorphic to the graph $\Gamma$, whose vertices are the lines of $PG(3, 2)$ where two vertices are adjacent if and only if the lines intersect.

**Proof.** Let $\mathcal{E}$ be the incidence structure whose points are the points of $\mathcal{D}$ and whose blocks are the classes of $\mathcal{D}$, incidence meaning that the point is in one of the blocks of the class. Suppose $\{x, x'\}$ and $\{y, y'\}$ are two classes of $\mathcal{D}$, then $x \cup x'$ and $y \cup y'$ are blocks of $\mathcal{E}$ and $|(x \cup x') \cap (y \cup y')| = 4 |x \cap y|$, on applying Result 2.1. On the other hand Lemma 7.1 implies that a point and a block of $\mathcal{D}$ are incident if and only if the corresponding pair from $GF(8)$ is contained in one of the two complementary 4-subsets of $GF(8)$. So the number of points that two blocks of $\mathcal{D}$ have in common only depends on the intersection pattern of the two pairs of complementary 4-subsets. On applying Result 7.3 it now follows that $G$ is isomorphic to $\Gamma$ or its complement. By looking at the valencies of $G$ and $\Gamma$ it follows that $G$ cannot be isomorphic to the complement of $\Gamma$. 

**Theorem 7.5.** $\mathcal{D}$ is embeddable in a symmetric 2-(71, 15, 3) design.

**Proof.** Let $\mathcal{D}$ be the 2-(15, 3, 1) design formed by the points and lines of $PG(3, 2)$. Then according to Lemma 7.4 the block graph of $\mathcal{D}$ is isomorphic to $G$. Now Theorem 6.2 applies.

Surprisingly the above construction leads to several nonisomorphic 2-(71, 15, 3) designs. For this and further properties of these designs we refer to [9].

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